数理解析研究所講究録
1181 巻 2001 年 199-209

Multiple solutions of inhomogeneous H-systems with zero Dirichlet boundary conditions

Futoshi Takahashi (高橋 太)
Department of mathematics, Faculty of Science,
Tokyo Institute of Technology

1. Introduction

This article is an abbreviated version of [TF1].

In this paper, we study the existence of multiple solutions to the Dirichlet problem of the inhomogeneous H-system:

$$\begin{cases} \Delta u = 2Hu_x \wedge u_y + f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded smooth domain, $H > 0$ is a given constant, and $f \in H^{-1}(\Omega; \mathbf{R}^3)$ is a given function. $a \wedge b$ denotes the usual vector product of $a, b \in \mathbf{R}^3$.

Solutions of (1.1) in $H_0^1(\Omega; \mathbf{R}^3)$ correspond to critical points of the energy functional:

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2H}{3}Q(u) + \int_\Omega f \cdot u,$$

where

$$Q(u) = \int_\Omega u \cdot u_x \wedge u_y$$

is the oriented volume functional.

This problem is interesting from the variational view point because the functional E does not satisfy the Palais-Smale(PS) compactness condition globally on $H_0^1(\Omega; \mathbf{R}^3)$. In the case $f \equiv 0$, it is known that the existence or the non-existence of multiple solutions of (1.1) depends on the topology of the domain. More precisely, it is known that when $f \equiv 0$ and Ω is simply-connected, then $u \equiv 0$ is the only solution of (1.1); on the other hand, when Ω is doubly-connected, there exists at least one non-trivial solution [W].

In [Ta1], G.Tarantello treated the following Dirichlet problem of semilinear elliptic equations involving critical Sobolev exponent:

$$\begin{cases} -\Delta u = u|u|^{2^*-2} + f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^N (N \geq 3)$ is a bounded smooth domain, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $f \in H^{-1}(\Omega)$. It is well known that when $f \equiv 0$ and Ω is star-shaped, (1.2) has the only solution $u \equiv 0$ [P]. On the other hand, there is a vast literature on the effect of the domain topology or geometry on the existence of multiple positive solutions of (1.2) when $f \equiv 0$; see [BaC], [Co], [Pa] and references therein. In spite of a possible lack of compactness, she obtained the existence of at least two non-trivial weak solutions of (1.2) for $f \not\equiv 0$ satisfying some suitable smallness condition.

Here, following her methods, we pursue the analogous results for the problem (1.1).

Before stating our results, we introduce a set of assumptions on the function f:

(f.1) $f \in H^{-1} \cap L^1(\Omega; \mathbf{R}^3)$,

(f.2) $-\int_\Omega f \cdot u < \frac{(\int_\Omega |\nabla u|^2)^2}{-8HQ(u)}$ for all $u \in H_0^1(\Omega; \mathbf{R}^3)$ with $Q(u) < 0$,

(f.3) $||f||_{H^{-1}} < \frac{\sqrt{3\pi}}{12H}$. ((f.3) implies (f.2))

We remark that by the isoperimetric inequality for H_0^1-mappings [BC]:

$$S|Q(u)|^{\frac{2}{3}} \leq \int_\Omega |\nabla u|^2 \qquad \text{for all} \quad u \in H_0^1(\Omega; \mathbf{R}^3),$$

where $S = (32\pi)^{1/3}$, it is easy to see that the assumption (f.2) always holds if $f \in H^{-1}(\Omega; \mathbf{R}^3)$ satisfies

$$||f||_{H^{-1}} < \frac{S^{3/2}}{8H} \left(= \frac{\sqrt{2\pi}}{2H} \right),$$

so, the assumption (f.2) appears essentially the smallness condition of f.

Our main results are the following:

Theorem 1. *Let $f \not\equiv 0$ satisfy the assumptions (f.1) and (f.2), then the problem (1.1) admits at least one solution $\underline{u}$ in $H_0^1(\Omega; \mathbf{R}^3)$.*

Theorem 2. *Let $f \not\equiv 0$ satisfy the assumptions (f.1) and (f.3), then, $\underline{u}$ obtained in Theorem 1 is a strict local minimum for the functional E in $H_0^1(\Omega; \mathbf{R}^3)$, and the problem (1.1) admits at least one more solution $\overline{u}$ in $H_0^1(\Omega; \mathbf{R}^3)$.*

This paper is organized as follows. In section 1, we prove Theorem 1 by using Ekeland's variational principle and Nehari variational method.

In section 2, we prove Theorem 2 by utilizing the strict local minimality of the first solution and the Mountain Pass Theorem.

2. Existence of the first solution

In this section, we prove Theorem 1 by considering a suitable minimization problem for the functional E. To this end, let us denote

$$\begin{aligned} \Lambda &= \{u \in H_0^1(\Omega; \mathbf{R}^3) : \langle E'(u), u \rangle = 0\} && (2.1) \\ &= \{u \in H_0^1(\Omega; \mathbf{R}^3) : \int_\Omega |\nabla u|^2 + 2HQ(u) + \int_\Omega f \cdot u = 0\}, && (2.2) \end{aligned}$$

where $\langle , \rangle$ denotes the usual dual pairing of H^{-1} and H_0^1, and

$$\Lambda_0 = \{u \in \Lambda : \int_\Omega |\nabla u|^2 + 4HQ(u) = 0\}, \tag{2.3}$$

$$\Lambda_+ = \{u \in \Lambda : \int_\Omega |\nabla u|^2 + 4HQ(u) > 0\}, \tag{2.4}$$

$$\Lambda_- = \{u \in \Lambda : \int_\Omega |\nabla u|^2 + 4HQ(u) < 0\}. \tag{2.5}$$

Recall that Q is analytic on $H_0^1(\Omega; \mathbf{R}^3)$ and $\langle Q'(u), u\rangle = 3Q(u)$. Λ is called the "Nehari manifold" and it contains all critical points for E in $H_0^1(\Omega; \mathbf{R}^3)$. Therefore, to obtain the solution of the problem (1.1), it is natural to consider the minimization problem:

$$c_0 = \inf_{u \in \Lambda} E(u). \tag{2.6}$$

We shall prove that under the assumptions (f.1) and (f.2), the infimum in (2.6) is achieved by some $\underline{u} \in \Lambda$ and $\underline{u}$ defines a critical point for E in $H_0^1(\Omega; \mathbf{R}^3)$.

We note that if we set

$$K(u) = \int_\Omega |\nabla u|^2 + 2HQ(u) + \int_\Omega f \cdot u, \quad u \in H_0^1(\Omega; \mathbf{R}^3),$$

then $\Lambda = \{u \in H_0^1(\Omega; \mathbf{R}^3) : K(u) = 0\}$ and Λ is in fact a smooth submanifold of $H_0^1(\Omega; \mathbf{R}^3)$ if $K'(u) \neq 0$ for any $u \in \Lambda$. Now we calculate

$$\langle K'(u), u\rangle = \int_\Omega |\nabla u|^2 + 4HQ(u), \quad \text{for } u \in \Lambda,$$

so, for the minimizer $\underline{u}$ for (2.6) (if it exists) to be a critical point of E in $H_0^1(\Omega; \mathbf{R}^3)$, we must ensure that $\Lambda_0 = \{0\}$.

We start with a lemma which shows the assumption (f.2) is indeed a sufficient condition for $\Lambda_0 = \{0\}$.

Lemma 2.1. *Suppose the assumption (f.2) holds, then for any $u \in \Lambda$, $u \not\equiv 0$, we have*

$$\int_\Omega |\nabla u|^2 + 4HQ(u) \neq 0.$$

Proof: Assume

$$\int_\Omega |\nabla u|^2 + 4HQ(u) = 0 \tag{2.7}$$

holds for some $u \in \Lambda, u \not\equiv 0$. Then $Q(u) < 0$ and, because u also satisfies

$$\int_\Omega |\nabla u|^2 + 2HQ(u) + \int_\Omega f \cdot u = 0, \tag{2.8}$$

we have

$$\int_\Omega f \cdot u = 2HQ(u) \tag{2.9}$$

by (2.7),(2.8).

Now from (f.2),(2.9) and (2.7) we derive:

$$\begin{aligned} 0 &< \int_\Omega f \cdot u + \frac{(\int_\Omega |\nabla u|^2)^2}{-8HQ(u)} = 2HQ(u) + \frac{(\int_\Omega |\nabla u|^2)^2}{-8HQ(u)} \\ &= |Q(u)| \cdot \{-2H + \frac{(-4HQ(u))^2}{8H|Q(u)|^2}\} = 0, \end{aligned}$$

which is a contradiction. □

Lemma 2.2. *Suppose the assumption (f.2) holds. Then for any $u \in \Lambda$, $u \not\equiv 0$, there exist an $\varepsilon > 0$ and a smooth function*

$$t : \{w \in H_0^1(\Omega; \mathbf{R}^3) : \|w\|_{H_0^1} < \varepsilon\} \to \mathbf{R}$$

such that

$$t(0) = 1, \quad t(w) \cdot (u - w) \in \Lambda \quad \textit{for} \quad \|w\|_{H_0^1} < \varepsilon,$$

and

$$\langle t'(0), w \rangle = \frac{2\int_\Omega \nabla u \cdot \nabla w + 6H \int_\Omega w \cdot u_x \wedge u_y + \int_\Omega f \cdot w}{\int_\Omega |\nabla u|^2 + 4HQ(u)}. \tag{2.10}$$

Proof: Define a smooth map $F : \mathbf{R} \times H_0^1(\Omega; \mathbf{R}^3) \to \mathbf{R}$ as

$$F(t, w) = t \int_\Omega |\nabla(u - w)|^2 + 2Ht^2 Q(u - w) + \int_\Omega f \cdot (u - w).$$

Since $F(1, 0) = 0$ for $u \in \Lambda$ and

$$F_t(1, 0) = \int_\Omega |\nabla u|^2 + 4HQ(u) \neq 0$$

by Lemma 2.1, we can apply the Implicit Function Theorem at the point $(1, 0) \in \mathbf{R} \times H_0^1(\Omega; \mathbf{R}^3)$ and the result follows. □

Lemma 2.3. *Let $f \not\equiv 0$ satisfy the assumption (f.1), then*

$$\mu_0 := \inf_{\substack{u \in H_0^1(\Omega;\mathbf{R}^3) \\ Q(u) = -1}} \left\{ \frac{1}{8H} \left(\int_\Omega |\nabla u|^2 \right)^2 + \int_\Omega f \cdot u \right\} \tag{2.11}$$

is achieved. In addition if f satisfies (f.2), then $\mu_0 > 0$.

The proof of Lemma 2.3 is a modification of that for the minimization problem treated in [TF2], so we omit it. (However, different from [TF2], the extra assumption that $f \in L^1(\Omega; \mathbf{R}^3)$ is needed in the current case.)

In the following, we proceed to the proof of Theorem 1 assuming that $f \not\equiv 0$ satisfies (f.1) and (f.2) simultaneously.

First, we give an upper and lower bound for c_0 in (2.6).

Proposition 2.1. *There exists a $t_0 > 0$ such that*

$$-\frac{2}{3}\|f\|_{H^{-1}}^2 \leq c_0 < -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2 \tag{2.12}$$

holds.

Proof: First, we show that E is bounded from below on Λ. Indeed, by definition (2.2),

$$\int_\Omega |\nabla u|^2 + 2HQ(u) + \int_\Omega f \cdot u = 0$$

for $u \in \Lambda$. Thus we have

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{2H}{3}Q(u) + \int_\Omega f \cdot u \\ &= \left(\frac{1}{2} - \frac{1}{3}\right)\int_\Omega |\nabla u|^2 + \left(1 - \frac{1}{3}\right)\int_\Omega f \cdot u \\ &\geq \frac{1}{6}\|\nabla u\|_{L^2}^2 - \frac{2}{3}\|f\|_{H^{-1}}\|\nabla u\|_{L^2} \geq -\frac{2}{3}\|f\|_{H^{-1}}^2 \end{aligned}$$

for any $u \in \Lambda$. In particular,

$$c_0 \geq -\frac{2}{3}\|f\|_{H^{-1}}^2.$$

In order to obtain an upper bound for c_0, let $v \in H_0^1(\Omega; \mathbf{R}^3)$ be the unique solution of $\Delta v = f$ in Ω.

Then for $f \not\equiv 0$, we have

$$\int_\Omega f \cdot v = -\int_\Omega |\nabla v|^2 < 0.$$

Now we divide the proof according to the sign of $Q(v)$.

If $Q(v) > 0$, then

$$\varphi(t) = \varphi^v(t) := t\int_\Omega |\nabla v|^2 + 2Ht^2 Q(v) \tag{2.13}$$

is a convex quadratic function in $t \in \mathbf{R}$ with $\varphi(0) = \varphi\left(\frac{\int_\Omega |\nabla v|^2}{-2HQ(v)}\right) = 0$. Note that, if $\varphi'(t) > 0$ at some $t \neq 0$ satisfying $\varphi(t) = -\int_\Omega f \cdot v$, then $tv \in \Lambda_+$.

Now we have $-\int_\Omega f \cdot v > 0$, so easy observation shows there exists a unique $t_0 > 0$ such that $t_0 v \in \Lambda_+$. Thus, by definition of Λ and Λ_+, we have

$$\begin{aligned} E(t_0 v) &= -\frac{1}{2}\int_\Omega |\nabla(t_0 v)|^2 - \frac{4H}{3}Q(t_0 v) \\ &< -\frac{1}{2}\int_\Omega |\nabla(t_0 v)|^2 + \frac{1}{3}\int_\Omega |\nabla(t_0 v)|^2 \\ &= -\frac{t_0^2}{6}\int_\Omega |\nabla v|^2 = -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2, \end{aligned}$$

which yields an upper bound of c_0 in this case.

Next if $Q(v) < 0$, then $\varphi(t)$ in (2.13) is a concave quadratic function in t and

$$\max_{t\in\mathbf{R}} \varphi(t) = \varphi(\frac{\int_\Omega |\nabla v|^2}{-4HQ(v)}) = \frac{(\int_\Omega |\nabla v|^2)^2}{-8HQ(v)}.$$

Now, by the assumption (f.2) we again obtain unique $t_0 > 0$ with $t_0 v \in \Lambda_+$, so the rest is the same as in the former case.

Finally if $Q(v) = 0$, then $v \in \Lambda_+$ and we can choose $t_0 = 1$. □

At this point, we are ready to apply the Ekeland's variational principle [AE] to the minimization problem (2.6).

Ekeland's variational principle. *Let M be a complete metric space with metric d, and let $E : M \to \mathbf{R} \cup +\infty$ be lower semicontinuous, bounded from below, and $\not\equiv \infty$.*

Then for any $\varepsilon, \delta > 0$, for any $u \in M$ with

$$E(u) \le \inf_M E + \varepsilon,$$

there exists an element $v \in M$ such that

(1) $E(v) \le E(u)$,

(2) $d(u,v) \le \delta$,

(3) $E(v) < E(w) + \frac{\varepsilon}{\delta} d(v,w)$, *for all $w \neq v$.*

Proposition 2.2. *There exists a minimizing sequence $\{u^n\} \subset \Lambda$ for (2.6) with the following properties:*

(a) $E(u^n) < c_0 + \frac{1}{n}$,

(b) $E(w) \ge E(u^n) - \frac{1}{n}\|\nabla(u^n - w)\|_{L^2}$, *for any $w \in \Lambda$,*

(c) $\frac{t_0^2}{4}\|f\|_{H^{-1}} < \|\nabla u^n\|_{L^2} < 4\|f\|_{H^{-1}}$, *where t_0 is given by Proposition 2.1, and*

(d) $\|E'(u^n)\|_{H^{-1}} \to 0$ *as $n \to \infty$.*

Sketch of Proof : Λ is closed with respect to the strong H_0^1-topology and E is bounded from below, continuous, and $\not\equiv \infty$ on Λ. Therefore we can apply Ekeland's variational principle to (2.6), and the statements (a),(b) are the direct consequences of this.

By taking n large enough, from (a) and (2.12) we have

$$E(u^n) = \frac{1}{6}\int_\Omega |\nabla u^n|^2 + \frac{2}{3}\int_\Omega f \cdot u^n < c_0 + \frac{1}{n} < -\frac{t_0^2}{6}\|f\|_{H^{-1}}^2 < 0. \tag{2.14}$$

This implies $u^n \not\equiv 0$ and

$$\frac{1}{6}\int_\Omega |\nabla u^n|^2 < -\frac{2}{3}\int_\Omega f\cdot u^n \leq \frac{2}{3}\|f\|_{H^{-1}}\|u^n\|_{H_0^1}.$$

Consequently, we have

$$\|u^n\|_{H_0^1} < 4\|f\|_{H^{-1}}.$$

On the other hand, (2.14) implies

$$0 < \frac{t_0^2}{6}\|f\|_{H^{-1}}^2 < -\frac{2}{3}\int_\Omega f\cdot u^n \tag{2.15}$$

for n large, which gives

$$\frac{t_0^2}{4}\|f\|_{H^{-1}} < \|\nabla u^n\|_{L^2}.$$

This proves (c).

Finally, to obtain (d), we shall argue by contradiction and assume $\|E'(u^n)\|_{H^{-1}} > 0$ for n large. Then we can get the contradiction, using Lemma 2.2 and Lemma 2.3. □

Proof of Theorem 1: From Proposition 2.2 we have obtained a minimizing Palais-Smale sequence $\{u^n\}$ for E, with a uniform H_0^1-bound. Let $\underline{u} \in H_0^1(\Omega;\mathbf{R}^3)$ be the weak limit of (a subsequence of) $\{u^n\}$. From (2.15) we note that $-\int_\Omega f\cdot\underline{u} > 0$.

By Proposition 2.2(d) and the fact that

$$\langle E'(u^n), w\rangle \to \langle E'(\underline{u}), w\rangle, \quad \forall w \in H_0^1(\Omega;\mathbf{R}^3)$$

(this follows from the weak continuity of $u_x^n \wedge u_y^n$:

$$u_x^n \wedge u_y^n \to \underline{u}_x \wedge \underline{u}_y \quad \text{in } \mathcal{D}'(\Omega;\mathbf{R}^3),$$

See [BC:Lemma A.9]), we have

$$\langle E'(\underline{u}), w\rangle = 0 \quad \text{for any } w \in H_0^1(\Omega;\mathbf{R}^3).$$

That is, $\underline{u}$ is a weak solution of (1.1) and in particular $\underline{u} \in \Lambda$.

Therefore

$$c_0 \leq E(\underline{u}) = \frac{1}{6}\int_\Omega |\nabla\underline{u}|^2 + \frac{2}{3}\int_\Omega f\cdot\underline{u} \leq \liminf_{n\to\infty} E(u^n) = c_0.$$

Consequently $u^n \to \underline{u}$ strongly in H_0^1 and $E(\underline{u}) = c_0 = \inf_\Lambda E$.

This proves Theorem 1. □

3. Existence of the second solution

In this section, we shall prove the existence of the second solution of problem (1.1) by the Mountain Pass Theorem of Ambrosetti-Rabinowitz [AR].

To this end, we first derive that $\underline{u}$ is a strict local minimum for E in $H_0^1(\Omega; \mathbf{R}^3)$, if f satisfies the assumption (f.3).

Proposition 3.1. *Let $f \not\equiv 0$ satisfy (f.1) and (f.3), then $\underline{u}$ obtained in Theorem 1.1 is a strict local minimum for E in $H_0^1(\Omega; \mathbf{R}^3)$.*
Proof: For any $v \in H_0^1(\Omega; \mathbf{R}^3)$ we expand:

$$\begin{aligned}
E(\underline{u}+v) &= \frac{1}{2}\int_\Omega |\nabla(\underline{u}+v)|^2 + \frac{2H}{3}Q(\underline{u}+v) + \int_\Omega f\cdot(\underline{u}+v) \\
&= \frac{1}{2}\int_\Omega |\nabla \underline{u}|^2 + \frac{2H}{3}Q(\underline{u}) + \int_\Omega f\cdot \underline{u} + \left[\int_\Omega \nabla\underline{u}\cdot\nabla v + 2H\int_\Omega \underline{u}_x \wedge \underline{u}_y\cdot v + \int_\Omega f\cdot v\right] \\
&+ \frac{1}{2}\int_\Omega |\nabla v|^2 + 2H\int_\Omega \underline{u}\cdot v_x\wedge v_y + \frac{2H}{3}Q(v) \\
&= E(\underline{u}) + \frac{1}{2}\int_\Omega |\nabla v|^2 + 2H\int_\Omega \underline{u}\cdot v_x\wedge v_y + \frac{2H}{3}Q(v).
\end{aligned}$$

Now, by Wente's L^2-estimate and the isoperimetric inequality, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla v|^2 + 2H\int_\Omega \underline{u}\cdot v_x\wedge v_y + \frac{2H}{3}Q(v) \\
&\geq \frac{1}{2}||\nabla v||_{L^2}^2 - 2HC_{L^2}\cdot||\nabla\underline{u}||_{L^2}||\nabla v||_{L^2}^2 - \left(\frac{2H}{3}\right)\left(\frac{1}{S}\right)^{3/2}||\nabla v||_{L^2}^3,
\end{aligned} \tag{3.1}$$

where $C_{L^2} = \sqrt{\frac{3}{16\pi}}$ is the best constant for Wente's L^2-estimate [Ge] and $S = (32\pi)^{1/3}$.

We denote

$$h(x) = \left(\frac{1}{2} - 2HC_{L^2}||\nabla\underline{u}||_{L^2}\right)x^2 - \left(\frac{2H}{3}\right)\left(\frac{1}{S}\right)^{3/2}x^3, \quad x \geq 0,$$

then it is easy to see that $h(x) > 0$ for sufficiently small $x > 0$ if $\frac{1}{2} - 2HC_{L^2}||\nabla\underline{u}||_{L^2} > 0$, that is,

$$||\nabla\underline{u}||_{L^2} < \frac{1}{4HC_{L^2}}. \tag{3.2}$$

Recall that $\underline{u}$ satisfies the estimate $||\nabla\underline{u}||_{L^2} \leq 4||f||_{H^{-1}}$ (by Proposition 2.2(c)), therefore if

$$||f||_{H^{-1}} < \frac{1}{16HC_{L^2}},$$

that is, under the assumption (f.3), we certainly have (3.2).

In conclusion, (f.3) implies that

$$E(\underline{u}+v) = E(\underline{u}) + h(||v||_{H_0^1}) > E(\underline{u})$$

for every sufficiently small $v \in H_0^1(\Omega; \mathbf{R}^3)$, so $\underline{u}$ is a strict local minimum for E. $\square$

Next, we study the compactness properties of the functional E. The following proposition is now more or less a standard result in this direction.

Proposition 3.2 (local compactness). *E satisfies the $(PS)_c$ condition for all $c < c_0 + \frac{4\pi}{3H^2}$. That is, every sequence $\{u^n\} \subset H_0^1(\Omega; \mathbf{R}^3)$ satisfying :*

(a) $E(u^n) \to c < c_0 + \frac{4\pi}{3H^2}$,

(b) $||E'(u^n)||_{H^{-1}} \to 0$,

has a strong convergent subsequence.

To proceed further, we need some definition. Let

$$\varphi^\varepsilon(x,y) = \frac{2\varepsilon}{\varepsilon^2 + x^2 + y^2} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix}, \qquad \varepsilon > 0 \tag{3.3}$$

be an extremal function for the isoperimetric inequality in $\mathbf{R}^2$.

For $a = (x_0, y_0) \in \Omega$, denote $\varphi^{\varepsilon,a}(x,y) = \varphi^\varepsilon(x - x_0, y - y_0)$, and let $\zeta_a \in C_0^\infty(\Omega)$ be the cut-off function with $0 \leq \zeta_a \leq 1$, $\zeta_a = 1$ near a. We set

$$v^{\varepsilon,a}(x,y) = \zeta_a(x,y)\varphi^{\varepsilon,a}(x,y). \tag{3.4}$$

Now, by calculating directly along the explicit path, we get the following proposition.

Proposition 3.3. *For every $R > 0$ and almost everywhere $a = (x_0, y_0) \in \{(x,y) \in \Omega : \nabla \underline{u}(x,y) \neq 0\}$, there exist an $\varepsilon_0 = \varepsilon_0(R,a) > 0$ and an orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$ in $\mathbf{R}^3$ having the same orientation as the canonical basis of $\mathbf{R}^3$, such that*

$$E(\underline{u} - Rv^{\varepsilon,a}) < c_0 + \frac{4\pi}{3H^2}$$

holds for every $0 < \varepsilon < \varepsilon_0$. Here we assume that $\varphi^{\varepsilon,a}$ is written with respect to $(\vec{i}, \vec{j}, \vec{k})$.

At this point, we recall the famous Mountain Pass Theorem of Ambrosetti-Rabinowitz [AR] in its standard form.

Mountain Pass Theorem. *Let F be a C^1-functional on a Banach space V. Suppose*

(1) *$F(0) = 0$;*

(2) *$\exists \rho, \alpha > 0$ such that $||v||_V = \rho \Longrightarrow F(v) \geq \alpha$;*

(3) *$\exists v^* \in V$ such that $||v^*||_V \geq \rho$ and $F(v^*) < 0$.*

Define

$$\Gamma = \{\gamma \in C^0([0,1];V) : \gamma(0) = 0, \gamma(1) = v^*\}$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} F(\gamma(t)) \ (\geq \alpha).$$

Then, there exists a sequence $\{v^n\} \subset V$ *such that*

$$F(v^n) \to c$$

and

$$F'(v^n) \to 0 \quad in\ V^*.$$

Further if F *satisfies the* $(PS)_c$ *condition, then there exists a critical point at the level* c.

Proof of Theorem 2 :

We only need to apply the Mountain Pass Theorem to the functional $F(v) = E(\underline{u}+v) - E(\underline{u})$ on $V = H_0^1(\Omega;\mathbf{R}^3)$. (1) is trivially satisfied and Proposition 3.1 implies (2). (3) is also verified because $E(\underline{u} - Rv^{\varepsilon,a}) \to -\infty$ as $R \to \infty$; we set $v^* = R_0(-v^{\varepsilon,a})$ for some $R_0 > 0$ large enough.

Proposition 3.2 and 3.3 implies the $(PS)_c$ condition for F. Therefore we have a critical point v^0 of F,$F(v^0) = c \geq \alpha > 0$, that is, we conclude there exists a critical point $\overline{u} := \underline{u} + v^0$ of E, $\overline{u} \not\equiv \underline{u}$.

The proof is completed. □

References

[AE] J.P. Aubin, and I. Ekeland. *Applied Nonlinear Analysis,* Pure and Applied Mathematics, Wiley Interscience Publications, 1984.

[AR] A. Ambrosetti, and P.H. Rabinowitz. *Dual variational methods in critical point theory and applications,* J. Funct. Anal. **14** (1973) 349-381.

[BaC] A. Bahri, and J.M. Coron. *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain,* Comm. Pure Apll. Math. **41** (1988) 253-294.

[BC] H. Brezis, and J.M. Coron. *Multiple solutions of H-systems and Rellich's conjecture,* Comm. Pure Apll. Math. **37** (1984) 149-187.

[Cha] J. Chabrowski. *Variational Methods for Potential Operator Equations,* (Studies in Math. 24) Walter de Gruyter, Berlin - New York 1997.

[Co] J.M. Coron. *Topologie et cas limite des injections de Sobolev,* C. R. Acad. Paris Ser. I. **299** (1984) 209-212.

[DW] Y. Deng, and G. Wang. *On inhomogeneous biharmonic equations involving critical exponents,* Proc. Royal Soc. Edinburgh. **129A** (1999) 925-946.

[Ge] Y. Ge. *Estimations of the best constant involving the L^2 norm in Wente's inequality,* (Preprint) 1996.

[Pa] D. Pasasseo. *Relative category and multiplicity of positive solutions for the equation $\Delta u + u^{2^*-1} = 0$,* Nonlinear Anal. T.M.A **33** (1998) 509-517.

[P] S. Pohozaev. *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$,* Soviet Math. Dokl. **6** (1965) 1408-1411.

[S] M. Struwe. *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 2nd edition,* Springer-Verlag, Berlin, 1996.

[TF1] F. Takahashi. *Multiple solutions of inhomogeneous H-systems with zero Dirichlet boundary conditions,* (Preprint) 2000.

[TF2] F. Takahashi. *A minimization problem related to the isoperimetric inequality for H_0^1-mappings,* (Preprint) 2000.

[Ta1] G. Tarantello. *On nonhomogeneous elliptic equations involving critical Sobolev exponent,* Ann. Inst. Henri Poincaré. **9**, No.3 (1992) 281-304.

[Ta2] G. Tarantello. *Multiplicity results for an inhomogeneous Neumann problem with critical exponent,* manuscripta math. **81** (1993) 57-78.

[W] H. Wente. *The differential equation $\Delta x = 2Hx_u \wedge x_v$ with vanishing boundary values,* Proc. Amer. Math. Soc. **50** (1975) 131-137.