

Navier-Stokes Flow Down a Vertical Column : an Axisymmetric Case

By

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1 Introduction

The Navier-Stokes equations describing an axially symmetric flow down a vertical column can be written in the following form:

$$(1.1) \quad \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{\mathcal{R}} \left(\left(\frac{\partial}{\partial r} \right)^2 v_r + \frac{1}{r} \frac{\partial v_r}{\partial r} + \left(\frac{\partial}{\partial z} \right)^2 v_r - \frac{1}{r^2} v_r \right),$$

$$(1.2) \quad \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\mathcal{R}} \left(\left(\frac{\partial}{\partial r} \right)^2 v_z + \frac{1}{r} \frac{\partial v_z}{\partial r} + \left(\frac{\partial}{\partial z} \right)^2 v_z \right) - \frac{2}{\mathcal{R}},$$

$$(1.3) \quad \frac{\partial}{\partial r} (r v_r) + \frac{\partial}{\partial z} (r v_z) = 0$$

The flow takes place in the asymmetric layer whose cross-section along the z -axis is given by

$$\{ (r, z) ; a < r < a + 1 + h(z, t) \}$$

at time $t > 0$. For the equations themselves, see [1, 8, 4].

Here we have used the dimensionless variables adopted in [8]. The positive constant a is the nondimensional radius of column and $r = a + 1 + h(z, t)$ is the unknown free surface. We take the mean thickness of the fluid layer as the

unit of length. v_r and v_z are the radial component and the z component of the velocity vector, respectively. p is the scalar pressure. The condition at the solid surface is the adhesion condition

$$(1.4) \quad v_r = v_z = 0 \quad \text{at} \quad r = a.$$

The stress balance conditions at the outer surface are the followings :

$$(1.5) \quad \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \left(1 - \left(\frac{\partial h}{\partial z} \right)^2 \right) + 2 \left(\frac{\partial v_r}{\partial r} - \frac{\partial v_z}{\partial z} \right) \frac{\partial h}{\partial z} = 0,$$

$$(1.6) \quad -p + \frac{2}{\mathcal{R}} \left\{ \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \left(\frac{\partial h}{\partial z} \right)^2 - \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \left(\frac{\partial h}{\partial z} \right) \right\} \left(1 + \left(\frac{\partial h}{\partial z} \right)^2 \right)^{-1} \\ = -p_0 - \gamma \left\{ \frac{1}{r} \left(1 + \left(\frac{\partial h}{\partial z} \right)^2 \right)^{-\frac{1}{2}} - \frac{\partial^2 h}{\partial z^2} \left(1 + \left(\frac{\partial h}{\partial z} \right)^2 \right)^{-\frac{3}{2}} \right\} \\ \text{at } r = a + 1 + h(z, t), \quad z \in \mathbf{R}, \quad t > 0.$$

Here p_0 is the atmospheric pressure and γ is the surface tension coefficient. The kinematic impermeability condition at $r = a + 1 + h(z, t)$, $z \in \mathbf{R}$ is written as follows

$$(1.7) \quad \frac{\partial h}{\partial t} - v_r + v_z \frac{\partial h}{\partial z} = 0.$$

A stationary solution to this problem is the following:

$$(1.8) \quad (v_r^{(0)}, v_z^{(0)}) = \left(0, \frac{1}{2}(r^2 - a^2) - (1 + a)^2 \log \frac{r}{a} \right),$$

$$(1.9) \quad p^{(0)} = p_0 + \frac{\gamma}{1 + a}, \quad h^{(0)} \equiv 0.$$

In this article we consider the evolution problem for downward periodic disturbances from this equilibrium state and announce some existence results.

2 Formulation

We first set

$$(v_r, v_z) = (u_r + v_r^{(0)}, u_z + v_z^{(0)}) , \quad p = q + p^{(0)}.$$

To reduce the problem to the one in a fixed domain, we use the method employed in [2]. Let Ω be the equilibrium domain

$$\{ (r', z') ; a < r' < 1 + a, 0 < z' < \ell \} .$$

Here ℓ denotes the periodicity in z - axis . Let $\tilde{h}$ be an extension of h to $\{a < r' < a + 1\}$. For each $t \geq 0$ we define diffeomorphisms

$$\Theta : \Omega \rightarrow \Omega(t) = \{(r, z); a < r < a + 1 + h(z, t), 0 < z < \ell\}$$

by

$$r = (r' - a) \left(1 + \tilde{h}(r', z', t) \right) + a , \quad z = z' .$$

For (u_1, u_2) on Ω we define (u_r, u_z) on $\Omega(t)$ by

$$(2.1) \quad u_r = \frac{r'}{r} u_1 + \frac{r'(r' - a)}{rJ} \frac{\partial \tilde{h}}{\partial z'} u_2$$

$$(2.2) \quad u_z = \frac{r'}{rJ} u_2 .$$

J denotes the jacobian of Θ . Set $q = q(\Theta(r', z'))$.

We next introduce the coordinates attached to the unperturbed flow :

$$r' = x_1 , \quad z' = x_2 + \left\{ \frac{1}{2} \left((a+1)^2 - a^2 \right) - (1+a)^2 \log \frac{1+a}{a} \right\} t' ; \quad t = t'.$$

We finally introduce an auxiliary unknown defined by

$$(2.3) \quad \eta = \frac{1}{2} \left((1+a+h)^2 - (1+a)^2 \right) .$$

In the same manner as in [Sect. 5 , 2], we can derive the problem for the new unknowns. Collecting linear terms in the left hand sides, we can rewrite our problems as follows

$$(2.4) \quad \frac{\partial \eta}{\partial t} - (1+a)u_1 = -\frac{\partial}{\partial x_2} \left[\frac{1}{2} \eta^2 - \frac{(1+a)^4}{4} \left(\left(1 + \frac{2\eta}{(1+a)^2} \right) \right. \right. \\ \left. \left. \times \left(\log \left(1 + \frac{2\eta}{(1+a)^2} \right) - 1 \right) + 1 \right) \right] \quad \text{on } x_1 = 1+a,$$

$$(2.5) \quad \begin{aligned} & \frac{\partial u_j}{\partial t} + \left(\frac{1}{2} (x_1^2 - (1+a)^2) - (1+a)^2 \log \frac{x_1}{1+a} \right) \frac{\partial u_j}{\partial x_2} \\ & - \frac{1}{\mathcal{R}} \left(\frac{\partial^2 u_j}{\partial x_1^2} + \frac{1}{x_1} \frac{\partial u_j}{\partial x_1} + \frac{\partial u_j}{\partial x_2^2} - \delta_{j1} \frac{u_1}{x_1^2} \right) \\ & + u_1 \delta_{j2} \left(x_1 - \frac{(1+a)^2}{x_1} \right) + \frac{\partial p}{\partial x_j} = f_j, j = 1, 2, \end{aligned}$$

$$(2.6) \quad \begin{aligned} & \frac{\partial}{\partial x_1} (x_1 u_1) + \frac{\partial}{\partial x_2} (x_1 u_2) = 0 \\ & \text{in } a < x_1 < 1+a, \end{aligned}$$

$$(2.7) \quad u_1 = u_2 = 0 \quad \text{on } x_1 = a,$$

$$(2.8) \quad (1+a) \frac{\partial u_1}{\partial x_2} + (1+a) \frac{\partial u_2}{\partial x_1} + 2\eta = f_3,$$

$$(2.9) \quad \begin{aligned} & p - \frac{2}{\mathcal{R}} \frac{\partial u_1}{\partial x_1} + \gamma \left(\frac{1}{(1+a)^3} \eta + \frac{1}{(1+a)} \frac{\partial^2 \eta}{\partial x_2^2} \right) = f_4, \\ & \text{on } x_1 = 1+a, \end{aligned}$$

$$(2.10) \quad \int_0^\ell \eta(x_2, t) dx_2 = 0, \quad t > 0.$$

Each f_j in the right hand sides consists of nonlinear terms.
Suppose that initial data

$$(2.11) \quad \eta_0(x_2), \quad 0 < x_2 < \ell,$$

$$(2.12) \quad u_0(x_1, x_2), \quad a < x_1 < 1+a, \quad 0 < x_2 < \ell$$

are given with certain compatibility conditions.

3 Results

Let $\Omega = (a, a+1) \times (0, \ell)$ in $\mathbf{R}^2$. Let $r \geq 0$. $H^r(\Omega)$ denotes the space of functions which belong to $H_{loc}^r((a, 1+a) \times \mathbf{R})$ and are periodic in x_2 with period ℓ . Let $S_F = \Omega \cap \{X_1 = 1+a\}$ and let $S_W = \Omega \cap \{X_1 = a\}$. We identify S_F with the interval $(0, \ell)$. Denote $H^r(S_F)$ by the space of functions which belong to $H_{loc}^r(\mathbf{R})$ and are periodic with period ℓ .

We now state our results obtained until now.

Theorem 3.1 Let $\mathcal{R} > 0$ and $\gamma > 0$ be arbitrary. Let $0 < \delta < \frac{1}{4}$. Let $u_0 \in H^{1+2\delta}(\Omega)$ and $\eta_0 \in H^{\frac{3}{2}+2\delta}(S_F)$. Assume that

$$(3.1) \quad \frac{\partial}{\partial x_1}(x_1 u_{0,1}) + \frac{\partial}{\partial x_2}(x_1 u_{0,2}) = 0 \quad \text{in } \Omega ,$$

$$(3.2) \quad u_0 = 0 \quad \text{on } S_W ,$$

$$(3.3) \quad \sup \left(|\eta_0| + \left| \frac{\partial \eta_0}{\partial x_2} \right| \right) < 1 \quad \text{on } S_F ,$$

$$(3.4) \quad \int_0^\ell \eta_0(x_2) dx_2 = 0 .$$

Then there is $T > 0$, so that the problem (2.4) – (2.12) has a solution (η, u, q) with

$$(3.5) \quad \eta \in H^0(0, T; H^{\frac{5}{2}+2\delta}(S_F)) \cap H^{\frac{5}{4}+\delta}(0, T; H^0(S_F)) ,$$

$$(3.6) \quad u \in H^0(0, T; H^{2+2\delta}(\Omega)) \cap H^{1+\delta}(0, T; H^0(\Omega)) ,$$

$$(3.7) \quad q \in H^0(0, T; H^{1+2\delta}(\Omega)) \cap H^{\frac{1}{2}+\delta}(0, T; H^0(\Omega)) .$$

Theorem 3.2 Let $T_1 > 0$ be given. Assume that (3.1)–(3.4). Then there exist $\mathcal{R}_0 > 0, \gamma_0 > 0, a_0 > 0$ and $\varepsilon > 0$ such that, if

$$\mathcal{R}_0 > \mathcal{R} \quad , \quad \gamma_0 < \gamma \quad , \quad a_0 < a$$

and

$$\|u_0\|_{1+2\delta} + |\eta_0|_{H^{\frac{3}{2}+2\delta}(S_F)} \leq \varepsilon ,$$

then the solution (η, u, q) obtained in Theorem 3.1 can be continued to any $t > 0$ and satisfies

$$\begin{aligned} u &\in C(T_1, \infty; H^2(\Omega)) , \\ \eta &\in C(T_1, \infty; H^3(S_F)) , \\ \nabla q &\in C(T_1, \infty; H^0(\Omega)) . \end{aligned}$$

To prove these we use the ideas developed in [2] and [7]. The integral identity below plays an important role. If $v, u \in H^2(\Omega)$, $p \in H^1(\Omega)$ and

$$\frac{\partial}{\partial x_1}(x_1 u_1) + \frac{\partial}{\partial x_2}(x_1 u_2) = 0 \quad \text{in } \Omega ,$$

then integration by parts gives

$$\begin{aligned}
 & \int_{\Omega} v_1 \left[-\frac{1}{\mathcal{R}} \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{x_1} \frac{\partial u_1}{\partial x_1} + \frac{\partial^2 u_1}{\partial x_2^2} - \frac{u_1}{x_1^2} \right) + \frac{\partial p}{\partial x_1} \right] x_1 d_1 d_2 \\
 & + \int_{\Omega} v_2 \left[-\frac{1}{\mathcal{R}} \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{1}{x_1} \frac{\partial u_2}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + \frac{\partial p}{\partial x_2} \right] x_1 d_1 d_2 \\
 & = \int_{\partial\Omega} v_1 \left(p - \frac{2}{\mathcal{R}} \frac{\partial u_1}{\partial x_1} \right) x_1 d_2 + \int_{\partial\Omega} v_2 \left(-\frac{2}{\mathcal{R}} \right) \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) x_1 d_2 \\
 & \quad + \frac{1}{2\mathcal{R}} \langle v, u \rangle,
 \end{aligned}$$

where

$$\begin{aligned}
 \langle v, u \rangle &= \sum_{j,k=1}^2 \int_{\Omega} \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right) \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) x_1 d_1 d_2 \\
 & \quad + 4 \int_{\Omega} \frac{v_1}{x_1} \frac{u_1}{x_1} x_1 d_1 d_2.
 \end{aligned}$$

Note that, in Ω , $0 < a < x_1 < 1 + a$. By Korn's inequality (see, for example, [3, 5]), we have

Proposition 3.3 *If $v \in H^1(\Omega)$ and $v = 0$ on S_W , then it holds that*

$$(3.8) \quad \|v\|_1 \leq C \langle v, u \rangle.$$

Here C is independent of v .

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