

Finite Type System of Partial
Differential Operators and
Decomposition of Solutions of
Partial Differential Equations^(*)

by

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§1. Introduction, $\mathcal{P}(D)$ -convexity.

First we fix some notations. Let Ω be an open set in the n -dimensional Euclidean space $\mathbb{R}^n$ whose points shall be denoted by their coordinates $x = (x_1, \dots, x_n)$. Let $\mathbb{C}[X_1, \dots, X_n]$ be the polynomial ring in n variables $X = (X_1, \dots, X_n)$ over the complex number field $\mathbb{C}$.

A partial differential operator with constant coefficients $P(D)$ is obtained from the polynomial $P(X) \in \mathbb{C}[X_1, \dots, X_n]$ just by replacing the variables $X = (X_1, \dots, X_n)$ by the differentiations $D = (D_1, \dots, D_n)$ with $D_j = \frac{\partial}{\partial x_j}$ ($j = 1, 2, \dots, n$).

Now let $\mathcal{P}(X) = (P_{jk}(X))$ be a matrix with q rows and p columns with coefficients in $\mathbb{C}[X_1, \dots, X_n]$. $\mathcal{Q}(X) = (Q_{jk}(X))$ be a relation matrix with r rows and q columns for $\mathcal{P}(X)$, i.e. the row vectors $(Q_{j1}(X), \dots, Q_{jq}(X))$ ($j = 1, \dots, r$) generate the relation module

(*) The content of this article was partly spoken in a slightly different form in Séminaire MALGRANGE (1964) à Orsay.

$$\{(Q_1(X), \dots, Q_q(X)) ; \sum_{j=1}^q Q_j(X) P_{jk}(X) = 0 \quad (k = 1, \dots, p)\} .$$

Then we get the following differential complex

$$(1.1) \quad [C^\infty(\Omega)]^p \xrightarrow{P(D)} [C^\infty(\Omega)]^q \xrightarrow{Q(D)} [C^\infty(\Omega)]^r ,$$

$$Q(D) P(D) = 0 .$$

DEFINITION. An open set Ω in $\mathbb{R}^n$ is called $P(D)$ -convex if the above sequence is exact.

EXAMPLES.

1) For the case of a single operator $P(D)$ ($p = q = 1$) , the exactness of (1.1) is reduced to the subjectivity

$$(1.2) \quad P(D) C^\infty(\Omega) = C^\infty(\Omega) ,$$

since the relation module is 0 . Thus the concept of $P(D)$ -convexity is a generalization of that of the usual $P(D)$ -convexity (see [3], [4]).

2) When $P(D) = \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}$ is the gradient operator, the exact-

ness is just the vanishing of the first de Rham cohomology group

$$(1.3) \quad H^1(\Omega, \mathbb{C}) = 0 .$$

3) If we identify the complex space $\mathbb{C}^m$ with $\mathbb{R}^n$ ($n = 2m$) in the usual manner, then the pseudo-convexity of an open set Ω can be characterized by the exactness of the following sequence

$$(1.4) \quad C_{(0,0)}^{\infty}(\Omega) \xrightarrow{\bar{\partial}} C_{(0,1)}^{\infty}(\Omega) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C_{(0,m)}^{\infty}(\Omega) \rightarrow 0$$

where $C_{(0,q)}^{\infty}$ denotes the space of all C^{∞} -forms of type $(0,q)$ defined on Ω (see [2], [5]). This can be stated as the exactness of the following sequence

$$(1.4)' \quad E_1 \xrightarrow{\bar{\partial}} E_2 \xrightarrow{\bar{\partial}} E_3$$

with $E_1 = \prod_{q=0}^{m-1} C_{(0,q)}^{\infty}(\Omega)$, $E_2 = \prod_{q=1}^m C_{(0,q)}^{\infty}(\Omega)$, $E_3 = \prod_{q=2}^{m+1} C_{(0,q)}^{\infty}(\Omega)$. Thus it can be written in the form (1.1).

4) When $\mathcal{P}(D)$ is quite general, we know only the following

THEOREM 1.1. (Ehrenpreis-Malgrange) *If Ω is convex, then Ω is $\mathcal{P}(D)$ -convex for any $\mathcal{P}(D)$ (See [1], [2], [5]).*

§2. Finite type systems of partial differential operators.

In the preceding section, we have explained the $\mathcal{P}(D)$ -convexity and some examples. But, other than these, no general results are known. Therefore it will have some meaning to state the following

THEOREM 2.1. *If Ω is simply connected, then Ω is $\mathcal{P}(D)$ -convex for any $\mathcal{P}(D)$ of finite type.^(*)*

Before giving a sketchy proof of the above theorem, we should explain some notions.

To a matrix $\mathcal{P}(X) = (P_{jk}(X))$ with q rows and p columns we associate the ideal $\mathcal{N} = \mathcal{N}(\mathcal{P})$ generated by all the (p,p) -minors of $\mathcal{P}(X)$. (If $p > q$, we only put $\mathcal{N} = 0$.) Let $V = V(\mathcal{P})$ be the algebraic variety defined by $\mathcal{N}(\mathcal{P})$, i.e.

$$V(\mathcal{P}) = \{z \in \mathbb{C}^n; P(z) = 0 \text{ for all } P \in \mathcal{N}(\mathcal{P})\}.$$

(*) See the definition below.

This is called the *variety attached* to the system of differential operators $\mathcal{P}(D)$.

DEFINITION. A system of partial differential operators $\mathcal{P}(D)$ is called of *finite type* if the attached variety is of dimension 0, i.e. $V(\mathcal{P})$ consists of only a finite number of points.

Now consider the following homogeneous equation

$$(2.1) \quad \mathcal{P}(D)U = 0$$

where U is an unknown element of $[C^\infty(\Omega)]^P$.

Then, using Hilbert's Nullstellensatz, it is not difficult to show the following

LEMMA The vector space over $\mathbb{C}$ of the solutions of (2.1) is of finite dimension if and only if the system $\mathcal{P}(D)$ is of finite type. And then the solutions of (2.1) consist only of entire functions, more precisely the exponential-polynomial solutions of (2.1).

To prove Theorem 2.1, first we fix a covering of Ω by its convex open subsets: $\Omega = \bigcup_i \Omega_i$, and consider the equation

$$(2.2) \quad \mathcal{P}(D)U = F$$

where F is an arbitrary given element in $[C^\infty(\Omega)]^q$ such that

$$(2.3) \quad \mathcal{Q}(D)F = 0.$$

In each convex open set Ω_i , we can apply Theorem 1.1. and we get a solution in Ω_i . In the intersection $\Omega_i \cap \Omega_j$ of two such convex open sets, the difference of the solutions should

satisfy the homogeneous equation (2.1). Hence, according to the above lemma, we get a solution of (2.2) in the union $\Omega_i \cup \Omega_j$ by adjusting those solutions by a certain entire function. Now starting at some fixed point we can proceed along curves repeating such adjustments in a manner similar to the usual analytic continuation in function theory. The resulting solution U should be univalent according to the assumption that Ω be simply connected. This shows that (1.1) is exact. This completes the proof.

§3. Decomposition of solutions of partial differential equations.

Since partial differential operators with constant coefficients operate on $C^\infty(\Omega)$ commutatively, we see clearly that

(I) For any pair of polynomials P_1, P_2 and for any pair of functions $u_1, u_2 \in C^\infty(\Omega)$ such that

$$(3.1) \quad P_1(D)u_1 = 0, \quad P_2(D)u_2 = 0,$$

the sum

$$(3.2) \quad u = u_1 + u_2$$

satisfies the equation for the product operator

$$(3.3) \quad P_1(D)P_2(D)u = 0;$$

(II) For any polynomial P and for any multiindexed family of solutions $u_\alpha \in C^\infty(\Omega)$ ($|\alpha| \leq \nu-1$)^(*) of the equation

(*) For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ (a sequence of non-negative integers) we set $|\alpha| = \alpha_1 + \dots + \alpha_n$. If $x = (x_1, \dots, x_n)$ is variable point, we set $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

$$(3.4) \quad P(D)u_\alpha = 0$$

the sum

$$(3.5) \quad u = \sum_{|\alpha| \leq v-1} x^\alpha u_\alpha$$

is a solution of the equation for the v -times interacted operator

$$(3.6) \quad P(D)^v u = 0$$

Our question is to ask when the converses of the above facts are true, naturally assuming always that

(i) P_1 and P_2 have no common factor

for the converse of (I), and that

(ii) P is irreducible

for the converse of (II).

THEOREM 3.1. If Ω is $\mathcal{P}(D)$ -convex with $\mathcal{P}(D) = \begin{pmatrix} P_1(D) \\ P_2(D) \end{pmatrix}$, then any solution $u \in C^\infty(\Omega)$ of (3.3) can be decomposed into the form (3.2) with (3.1). Conversely, if each solution u of (3.3) can be written in the form (3.2) with (3.1) and if Ω is $P_1(D)P_2(D)$ -convex, then Ω is $\mathcal{P}(D)$ -convex.

DEFINITION. We say that an open set Ω is a *decomposing domain* if the converses of (I) and (II) are true for all P_1, P_2 and P satisfying the conditions (i) and (ii).

Application of Ehrenpreis-Malgrange's theorem thus shows

THEOREM 3.2. If Ω is convex, then Ω is a decomposing domain.

In the plane $\mathbb{R}^2$, we can prove a more precise result, using Theorem 2.1.

THEOREM 3.3. An open set Ω in $\mathbb{R}^2$ is a decomposing domain if and only if Ω is simply connected.

The "only if" part comes from the fact that, for an open set Ω in the plane $\mathbb{R}^2$, the vanishing of the first de Rham cohomology group (1.3) is equivalent to the simple-connectedness of Ω . (*)

Theorem 3.3 is no longer true for $n \geq 3$.

If we restrict ourselves to the polynomial or exponential polynomial solutions then we get, by a purely algebraic argument, the following

THEOREM 3.4. *If we assume (i) and (ii), then any polynomial (resp. exponential-polynomial) solution of (3.3) or (3.6) can be decomposed into the form (3.2) or (3.5) with u_1, u_2 or u_α polynomials (resp. exponential-polynomials).*

The content of the present article can be considered as a completion of our previous work [6]. The more detailed treatment shall be published elsewhere.

References.

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(*) This can be proved by a combinatorial argument, for that I thank Prof. H. TODA.

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