

Correspondence functors

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Abstract: This is a report on some recent joint work with Jacques Thévenaz, which appears in [1] and [2]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

The first part of this joint work is presented in Thévenaz's report, in these proceedings.

1. Introduction

1.1. This is an exposition of a joint work in progress with Jacques Thévenaz¹, on the *representation theory of finite sets*, by which we mean the following: let $\mathcal{C}$ denote the category in which objects are finite sets. For any two finite sets X and Y , the set of morphisms from X to Y in $\mathcal{C}$ is the set of all *correspondences* from X to Y , i.e. the set of subsets of $Y \times X$. We denote² this set by $\mathcal{C}(Y, X)$. A correspondence from X to itself is called a *relation* on X . The composition of correspondences is defined as follows: for finite sets X, Y, Z , for $R \subseteq Y \times X$ and $S \subseteq Z \times Y$

$$S \circ R (= SR) = \{(z, x) \in Z \times X \mid \exists y \in Y, (z, y) \in S \text{ and } (y, x) \in R\} .$$

The identity morphism of the finite set X is the diagonal

$$\Delta_X = \{(x, x) \mid x \in X\} \subseteq X \times X .$$

We now fix a commutative ring k (with identity element 1), and we consider functors from $\mathcal{C}$ to the category $k\text{-Mod}$ of k -modules. Equivalently, we first introduce the k -linearization $k\mathcal{C}$ of $\mathcal{C}$, i.e. the category with the same objects as $\mathcal{C}$, but in which the set of morphisms from X to Y is the free k -module $k\mathcal{C}(Y, X)$ on the set $\mathcal{C}(Y, X)$, and composition is k -linearly extended from composition in $\mathcal{C}$. Then we consider *correspondence functors* over k , i.e. k -linear functors from $k\mathcal{C}$ to $k\text{-Mod}$. These functors are the objects of a category $\mathcal{F}_k$, in which morphisms are natural transformations of functors. The category $\mathcal{F}_k$ is an abelian k -linear category.

¹cf. Jacques Thévenaz's report in these Proceedings.

²We emphasize that our notation is opposite to the usual notation $\mathcal{C}(X, Y)$ of category theory.

1.2. Examples : For any finite set E , the representable functor $Y_{E,k}$ sending a finite set X to the set $\text{Hom}_{k\mathcal{C}}(E, X) = k\mathcal{C}(X, E)$ is a *projective* object of $\mathcal{F}_k$, by the Yoneda Lemma. In particular:

- When $E = \emptyset$, then $Y_{E,k}(X) \cong k$ for any finite set X , and for any correspondence $U \subseteq Y \times X$ from X to a finite set Y , the map $Y_{E,k}(U) : Y_{E,k}(X) \rightarrow Y_{E,k}(Y)$ is the identity map of k . In other words, the functor $Y_{\emptyset,k}$ is *the constant functor* equal to k everywhere.
- When $E = \bullet$ is a set of cardinality one, then for any finite set X , the module $Y_{E,k}(X)$ is the free k -module with basis the set 2^X of subsets of X . Hence $Y_{\bullet,k}$ is *the functor of subsets*.
- The Yoneda Lemma implies that $\text{End}_{\mathcal{F}_k}(Y_{E,k})$ is isomorphic to the algebra $k\mathcal{C}(E, E)$ of all relations on E . In particular, when R is a preorder on E , i.e. R is a reflexive and transitive relation on E , or equivalently $\Delta_E \subseteq R = R^2$, then we get a direct summand $Y_{E,k}R$ of $Y_{E,k}$ defined on a finite set X by $Y_{E,k}R(X) = k\mathcal{C}(X, E)R$. The functor $Y_{E,k}R$ is a projective object of $\mathcal{F}_k$.

2. Functors associated to lattices

2.1. The previous examples are special cases of a more general construction that we now introduce. Recall that a lattice $T = (T, \vee, \wedge)$ is a poset in which any pair $\{x, y\}$ of elements has an least upper bound $x \vee y$ (called the *join* of x and y) and a greatest lower bound $x \wedge y$ (called the *meet* of x and y). A finite lattice T admits a smallest element 0_T (the meet of all elements of T) and a largest element 1_T (the join of all elements of T).

2.2. Definition : Let T be a finite lattice.

- When X is a finite set, let $F_T(X) = k(T^X)$ denote the free k -module with basis the set T^X of all maps from X to T .
- When $U \subseteq Y \times X$ is a correspondence from X to a finite set Y , let $F_T(U) : F_T(X) \rightarrow F_T(Y)$ be the k -linear map sending $\varphi : X \rightarrow T$ to the map $F_T(U)(\varphi) : Y \rightarrow T$, also denoted by $U\varphi$, defined by

$$\forall y \in Y, \quad (U\varphi)(y) = \bigvee_{(y,x) \in U} \varphi(x) .$$

Recall that a lattice T is called *distributive* if $\vee$ is distributive with respect to $\wedge$ or, equivalently, if $\wedge$ is distributive with respect to $\vee$.

2.3. Theorem : *Let T be a finite lattice. Then F_T is a correspondence functor. Moreover F_T is projective in $\mathcal{F}_k$ if and only if T is distributive.*

This result motivates the following definition:

2.4. Definition : *Let $k\mathcal{L}$ denote the following category:*

- *The objects of $k\mathcal{L}$ are the finite lattices.*
- *For two finite lattices T and T' , the set of morphisms from T to T' in $k\mathcal{L}$ is the free k -module with basis the set of all maps $f : T \rightarrow T'$ which respect the join operation, i.e. such that*

$$\forall A \subseteq T, \quad f\left(\bigvee_{t \in A} t\right) = \bigvee_{t \in A} f(t) .$$

- *The composition of morphisms in $k\mathcal{L}$ is the k -linear extension of the composition of maps.*

2.5. Remark : Note that a map from a finite lattice T to a finite lattice T' which respects the join operation need not respect the meet operation. On the other hand, it has to send the smallest element 0_T of T (which is equal to the join $\bigvee_{t \in \emptyset} t$) to the smallest element $0_{T'}$ of T' .

2.6. Theorem : *The assignment $T \mapsto F_T$ is a fully faithful k -linear functor from $k\mathcal{L}$ to $\mathcal{F}_k$.*

2.7. We will conclude this section by introducing a canonical subfunctor H_T of F_T , for any finite lattice T , which will be fundamental in the explicit description of simple correspondence functors.

First recall that an element e of a finite lattice T is called *irreducible* if for any subset A of T , the equality $e = \bigvee_{t \in A} t$ implies that $e \in A$. In other words $e \neq 0_T$, and if $e = x \vee y$ for $x, y \in T$, then $e = x$ or $e = y$. We denote by $\text{Irr}(T)$ the set of irreducible elements of T , viewed as a full subposet of T .

2.8. Definition : *Let T be a finite lattice. For a finite set X , let $H_T(X)$ denote the k -submodule of $F_T(X) = k(T^X)$ generated by all maps $\varphi : X \rightarrow T$ such that $\varphi(X) \not\subseteq \text{Irr}(T)$.*

2.9. Lemma :

1. Let Y, X be finite sets, let $U \in \mathcal{C}(Y, X)$, and let $\varphi : X \rightarrow T$. Then $(U\varphi)(Y) \cap \text{Irr}(T) \subseteq \varphi(X) \cap \text{Irr}(T)$.
2. The assignment $X \mapsto H_T(X)$ is a subfunctor of F_T .

Proof : Let $U \in \mathcal{C}(Y, X)$, let $\varphi : X \rightarrow T$, let $e \in (U\varphi)(Y) \cap \text{Irr}(T)$, and $y \in Y$ such that $e = (U\varphi)(y)$. Then $e = \bigvee_{(y,x) \in U} \varphi(x)$, so there exists x such that $(y, x) \in U$ and $e = \varphi(x)$. Hence $e \in \varphi(X) \cap \text{Irr}(T)$, proving Assertion 1. Assertion 2 follows trivially. $\square$

3. Simple functors

3.1. Let S be a simple object of $\mathcal{F}_k$, that is, a correspondence functor admitting exactly two subfunctors. Then S is non zero, so there is a set E of minimal cardinality such that $S(E) \neq \{0\}$. As explained in Jacques Thévenaz's report in these proceedings, the evaluation $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E , defined by

$$\mathcal{E}_E = k\mathcal{C}(E, E) / \sum_{|F| < |E|} k\mathcal{C}(E, F)\mathcal{C}(F, E) .$$

It follows from [1] that the simple $\mathcal{E}_E$ modules (up to isomorphism) are parametrized by pairs (R, W) of a partial order R on E and a simple $k\text{Aut}(E, R)$ -module W (up to permutation of E), where $\text{Aut}(E, R)$ is the automorphism group of the pair (E, R) , i.e. the group of permutations of E which preserve R .

Conversely, if E is a finite set, if R is a partial order on E , and if W is a simple $k\text{Aut}(E, R)$ -module, then there is a unique simple correspondence functor $S = S_{E,R,W}$ such that E is minimal with $S(E) \neq \{0\}$ and $S(E) \cong W$ as $\mathcal{E}_E$ -modules. This gives the following:

3.2. Theorem : *The simple correspondence functors over k (up to isomorphism) are parametrized by triples (E, R, W) consisting of a finite set E , a partial order R on E , and a simple $k\text{Aut}(E, R)$ -module W (up to identification of triples (E, R, W) and (E', R', W') for which there exists an isomorphism of posets $\varphi : (E, R) \rightarrow (E', R')$ sending W to W').*

3.3. Examples : Assume that k is a field.

- The representable functor $Y_{\emptyset, k}$ (see 1.2) is simple, projective, and injective in $\mathcal{F}_k$. The corresponding triple is $(\emptyset, tot, k)$, where tot is the unique (order) relation on $\emptyset$, and k is the unique simple module for $k\text{Aut}(\emptyset, tot) \cong k$.
- The representable functor $Y_{\bullet, k}$ is not simple, but one can show that it is isomorphic to the direct sum of the previous one $Y_{\emptyset, k}$ and the simple functor $S_{\bullet, tot, k}$, where tot is the unique order relation on the set $\bullet$, and k is the unique simple module for $k\text{Aut}(\bullet, tot) \cong k$. This functor $S_{\bullet, tot, k}$ is also simple, projective and injective in $\mathcal{F}_k$.

3.4. The two previous examples deal with a total order on a set of cardinality 0 and 1. We now consider the general case of a total order.

For this, we chose a non negative integer n , and we denote by $\underline{n}$ the totally ordered set $\{0, 1, \dots, n\}$. Then $\underline{n}$ is a lattice, in which $x \vee y = \text{Max}(x, y)$ and $x \wedge y = \text{Min}(x, y)$. We denote by $[n]$ the set $\text{Irr}(T)$. Clearly $[n] = \underline{n} - \{0\} = \{1, 2, \dots, n\}$.

3.5. Theorem : For $n \in \mathbb{N}$, set $S_{[n]} = F_{\underline{n}}/H_{\underline{n}}$. Then:

1. The surjection $F_{\underline{n}} \rightarrow S_{[n]}$ splits. The functor $S_{[n]}$ is projective.
2. If X is a finite set, then $S_{[n]}(X)$ is a free k -module of rank $\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} (i+1)^{|X|}$.
3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S_{[|A|]} \cong \bigoplus_{j=0}^n S_{[j]}^{\oplus \binom{n}{j}}$.
4. $\text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^n M_{\binom{n}{j}}(k)$.
5. If k is a field, then $S_{[n]}$ is simple (and projective, and injective), isomorphic to $S_{[n], tot, k}$.

3.6. In order to deal with the general case of simple functors, we need to introduce some notation. We start with a finite poset (E, R) , and we first choose a finite lattice T with the following two properties:

- (1) The poset $\text{Irr}(T)$ is isomorphic to (E, R) .
- (2) The natural restriction map $\text{Aut}(T) \rightarrow \text{Aut}(E, R)$ is an isomorphism.

Using Condition (1), we will identify (E, R) with the subposet $\text{Irr}(T)$ of T . In Condition (2), we denote by $\text{Aut}(T)$ the group of automorphisms of the

poset T (one can show that this is equal to the group of *bijections* of T which respect the join operation - see Definition 2.4). An automorphism of T clearly maps an irreducible element to an irreducible element, so we have a restriction map $\text{Aut}(T) \rightarrow \text{Aut}(\text{Irr}(T))$. This map is injective, because any element t of T is equal to the join $\bigvee_{\substack{e \in \text{Irr}(t) \\ e \leq_T t}} e$ of those irreducible elements smaller than t

in T , thus any automorphism of T is determined by its restriction to $\text{Irr}(T)$. So Condition (2) above amounts to requiring that any automorphism of the poset (E, R) can be extended to an automorphism of T .

The poset (E, R) being given, it is always possible to choose a finite lattice T with the above two properties, e.g. the lattice $I_\downarrow(E, R)$ consisting of lower ideals of (E, R) (i.e. subsets A of E such that $(x, y) \in R$ and $y \in A$ implies $x \in A$, for any $x, y \in E$), ordered by inclusion of subsets (the join operation on $I_\downarrow(E, R)$ is union of subsets, and the meet operation is intersection of subsets).

3.7. When T is a finite poset, and $t \in T$, we set

$$r(t) = \bigvee_{\substack{x \in T \\ x <_T t}} x .$$

Thus $r(t) = t$ if $t \notin \text{Irr}(T)$, and if $t \in \text{Irr}(T)$, then $r(t)$ is the largest element of T strictly smaller than t .

When $A \subseteq T$, we denote by $\gamma_A : E \rightarrow T$ the map defined by

$$\forall e \in E, \gamma_A(e) = \begin{cases} e & \text{if } e \notin A \\ r(e) & \text{if } e \in A \end{cases} .$$

We define moreover an element γ of $k(T^E)$ by

$$\gamma = \sum_{A \subseteq E} (-1)^{|A|} \gamma_A ,$$

and we view $k(T^E)$ as the evaluation at E of the functor $F_{T^{op}}$, where T^{op} is the opposite lattice to T (i.e. the lattice obtained by replacing the order relation on T by its opposite, or equivalently, by switching the join and meet operations of T).

Finally we denote by $\mathbb{S}_{E,R}$ the subfunctor of $F_{T^{op}}$ generated by the element γ of $F_{T^{op}}(E)$, i.e. the intersection of all subfunctors M of $F_{T^{op}}$ such that $\gamma \in M(E)$.

3.8. Theorem :

1. The functor $\mathbb{S}_{E,R}$ doesn't depend on the choice of T , up to isomorphism.
2. There exists a positive integer $f = f_{E,R}$ (explicitly computable) such that, for any finite set X , the k -module $\mathbb{S}_{E,R}(X)$ is free of rank

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} (i + f)^{|X|} .$$

Moreover $\mathbb{S}_{E,R}(X)$ is a free right $k\text{Aut}(E, R)$ -module.

3. Let W be a $k\text{Aut}(E, R)$ -module. For a finite set X , define

$$\mathbb{S}_{E,R,W}(X) = \mathbb{S}_{E,R}(X) \otimes_{k\text{Aut}(E,R)} W .$$

Then the assignment $X \mapsto \mathbb{S}_{E,R,W}(X)$ is a correspondence functor.

4. If k is a field and W is simple, then $\mathbb{S}_{E,R,W} \cong S_{E,R,W}$.

Proof : (Sketch) • First we introduce a non-degenerate functorial bilinear pairing $F_T \times F_{T^{op}} \rightarrow k$, in the following way: if X is a finite set, if $\varphi : X \rightarrow T$ and $\psi : X \rightarrow T^{op}$, we set

$$(\varphi, \psi)_X = \begin{cases} 1 & \text{if } \phi(x) \leq_T \psi(x) \ \forall x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is functorial in the sense that for any correspondence $U \subseteq Y \times X$ from X to a finite set Y , for any $\varphi : X \rightarrow Y$ and any $\psi : Y \rightarrow T^{op}$, we have that

$$(U\varphi, \psi)_Y = (\varphi, U^{op} \star \psi)_X ,$$

where $U^{op} = \{(x, y) \in X \times Y \mid (y, x) \in U\}$ denotes the opposite correspondence, and $U^{op} \star \psi = F_{T^{op}}(U^{op})(\psi) \in F_{T^{op}}(X)$ is the image of ψ under U^{op} .

This pairing is non degenerate in the strong sense that it induces an isomorphism between $F_T(X)$ and the k -dual of $F_{T^{op}}(X)$, for any finite set X (so it induces an isomorphism between $F_{T^{op}}$ and the dual functor $(F_T)^\natural$).

- We show that there exists a surjective homomorphism of correspondence functors

$$\Theta_T : F_T/H_T \rightarrow \mathbb{S}_{E,R^{op}} ,$$

where R^{op} is the opposite partial order to R on E .

- We define a subset G of T , containing E , and invariant under $\text{Aut}(E, R)$, with the property that for any finite set X , the image under $\Theta_{T,X} \circ \pi_{T,X}$ of

the set

$$\{\varphi : X \rightarrow T \mid E \subseteq \varphi(X) \subseteq G\}$$

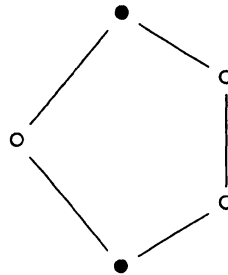
of elements of $F_T(X)$ is a k -basis of $\mathbb{S}_{E,R^{op}}(X)$, where $\pi_T : F_T \rightarrow F_T/H_T$ is the quotient morphism. Then the integer $f = f_{E,R}$ appearing in Theorem 3.8 is equal to $|G| - |E|$. $\square$

3.9. Corollary : *Let k be a field. Let (E, R) be a finite poset, and W be a simple $k\text{Aut}(E, R)$ -module. Then for any finite set X ,*

$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|\text{Aut}(E, R)|} \sum_{i=0}^{|E|} (-1)^{|E|-i} \binom{|E|}{i} (i + f_{E,R})^{|X|} .$$

4. Examples

4.1. Let D denote the following lattice:



where the white dots are the irreducible elements. Then over a field of odd characteristic, the functor F_D is semisimple: it splits as

$$F_D \cong \mathbb{S}_{[0]} \oplus 4\mathbb{S}_{[1]} \oplus 4\mathbb{S}_{[2]} \oplus \mathbb{S}_{[3]} \oplus 2\mathbb{S}_{\bullet\bullet} \oplus \mathbb{S}_{\begin{smallmatrix} \bullet \\ \bullet \\ \bullet \end{smallmatrix}} ,$$

where $\mathbb{S}_{\bullet\bullet}$ denotes the functor $\mathbb{S}_{E,\Delta}$ for a set E of cardinality 2, ordered by the equality relation, and $\mathbb{S}_{\begin{smallmatrix} \bullet \\ \bullet \\ \bullet \end{smallmatrix}}$ is the functor $\mathbb{S}_{F,R}$ associated to a poset (F, R) of cardinality 3 with 2 connected components.

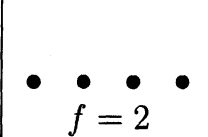
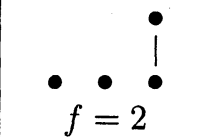
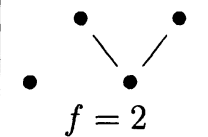
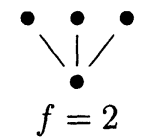
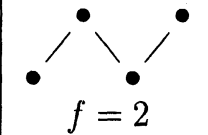
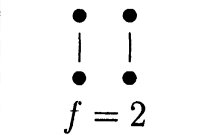
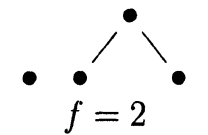
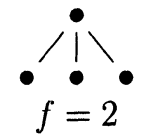
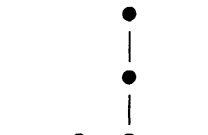
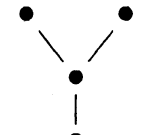
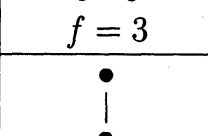
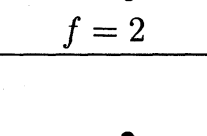
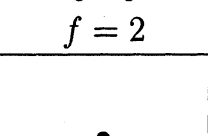
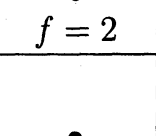
Observe that for any $i \in \mathbb{N}$, the multiplicity of the functor $\mathbb{S}_{[i]}$ as a summand of F_D is equal to the number of increasing sequences

$$0_D = x_0 < x_1 < \dots < x_i$$

in D . This statement holds more generally for an arbitrary finite lattice T .

4.2. There are 16 posets up to isomorphism on a set of cardinality 4. The

following table displays the Hasse diagrams of these posets, together with the corresponding value of the integer f appearing in Theorem 3.8:

 $f = 2$	 $f = 2$	 $f = 2$	 $f = 2$
 $f = 2$	 $f = 2$	 $f = 2$	 $f = 2$
 $f = 3$	 $f = 2$	 $f = 2$	 $f = 2$
 $f = 1$	 $f = 2$	 $f = 2$	 $f = 2$

The only poset for which $f = 1$ is the total order. This is a general phenomenon: if (E, R) is a finite poset, then $f_{E,R} = 1$ if and only if R is a total order.

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