

A sufficient condition for splitting of arithmetically Cohen-Macaulay bundles on general hypersurfaces

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Introduction

k : an algebraically closed field,

$X \subset \mathbb{P}_k^n$: a hypersurface,

$\mathcal{O}_X(1)$: the very ample line bundle induced by the natural embedding.

We say that a vector bundle $\mathcal{E}$ on X *splits* if it is isomorphic to $\bigoplus \mathcal{O}_X(l_i)$ ($l_i \in \mathbb{Z}$), and is *arithmetically Cohen-Macaulay (ACM)* if $H^i(X, \mathcal{E}(l)) = 0$ for $1 \leq i \leq n - 2$ and all $l \in \mathbb{Z}$.

Various criteria for vector bundles on hypersurfaces to split have been studied by many authors. For vector bundles on projective spaces, Horrocks has established the following criterion:

Theorem(Horrocks). A vector bundle on $\mathbb{P}_k^n$ ($n \geq 2$) splits if and only if it is ACM.

This theorem fails on hypersurfaces of higher degree in general.

Question. When do ACM bundles split?

Main theorem

Theorem. $n, d, r \in \mathbb{Z}$ ($n \geq 3, d \geq 2, r \geq 1$).

$$rd \left(n + 1 + \binom{n + d - 1}{d - 1} \right) + 2 \leq \binom{n + d}{d}$$

$\Rightarrow$ all ACM bundles of rank r on general hypersurfaces of degree d in $\mathbb{P}_k^n$ split.

In particular the condition is satisfied if $r \ll n$.

In the proof, the matrix factorization introduced by Eisenbud plays an essential role.

Outline of the proof

$R := k[x_0, x_1, \dots, x_n]$ ($n \geq 3$),

$f \in R$: a nonsing. homog. poly. of degree d ,

$X := \text{Proj } R/(f)$.

A **(reduced)** graded matrix factorization of f is a triple $((\varphi, \psi), a)$ consisting of a pair of square matrices (φ, ψ) over R of the same size which

give rise to homogeneous homomorphism between graded free R -modules and $a \in \mathbb{Z}^l$ s.t.

- (i) all entries of φ **(and ψ)** are non-units,
- (ii) $\varphi\psi = \psi\varphi = fE$, and
- (iii) φ defines a homogeneous homomorphism $\bigoplus R(a_i) \rightarrow \bigoplus R(b_i)$ for some $b \in \mathbb{Z}^l$.

Then there are the following one-to-one correspondences:

{ ACM bundles on X **(without a free summ.)** }

$$\begin{array}{ccc} \mathcal{E} & & \widetilde{M} \\ \downarrow & \begin{array}{c} 1 \downarrow : \uparrow 1 \end{array} & \uparrow \\ \Gamma_* \mathcal{E} = \bigoplus_i H^0(X, \mathcal{E}(i)) & & M \end{array}$$

 { graded maximal CM $R/(f)$ -modules **(without a free summ.)** }

For M take a free resol.
$$0 \rightarrow \bigoplus R(a_i) \xrightarrow{\varphi} \bigoplus R(b_i) \rightarrow M \rightarrow 0$$

 { **(reduced)** graded matrix factorizations of f }

Suppose that there is an ACM bundle $\mathcal{E}$ without a free summand on X . Let $((\varphi, \psi), a)$ be the reduced gmf corresponding to $\mathcal{E}$. Looking at the $(1, 1)$ entry of $\varphi\psi = fE$, we have the equality

$$\sum_{k=1}^{\text{size } \varphi} \varphi_{1k} \psi_{k1} = f.$$

Since $((\varphi, \psi), a)$ is reduced, this means that f can be expressed as a sum of at most $\text{size } \varphi = rd$ products of two homogeneous polynomials of positive degree. Therefore it suffices to show the following claim:

claim. $r, d \ll n \Rightarrow$ general $f \in R$ of deg d cannot be expressed as a sum of at most rd products of two homogeneous polynomials.

The statement is obvious. We complete the proof by estimating “ $r, d \ll n$ ” strictly.