Fractional integral operators on Herz spaces

東海大学・開発工学部　古谷康雄 (Yasuo Komori-Furuya)
School of High Technology for Human Welfare
Tokai University

This article is organized in the following way. In Section 1 we state a brief history of Herz space. In Section 2 we summarize our recent results in [7].

1 Introduction

We consider the boundedness of fractional integral operators on Herz space. First we state a brief history of Herz space. About the precise definition, see the next section.

Let $T = [0, 2\pi)$ and define

$$A(T) = \Big\{ f \in L^1(T);\ \sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty \Big\},$$

where $\hat{f}(k) = (2\pi)^{-1} \int_T f(t) e^{-ikt} dt$. $A(T)$ is called the Wiener algebra and it is difficult to characterize this algebra. About this problem we know the following theorem (see, for example, [6]).

Theorem (Bernstein). *If $f \in Lip_\varepsilon(T)$ for some $\varepsilon > 1/2$, then $f \in A(T)$.*

We also know a non-periodic version of this theorem. We define generalized Lipschitz spaces as follows (see, for example, [4] and [12]).

Definition . Let $0 < p < \infty$ and $0 < \alpha < 1$. We say $f \in \Lambda(\alpha; 2, p)(\mathbb{R}^1)$ if $f \in L^2(\mathbb{R}^1)$ and

$$\int_{\mathbb{R}^1} \frac{\|f(\cdot + h) - f(\cdot)\|_{L^2}^p}{|h|^\alpha} \frac{dh}{|h|} < \infty.$$

Theorem (a non-periodic version of Bernstein's theorem). *Let $\mathcal{F}$ be the Fourier transform. If $f \in \Lambda(1/2; 2, 1)(\mathbb{R}^1)$ then $\mathcal{F}f \in L^1(\mathbb{R}^1)$.*

Beuring [1] introduced a new function space $A^2(\mathbb{R}^1)$, and proved the next result. Nowadays this function space is called the Beurling algebra, and we know that this is the special case of Herz spaces, that is, $A^2(\mathbb{R}^1) = K_2^{1/2,1}(\mathbb{R}^1)$.

Definition . We say $f \in A^2(\mathbb{R}^1)$ if

$$\|f\|_{A^2}^2 = \inf_{\omega \in \Omega} \int_{\mathbb{R}^1} \omega(x)dx \int_{\mathbb{R}^1} \frac{|f(x)|^2}{\omega(x)} dx < \infty,$$

where Ω consists of integrable positive ω which are nonincreasing functions of $|x|$.

Definition . We say $f \in \widetilde{A}^2(\mathbb{R}^1)$ if there exists $g \in A^2(\mathbb{R}^1)$ such that $f = \mathcal{F}g$.

Theorem (Beurling)**.** *$f \in \widetilde{A}^2(\mathbb{R}^1)$ if and only if f satisfies the following three conditions.*

$$\begin{aligned} &f \text{ is continuous}, \\ &\lim_{|x| \to \infty} f(x) = 0, \\ &f \in \Lambda(1/2; 2, 1)(\mathbb{R}^1). \end{aligned}$$

Herz [5] obtained the following.

Theorem (Herz)**.** *Let $0 < \alpha < 1$ and $0 < p < \infty$. If $f \in \Lambda(\alpha; 2, p)(\mathbb{R}^1)$ then $\mathcal{F}f \in K_2^{\alpha,p}(\mathbb{R}^1)$.*

Corollary . *If $f \in \Lambda(1/p - 1/2; 2, 1)(\mathbb{R}^1)$ then $\mathcal{F}f \in L^p(\mathbb{R}^1)$, where $1 \le p < 2$.*

If we take $p = 1$ in Corollary, then we can obtain the non-periodic version of Bernstein's theorem above.

2 Results

We define homogeneous Herz spaces and fractional integral. Let $0 < p < \infty, 1 \le q < \infty$ and $\alpha \in \mathbb{R}^1$.

Definition .

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}); \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where
$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\int_{2^{k-1}\le|x|<2^k} |f(x)|^q dx \right)^{p/q} \right\}^{1/p}.$$

Remark . $K_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$.

Definition .
$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\beta}} f(y)dy \quad \text{for} \quad 0<\beta<n.$$

The next proposition is well-known (see, for example, [13]).

Proposition 1. *I_β is bounded from $L^{q_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ where* $1/q_2 = 1/q_1 - \beta/n > 0$.

Li and Yang [8] proved the following.

Proposition 2. *I_β is bounded from $\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$ where*
$$1/q_2 = 1/q_1 - \beta/n > 0,\ \beta - n/q_1 < \alpha < n(1-1/q_1) \quad \textit{and} \quad 0<p<\infty.$$

When $q_1 \ge n/\beta$, we know the following proposition (see [11] and [13]). We define Lipschitz spaces and modified fractional integral. Let $0 \le \varepsilon < 1$.

Definition .
$$Lip_\varepsilon(\mathbb{R}^n) = \Big\{ f; \|f\|_{Lip_\varepsilon} = \sup_Q \inf_c \frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |f(x)-c|dx < \infty \Big\},$$
where the supremum is taken over all balls $Q \subset \mathbb{R}^n$.

We denote $Lip_0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

Proposition 3. *When $0<\varepsilon<1$,*
$$\|f\|_{Lip_\varepsilon} \approx \sup_{x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\varepsilon}.$$

Definition .
$$\widetilde{I}_\beta f(x) = \int_{\mathbb{R}^n} \Big\{ \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|y|^{n-\beta}} \chi_{\{|y|\ge 1\}} \Big\} f(y)dy \quad \text{for} \quad 0<\beta<n.$$

Proposition 4. *$\widetilde{I}_\beta$ is bounded from $L^q(\mathbb{R}^n)$ to $Lip_{\beta-n/q}(\mathbb{R}^n)$ where*

$$0 \le \beta - n/q < 1.$$

Especially $\widetilde{I}_\beta$ is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

We consider the boudedness of I_β on $\dot{K}_q^{\alpha,p}$ where $q \ge n/\beta$.

For our purpose we consider a variant of Lipschitz space. Our definition is the following. Let $0 \le \varepsilon < 1$ and $\lambda \in \mathbb{R}^1$.

Definition .

$$Lip_\varepsilon^\lambda(\mathbb{R}^n)$$
$$= \left\{ f; \|f\|_{Lip_\varepsilon^\lambda} = \sup_{\substack{x\in\mathbb{R}^n \\ R>0}} \inf_c \frac{1}{(|x|+R)^\lambda} \frac{1}{|B(x,R)|^{1+\varepsilon/n}} \int_{B(x,R)} |f(y)-c| dy < \infty \right\},$$

where $B(x,R) = \{y \in \mathbb{R}^n; |x-y| < R\}$.

Remarks . $Lip_\varepsilon^0(\mathbb{R}^n) = Lip_\varepsilon(\mathbb{R}^n)$ and $Lip_0^0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

$$\begin{array}{lll} Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n) \subset Lip_{\beta-n/q-\alpha}(\mathbb{R}^n) & \text{if} & \alpha > 0, \\ Lip_{\beta-n/q-\alpha}(\mathbb{R}^n) \subset Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n) & \text{if} & \alpha < 0. \end{array}$$

$$\|f\|_{Lip_\varepsilon^\lambda} \approx \sup_{\substack{x\in\mathbb{R}^n \\ R>0}} \frac{1}{(|x|+R)^\lambda} \frac{1}{|B(x,R)|^{1+\varepsilon/n}} \int_{B(x,R)} |f(y)-f_{B(x,R)}| dy < \infty,$$

where $f_{B(x,R)} = \frac{1}{|B(x,R)|}\int_{B(x,R)} f(y)dy$.

We give some examples.

$$\begin{array}{l} |x|^{1+\varepsilon} \in Lip_\varepsilon^1(\mathbb{R}^n), \quad |x|^{-\varepsilon} \in Lip_0^{-\varepsilon}(\mathbb{R}^n) \quad \text{where} \quad 0 \le \varepsilon < 1. \\ (\log|x|)^2 \chi_{\{|x|\ge 1\}} \in Lip_0^1(\mathbb{R}^n) \setminus BMO(\mathbb{R}^n), \\ (\log|x|)^2 \chi_{\{|x|\le 1\}} \in Lip_0^{-1}(\mathbb{R}^n) \setminus BMO(\mathbb{R}^n). \end{array}$$

Such function spaces are introduced by Nakai et al. [9], [10], [2]. They consider more general function spaces, that is, generalized Campanato spaces:

$$Lip_\varepsilon^\varphi(\mathbb{R}^n)$$
$$= \left\{ f; \|f\|_{Lip_\varepsilon^\varphi} = \sup_{\substack{x\in\mathbb{R}^n \\ R>0}} \inf_c \frac{1}{\varphi(x,R)} \frac{1}{|B(x,R)|^{1+\varepsilon/n}} \int_{B(x,R)} |f(y)-c| dy < \infty \right\}.$$

Our result is the following.

Theorem ([7]). *Let $q \geq n/\beta$, $0 < p < \infty$ and $\beta - n/q - 1 < \alpha < n - n/q$. If $0 \leq \beta - n/q < 1 + \min(0, \alpha)$, then $\widetilde{I}_\beta$ is bounded from $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ to $Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n)$.*

Corollary .

$$\begin{aligned} &\widetilde{I}_\beta : \dot{K}_q^{0,p}(\mathbb{R}^n) \to Lip_{\beta-n/q}(\mathbb{R}^n) \qquad \textit{if} \qquad 0 < \beta - n/q < 1, \\ &\widetilde{I}_\beta : \dot{K}_{n/\beta}^{0,p}(\mathbb{R}^n) \to BMO(\mathbb{R}^n). \end{aligned}$$

Since $L^q \subset K_q^{0,p}$ when $q \leq p$, Corollary is an extension of Proposition 4.

An outline of the proof of Theorem. Let $\varepsilon = \beta - n/q$. Fix a ball $Q = B(x_0, R)$ and we estimate

$$\inf_c \frac{(|x_0| + R)^\alpha}{|Q|^{1+\varepsilon/n}} \int_Q |\widetilde{I}_\beta f(x) - c| dx.$$

Let k be the least integer such that $Q \subset B(0, 2^k)$. Note that

$$|x_0| + R \approx 2^k.$$

We consider three cases:

(i) $Q \cap B(0, 2^{k-2}) \neq \emptyset$,

(ii) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R \geq 2^{k-4}$,

(iii) $Q \cap B(0, 2^{k-2}) = \emptyset$ and $R < 2^{k-4}$.

The case (i) or (ii). Note that $|Q| \geq C2^{kn}$ in both cases. We write

$$f(x) = f(x)\chi_{B(0,2^{k+1})} + f(x)\chi_{\complement B(0,2^{k+1})} =: f_1(x) + f_2(x).$$

First we estimate $\widetilde{I}_\beta f_1$. Let $c_1 = -\int_{|y|\geq 1} f_1(y)/|y|^{n-\beta} dy$. Then $\widetilde{I}_\beta f_1(x) - c_1 = I_\beta f_1(x)$, and we have

$$\int_Q |I_\beta f_1(x)| dx \leq C2^{k(n(1-1/q)-\alpha)} |Q|^{\beta/n} \|f\|_{\dot{K}_q^{\alpha,p}},$$

and obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |\widetilde{I}_\beta f_1 - c_1| dx \leq C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Next we estimate $\widetilde{I}_\beta f_2$. Let $c_2 = \widetilde{I}_\beta f(x_0)$. For any $x \in Q$,

$$|\widetilde{I}_\beta f_2(x) - c_2| \le CR\, 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}},$$

and we have

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |\widetilde{I}_\beta f_2 - c_2| dx \le C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

The case (iii). We write

$$\begin{aligned} f(x) &= f(x)\chi_{B(x_0,2R)} + f(x)\chi_{\complement\{2^{k-3}<|x|\le 2^{k+1}\}} + f(x)\chi_{\{2^{k-3}<|x|\le 2^{k+1}\}\setminus B(x_0,2R)} \\ &=: f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

First we estimate $\widetilde{I}_\beta f_1$. Let $c_1 = -\int_{|y|\ge 1} f_1(y)/|y|^{n-\beta} dy$. Then we have

$$\int_Q |\widetilde{I}_\beta f_1(x) - c_1| dx \le C 2^{-k\alpha} |Q|^{1-1/q+\beta/n} \|f\|_{\dot{K}_q^{\alpha,p}},$$

because $B(x_0, 2R) \subset \{x; 2^{k-3} < |x| \le 2^{k+1}\}$. Therefore we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_1(x) - c_1| dx \le C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Next we estimate $\widetilde{I}_\beta f_2$. Let $c_2 = \widetilde{I}_\beta f_2(x_0)$. It follows that for any $x \in Q$,

$$\begin{aligned} |\widetilde{I}_\beta f_2(x) - c_2| &\le CR \Big(\int_{|y|\ge 2^{k+1}} \frac{|f(y)|}{|x_0 - y|^{n+1-\beta}} dy + \int_{|y|\le 2^{k-3}} \frac{|f(y)|}{|x_0 - y|^{n+1-\beta}} dy \Big) \\ &\le CR\, 2^{k(\beta-n/q-\alpha-1)} \|f\|_{\dot{K}_q^{\alpha,p}}. \end{aligned}$$

Since $\beta - n/q - 1 < 0$, we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_2(x) - c_2| dx \le C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}.$$

Finally we estimate $\widetilde{I}_\beta f_3$. Let $c_3 = \widetilde{I}_\beta f_3(x_0)$. It follows that for any $x \in Q$,

$$|\widetilde{I}_\beta f_3(x) - c_3| \le CR^{\beta-n/q}\, 2^{-k\alpha} \|f\|_{\dot{K}_q^{\alpha,p}},$$

and we obtain

$$\frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |I_\beta f_3(x) - c_2| dx \le C(|x_0| + R)^{-\alpha} \|f\|_{\dot{K}_q^{\alpha,p}}. \qquad \square$$

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Yasuo Komori-Furuya
School of High Technology for Human Welfare, Tokai University
317 Nishino Numazu Shizuoka 410-0395, Japan
komori@wing.ncc.u-tokai.ac.jp