

Curves on Rational Surfaces with Hyperelliptic Hyperplane Sections

Ovidiu Păşărescu*

Abstract

In this article we study, given a pair of integers (d, g) , the problem of existence of a smooth, irreducible, non-degenerate curve in $\mathbf{P}^n$ of degree d and genus g (the Halphen-Castelnuovo problem). We define two domains from the (d, g) -plane, D_1^n and D_2^n , and we prove that there is no gap in D_1^n . This follows constructing curves on some rational surfaces with hyperelliptic hyperplane sections and from some previous theorems of Ciliberto, Sernesi and of the author. Moreover, in the last section, based on some results of Horowitz, Ciliberto, Harris, Eisenbud, we conjecture that D_2^n is the right lacunary domain.

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1 Introduction

In this article we work over an algebraically closed field K of characteristic zero (e.g. $K = \mathbf{C}$ the complex field); see the section 4 also for arbitrary characteristic. We'll use the standard notations, see [Ha1].

By a *curve* (resp. a *surface*) we mean a K -algebraic integral scheme of dimension 1 (resp. 2). If not otherwise specified the curves and surfaces appearing in this paper will be supposed non-singular.

A (possible singular) curve $C \subset \mathbf{P}^n$ (= the projective n -space) is called *non-degenerate* if it is not contained in any hyperplane.

Given a property $\mathcal{P}$ we call *$\mathcal{P}$ -Halphen Castelnuovo Theory in $\mathbf{P}^n$* the study of existence - for arbitrary, but fixed triplets of integers (n, d, g) , $n \geq 2$, $d \geq n$, $g \geq 0$ - of non-degenerate (possible singular) curves $C \subset \mathbf{P}^n$ of degree d and geometric genus g , and having the property $\mathcal{P}$. Then it is natural to study the (non-empty) families $\mathcal{F}_{d,g}^{n,\mathcal{P}}$ of curves as before.

The property $\mathcal{P}$ could be, for instance, smootheness, linear normality, projective normality, maximal rank, expected number of moduli, etc. If $\mathcal{P}$ =smootheness and $H_{d,g}^n$ is the Hilbert scheme of smooth curves from $\mathbf{P}^n$ (i.e. the closure in the general Hilbert scheme of the locus of smoth, irreducible, non-degenerate curves from $\mathbf{P}^n$), then one could take $\mathcal{F}_{d,g}^{n,\mathcal{P}} = H_{d,g}^n$.

When studying the non-degenerate *embedded curves* $C \subset \mathbf{P}^n$ for some $n \geq 2$ one usually study the previous schemes $H_{d,g}^n$, the first step being to find the triplets of integers (n, d, g) for which $H_{d,g}^n \neq \emptyset$. Let's now denote, as usual, by $\mathcal{M}_g$ the moduli space of curves of genus g . Then, there is a natural rational map $\pi = \pi_{d,g}^n : H_{d,g}^n \dashrightarrow \mathcal{M}_g$ whose fibers over the $[C] \in \mathcal{M}_g$ are related to the determinantal varieties $W_d^n = W_d^n(C) := \{g_d^s | g_d^s \text{ a base-point free linear system of degree } d \text{ and dimension } s, s \geq n\}$ appearing in the Brill-Noether Theory for *polarized curves* (C, g_d^s) . As explained by J. Harris in [H], the rational map π is very useful when studying the connections between the *extrinsic geometry* (represented by the properties of projective embeddings and of the Hilbert scheme) and the *intrinsic geometry* (represented by the properties of the *abstract curves* and of the moduli space) of the families of

curves. Certainly, this is possible when $\pi \neq \emptyset$ (so, again when $H_{d,g}^n \neq \emptyset$).

We recall now the *Castelnuovo bound* (found in 1893, for a modern proof, see [H], ch.3):

Theorem 1.1 (Castelnuovo [C]). *If $n \in \mathbf{Z}$, $n \geq 3$ and $C \subset \mathbf{P}^n$ is a non-degenerate (possibly singular) curve of degree d and geometric genus g , then $d \geq n$ and $0 \leq g \leq \pi_0(d, n)$.*

This result has been a generalization of a previous similar result, obtained by Halphen and Noether in 1882 for $n = 3$ ([Hl], [N]).

Here $\pi_0(d, n)$ is the first one of the Harris-Eisenbud numbers $\pi_p(d, n)$, introduced in [H], ch.3 (for $p \geq 1$), and given (for $0 \leq p \leq n - 2$) by:

$$(1.1) \quad \pi_p = \pi_p(d, n) = \frac{m_p(m_p - 1)}{2}(n + p - 1) + m_p(\varepsilon_p + p) + \mu_p, \quad \text{where}$$

$$(1.2) \quad m_p = m_p(d, n) = [(d - 1)/(n + p - 1)]_*$$

(we denote, during this article, by $[x]_*$ the integer part of $x \in \mathbf{R}$ = the field of the real numbers)

$$(1.3) \quad \varepsilon_p = \varepsilon_p(d, n) = d - 1 - m_p(n + p - 1)$$

$$(1.4) \quad \mu_p = \mu_p(d, n) = \max(0, [(p - n + 2 + \varepsilon_p)/2]_*).$$

We remark that $\mu_0 = 0$ and $\pi_p = d^2/(2(n + p - 1)) + O(d)$.

We recall, as an example, that if $\mathcal{P}$ =nodality, the complete answer in $\mathcal{P}$ -Halphen-Castelnuovo Theory is given by the following:

Theorem 1.2 (Tannenbaum [T₁], [T₂]). *For $n \geq 2$, $n \in \mathbf{Z}$ and any $d, g \in \mathbf{Z}$, $d \geq n$ and $0 \leq g \leq \pi_0(d, n)$ there is a non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and geometric genus g having only nodes as singularities.*

This Theorem generalizes a similar result of Severi from 1915 for $n = 2$ ([Sv]).

If $\mathcal{P}$ = smoothness we arrive to the *Halphen-Castelnuovo Problem* (related to the classification of the abstract curves and to the comparison between the intrinsic and extrinsic geometry of curves), denoted by $HC(n)$, for ≥ 2 , $n \in \mathbf{Z}$:

HC(n): For which pairs of integers (d, g) , $d \geq n$, $0 \leq g \leq \pi_0(d, n)$ do we have $H_{d,g}^n \neq \emptyset$ (equivalently, for which pairs (d, g) as before there is a (smooth, irreducible) non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g ?).

This is the Problem which we'll consider in this article.

Definition 1.3. A pair of integers (d, g) , $d \geq n$, $0 \leq g \leq \pi_0(d, n)$ is called a **gap** for $HC(n)$ if $H_{d,g}^n = \emptyset$ (i.e. there is no non-degenerate (smooth, irreducible) curve $C \subset \mathbf{P}^n$ of degree d and genus g).

From the Theorem 1.2 follows that there is no gap in $\mathcal{P}$ -Halphen-Castelnuovo Theory when $\mathcal{P} = \text{nodality}$. However, the situation is different for $HC(n)$, so the Problem is more complicated.

Indeed, $HC(2)$ is simple: $H_{d,g}^2 \neq \emptyset \Leftrightarrow d \geq 2$ and $g = (d-1)(d-2)/2$. But, for $n \geq 3$, $HC(n)$ becomes highly non-trivial. A (correct) answer to $HC(3)$ has been stated by Halphen in 1882 ([H1]), but his proof has been, partially, incorrect. A complete proof has been given, 100 years later, by Gruson and Peskine ([GP1], [GP2]). The solution of $HC(3)$ is the following:

Theorem 1.4 (Halphen-Gruson-Peskine). For $HC(3)$ there are two domains in the (d, g) -plane: the **non-lacunary** domain D_1^3 , where there is no gap for $HC(3)$ and the **lacunary** domain D_2^3 , where there are gaps. Here

$$D_1^3 : 0 \leq g \leq \pi_1(d, 3), \quad d \geq 3$$

$$D_2^3 : \pi_1(d, 3) < g \leq \pi_0(d, 3), \quad d \geq 3.$$

Moreover, a pair $(d, g) \in D_2^3$ is **not** a gap for $HC(3)$ iff $(d-2)^2 - 4g$ is the square of some integer (see the Theorem 1.1 also).

$HC(n)$ has been also solved for $n = 4, 5$ by Rathmann ([Ra]), $n = 6$ by Ciliberto ([Ci]) and "almost" solved by Ciliberto if $n = 7$ ([Ci]). The situation is similar to the case $n = 3$ in the sense that there are two domains: D_1^n , where there is no gap, and D_2^n , where there are gaps. The domains D_1^n and D_2^n ($n \in \{4, 5, 6, 7\}$) are explicitly defined and the gaps from D_2^n are also determined (with the exception of 24 pairs for $n = 7$ for which is not yet known if they are gaps or not). Results for the general $HC(n)$ belongs to Ciliberto, Sernesi ([Ci], [CS]), Harris, Eisenbud ([H]), Rathmann ([R]), Păsarescu ([P1], [P2], [P3]) and others. We'll use some of them in the next section (see Section 4 also).

We end this section remarking that the case $n = 3$ of $HC(n)$ has some intrinsic importance, because any (smooth, irreducible) curve from some $\mathbf{P}^n$, $n \geq 4$, can be projected (so, preserving the degree) isomorphically (so, preserving the genus) on $\mathbf{P}^3$. But, for more sophisticated reasons, consisting in the comparison between the extrinsic and intrinsic properties, the natural problem is the general $HC(n)$, $n \geq 3$.

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2 The functions $\alpha_p(d, n)$; the domains D_1^n, D_2^n and A_p^n ; the Main Theorem: absence of gaps in D_1^n

Before stating the Main Theorem which will be proved in this article, we need a number of definitions.

For $p, n \in \mathbf{Z}$, $p \geq 0$, $n \geq 3$ we *define* the following numerical functions $\alpha_p(d, n) = d^2/(2(n + p - 1)) + O(d)$ given by:

$$(2.1) \quad \alpha_p = \alpha_p(d, n) = \frac{x_p(x_p - 1)}{2}(n + p - 1) + x_p(t_p + p) + u_p, \quad \text{where}$$

$$(2.2) \quad x_p = x_p(d, n) := [(d - a_p^n)/(n + p - 1)]_*$$

$$(2.3) \quad a_p = a_p^n := [(n - p)/2]_* + 1$$

$$(2.4) \quad t_p = t_p(d, n) := d - 1 - x_p(n + p - 1)$$

$$(2.5) \quad u_p = u_p(d, n) := [(p - n + 1 + t_p)/2]_*.$$

We also consider the functions

$$(2.6) \quad d_1(n) := \begin{cases} \max(2n + 1, \frac{1}{6}(3 + (4k - 1)\sqrt{24k - 33})), & n \equiv 0 \pmod{3} \\ \max(2n + 1, \frac{4k}{4k + 1}(-4k + \sqrt{32k^3 + 16k^2 - 2k - 1})), & n \equiv 1 \pmod{3} \\ \max(2n + 1, 5k + 3 + (2k + 1)\sqrt{48k + 6}), & n \equiv 2 \pmod{3} \end{cases}$$

where $k = [n/3]_*$, $n \geq 4$.

We'll need also (with k from before):

$$(2.7) \quad A(d, n) := \begin{cases} \pi_k(d, n), & \text{if } n \equiv 0, 1 \pmod{3} \\ \pi_k(d, n) - \mu_k + \max(0, \varepsilon_k - 3k - 1), & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

(see (1.1)-(1.4)).

$$(2.8) \quad B(d, n) := \begin{cases} \alpha_{k+1}(d, n), & \text{if } n \equiv 1, 2 \pmod{3} \\ \alpha_k(d, n), & \text{if } n \equiv 0 \pmod{3} \end{cases}, \quad \text{if } 2n + 1 \leq d < d_1(n) \\ A(d, n), & \text{if } d \geq d_1(n)$$

We consider now the following graphs (contained in the (d, g) -plane), of equations:

$$\begin{aligned} B_0^n &: g = \pi_0(d, n) \\ B_k^n &: g = \begin{cases} \pi_0(d, n), & n \leq d \leq 2n \\ B(d, n), & d \geq 2n + 1 \end{cases}, \quad k = [n/3]_* \\ C_p^n &: g = \alpha_p(d, n), \quad p \geq n/3, \quad p, n \in \mathbf{Z}, \quad n \geq 3 \end{aligned}$$

Now we are ready to *define* the domains D_1^n (bounded by $g = 0$ and B_k^n) and D_2^n (bounded by B_k^n and B_0^n), contained in the (d, g) -plane:

$$(2.9) \quad D_1^n : 0 \leq g \leq \begin{cases} \pi_0(d, n), & n \leq d \leq 2n \\ B(d, n), & d \geq 2n + 1 \end{cases}$$

$$(2.10) \quad D_2^n : B(d, n) < g \leq \pi_0(d, n), \quad d \geq 2n + 1$$

The Main Theorem belonging to Halphen-Castelnuovo Theory which we'll prove in this article is:

MAIN THEOREM. *In the domain D_1^n there is no gap for $HC(n)$, $n \geq 4$, $n \in \mathbf{Z}$.*

Remark 2.1. *The Main Theorem is known to be true for $d > D(n) =$ a quadratic function in n (see [Ci], [CS], [P1]). So, our main contribution is for "small" degrees, $2n + 1 \leq d < D(n)$. Moreover, as we recalled in section 1 the Main Theorem is known to be true for $n \in \{4, 5, 6, 7\}$, due to*

the contributions of Rathmann ([Ra]), Ciliberto, Sernesi ([Ci], [CS]), and of the author ([P1]). So, we'll suppose that $n \geq 8$ (but our arguments work, in principle, for $5 \leq n \leq 7$ also).

During the proof of the Main Theorem, the domain D_1^n will be divided in subdomains A_p^n , defined by:

$$(2.11) \quad A_{n-3}'^n : 0 \leq g \leq \alpha_{n-3}(d, n), \quad d \geq 2n + 1 \text{ (between } g = 0 \text{ and } C_{n-3}^n)$$

$$(2.12) \quad A_p^n : \alpha_{p+1}(d - 1, n) \leq g \leq \alpha_p(d, n), \quad d \geq 2n + 1, p \geq n/3$$

(containing the domain between C_{p+1}^n and C_p^n for $p \leq n - 1$: see subsection 3.3 from section 3).

$$(2.13) \quad \tilde{A}_k^n : \alpha_{p+1}(d - 1, n) < g \leq B(d, n), \quad d \geq 2n + 1, k = [n/3]_*$$

(containing the domain between C_{k+1}^n and B_k^n)

$$(2.14) \quad A_0^n : 0 \leq g \leq \pi_0(d, n), \quad n \leq d \leq 2n.$$

Lemma 2.2. *In the domain A_0^n there is no gap for $HC(n)$, $n \geq 8$.*

Proof. We use projections of non-special curves. For $d = 2n$ and $g = \pi_0(d, n) = \pi_0(2n, n)$, we consider the extremal curve.

q.e.d.

The following equality holds (see Lemma 3.3.1b)):

$$(2.15) \quad D_1^n = A_0^n \cup A_{n-3}'^n \cup \left(\bigcup_{\frac{n}{3} \leq p \leq n-4} A_p^n \right) \cup \tilde{A}_k^n$$

We just solved the elementary step (curves in A_0^n) of $HC(n)$, $n \geq 8$ in the previous lemma. By (2.15), the Main Theorem is now the consequence of the following three Theorems:

Theorem A. *If $n \in \mathbf{Z}$, $n \geq 8$, then there is no gap for $HC(n)$ in the domain $\tilde{A}_k^n$.*

The *proof* of this Theorem essentially belongs to Ciliberto ([Ci], Theorem 1.1). However the statement of the Theorem A differs slightly from the statement of the Theorem of Ciliberto. The proof of the Theorem A can be found in [P2].

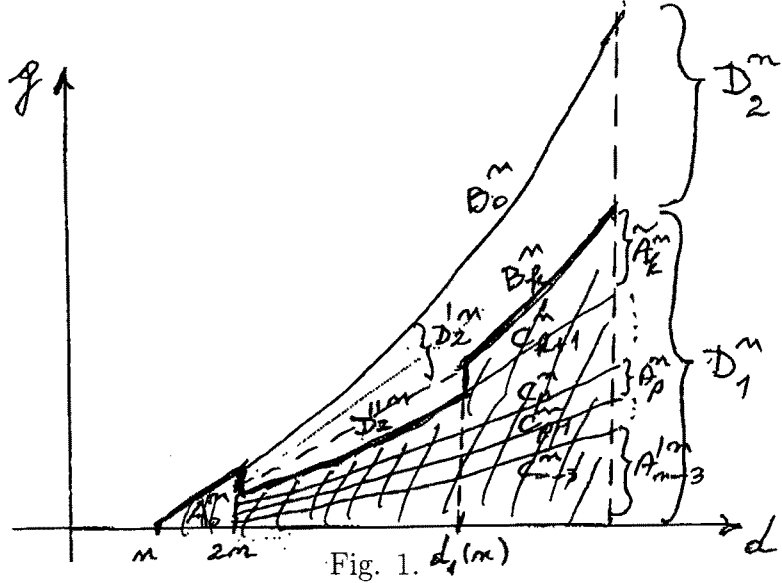
Theorem B. *If $n \in \mathbf{Z}$, $n \geq 8$, then there is no gap for $HC(n)$ in the domain $A_{n-3}'^n$.*

The *proof* of this Theorem essentially belongs to Ciliberto and Sernesi ([CS], Main Theorem), see also [P1]. However the statement of the Theorem of Ciliberto and Sernesi differs slightly from the statement of the Theorem B. The proof of the Theorem B can be found in [P3].

Theorem C. *If $n \in \mathbf{Z}$, $n \geq 8$, then there is no gap for $HC(n)$ in the domain $\bigcup_{\frac{n}{3} \leq p \leq n-4} A_p^n$.*

This Theorem C represents our main contribution. It has been already used during the proof of the Theorem A in [P2]. The complete proof of the Theorem C is given in the next section 3. It is done constructing curves on some rational surfaces $X_p^n \subset \mathbf{P}^n$ with hyperelliptic hyperplane sections.

We end this section with a picture:



The hashurated domain in D_1^n (a non-lacunary domain).

3 The proof of Main Theorem

3.1 Methods

Let's fix some notations. Let there be $\Sigma \subset \mathbf{P}^2$ a finite set of (distinct) points $\Sigma = \{P_0, P_1, \dots, P_s\}$. We denote by $S = BL_\Sigma(\mathbf{P}^2) \rightarrow \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ in Σ . Then $\text{Pic}(S) \cong \mathbf{Z} \oplus \mathbf{Z}^{s+1}$ with $(l; -e_0, -e_1, \dots, -e_s)$ a $\mathbf{Z}$ -basis (here l is the class of the inverse image in S of a line $L \subset \mathbf{P}^2$ and e_i are the classes of

the exceptional divisors $E_i \subset S$ corresponding to the points P_i , $0 \leq i \leq s$). We recall that the intersection form on $\text{Pic}(S)$ is given by:

$$(l \cdot e_i) = 0, (e_i^2) = -1, i = \overline{0, s}, (e_i \cdot e_j) = 0, (\forall) i \neq j, (l^2) = 1.$$

If $\mathcal{L} = al = \sum_{i=0}^s b_i e_i \in \text{Pic}(S)$, we write $\mathcal{L} = (a; b_0, b_1, \dots, b_s)$.

If $\mathcal{L} = \mathcal{O}_S(D)$, $D \in \text{Div}(S)$, we denote by $[\mathcal{L}] = |D|$ the complete linear system associated to $\mathcal{L}$ (resp. D). We write $[(a; b_0, b_1, \dots, b_s)] := [a; b_0, b_1, \dots, b_s]$. We say that $a, b_0, b_1, \dots, b_s$ are the *coefficients* of $\mathcal{L}$ (or $[\mathcal{L}]$).

Let's now analyse the Gruson-Peskine type construction given by Ciliberto ([Ci]). The surfaces X_k^n used by Ciliberto, containing the necessary curves ($k = [n/3]_*$) are obtained blowing up $3k - n + 6$ points from $\mathbf{P}^2$ in *general position* (i.e. any 3 non-collinear and not all of them on a smooth conic) and embedding the abstract surface obtained in such a way using the very ample invertible sheaf $(k + 2; k, \underbrace{1^{3k-n+5}}_{s \text{ times}})$ (we denoted $(a; b, \underbrace{c, \dots, c}_{s \text{ times}})$ by $(a; b, c^s)$).

Because $3k - n + 6 \in \{4, 5, 6\}$ there is an well-known criterion giving sufficient conditions for a linear system in order to contain (smooth, irreducible) curves. Precisely, if $\mathcal{L} = (a; b_0, b_1, \dots, b_{s_k^n}) \in \text{Pic}(X_k^n)$, $s_k^n = 3k - n + 5$ and

$$(3.1.1) \quad a \geq b_0 + b_1 + b_2, b_0 \geq b_1 \geq \dots \geq b_{s_k^n} \geq 0, a > b_0$$

then $[\mathcal{L}] = [a; b_0, b_1, \dots, b_{s_k^n}]$ contains a (smooth, irreducible) curve.

Due to this criterion (depending only on coefficients), the construction of curves on the surfaces X_k^n has two parts: an *arithmetical part*, where the necessary degrees and arithmetical genera are obtained using the well-known formulae of genus and degree for various $\mathcal{L} \in \text{Pic}(X_k^n)$, and a *geometrical part*, where the necessary (smooth, irreducible) curves are obtained applying (3.1.1) to the sheaves $\mathcal{L}$ used in order to solve the arithmetical part.

The main contribution of this paper to $HC(n)$ is the construction of invertible sheaves (and then curves) having the degree and genus in the domains A_p^n (see (2.12)), on some surfaces X_p^n . Precisely, let's denote by $S_p = S_p^n := \text{Bl}_{\Sigma_p}(\mathbf{P}^2)$, $\Sigma_p = \Sigma_p^n = \{P_0, P_1, \dots, P_{s_p^n}\} \subset \mathbf{P}^2$, $s_p^n = 3p - n + 5$, Σ_p^n containing general points. Then $\mathcal{H}_p^n := (p + 2; p, 1^{s_p^n}) \in \text{Pic}(S_p^n)$ is very ample (see subsection 3.2, Prop. 3.2.2) and then $X_p^n := \text{Im} \varphi_{[\mathcal{H}_p^n]} \subset \mathbf{P}^n$, $\deg X_p^n = n + p - 1$. If $p = k$ one obtains the surfaces X_k^n used in [Ci].

If we try to use an argument similar to the argument from [GP2] or [Ci] for X_p^n , so to divide the proof in two parts, an arithmetical and a geometrical one (the smoothing) we need a criterion similar to (3.1.1) for an *arbitrary* number

of points from Σ_p . But this is, anyway, very complicated by itself (see, for instance [Hi]). On the other hand, such a general criterion must include the conditions (3.1.1) (*normalisation*, using quadratic transformations, see [Hi]); but these conditions allow us to construct only curves whose degree d is $d > a$ function of degree $3/2$ in n , as it can be seen if we try to apply an argument as in [Ci]. So, in order to produce the necessary curves *for small degrees*, we need a new technique, which we shall explain here.

The main idea is to consider the arithmetical and the geometrical parts of the construction of curves on X_p^n *entirely linked*, i.e. to construct linear systems containing the necessary curves directly and explicitly enough using combinations between some simple sheaves (so, somehow easy to understand them).

Explicitly, we'll proceed as follows (in order to construct the necessary curves on X_p^n whose (degree, genus) belongs to the domain A_p^n): we start with a simple *initial family* $\mathcal{D}_0$ of sheaves

$$(3.1.2) \quad \mathcal{D}_0 = (a + 2; a, 1^t, 0^{s_p^n - t}) \in \text{Pic}(X_p^n), \quad 0 \leq t \leq s_p^n, \quad a \in \mathbf{Z}$$

of arithmetical genus a and some degrees, realising the necessary genera in some initial intervals in d (like $x_p(d, n) = 0$, for instance - see (2.2) - but not only). The families (3.1.2) are the only one which are good for the initial intervals in our method (we'll explain this later). After the initial construction we continue using a number of inductive arguments (after $x_p(d, n)$, for instance, but not only), adding repeatedly to $\mathcal{D}_0$ (by tensorisation) some simple (and well understood) invertible sheaves, similar to the class of hyperplane section (see Lemma 3.2.4). Let's explain, shortly, why the inductive processes works. Let's suppose that we need to construct curves on the surfaces X_p^n of degree d and genus $g = \pi_p(d, n)$ (see (1.1)). Let's suppose that we already constructed the necessary curves for $m_p(d, n) = 0$ (see (1.2)). Because

$$(3.1.3) \quad \begin{aligned} \pi_p(d + (n_p - 1)) &= \pi_p(d) + (d + p - 1); \\ m_p(d + (n + p - 1)) &= m_p(d) + 1 \end{aligned}$$

in order to construct curves C with $(\deg(C), g(C)) = (d, g)$, we proceed as follows: if the curves C have degrees d and genera $g = \pi_p(d, n)$ in the domain $m_p(d, n) = m$, then the (smooth, irreducible!) curves $C' \in |C + H_p^n|$, $H_p^n \in [p + 2; p, 1^{s_p^n}]$ have degrees $d' = d + (n + p - 1)$ and genera $g' = \pi_p(d', n)$ in the next domain $m_p(d, n) = m + 1$ (use (3.1.3)). Hence, we obtain invertible sheaves for all pairs (d, g) , $g = \pi_p(d, n)$ as far as we succeed to construct these

invertible sheaves in the domain $m_p(d, n) = 0$. Moreover, we can test when the associated linear system contain (smooth, irreducible) curves, because the appearing linear systems $|D_0 + tH_p^n|$, $t \geq 0$ are simple enough (specialize the points from Σ_p^n on a rational plane curve of degree $p + 1$, having one ordinary singular point P_0 with multiplicity p , see Prop. 3.2.2 and Lemma 3.2.4).

In our case, in order to construct the necessary curves covering by (degree, genus) the domains A_p^n it is necessary to change in some way the functions π_p so that the initial verifications (corresponding to $m_p(d, n) = 0$) to be made using sheaves as in (3.1.2) and a property as in (3.1.3) to still hold (and some others appearing, see subsection 3.3, Lemma 3.3.1). We get in such a way the (unique) functions $\alpha_p(d, n)$ from section 2 (see (2.1)).

Moreover, the method shortly explained here works only if $\#\Sigma_p^n = 3p - n + 6 \geq 12$ ($\#A$ means the number of elements of the finite set A) so, if $p \geq n/3 + 2$. So, for $\#\Sigma_p^n \leq 11$ we need *another method* in order to construct curves on X_p^n . Because we need such curves for *small degrees* also, the inductive argument is used again, as possible. But in this second method of construction we'll be, partially inspired from [GP2], because in this case a smoothing criterion of type (3.1.1) will be good enough (it is obtained specializing the points from Σ_p^n on a smooth plane cubic curve, see Prop. 3.2.1 and 3.2.3). In this case, a *suplimentary property* of the functions α_p will be necessary, namely that these functions must be related in a "good" way with the genus formula (see subsection 3.3, Lemma 3.3.2); and this is possible exactly because one uses initial families of type (3.1.2) (so these families are obligatory).

The smoothing criteria used in the methods are collected in the subsection 3.2, in Prop. 3.2.3 and Lemma 3.2.4. The numerical properties of the functions α_p are presented in the subsection 3.3. In the subsection 3.4 we'll construct invertible sheaves from $\text{Pic}(X_p^n)$ covering by (degree, arithmetical genus) the domain A_p^n , using both methods. In subsection 3.5 we'll apply the two smoothing criteria to sheaves from the subsection 3.4 in order to get (smooth, irreducible) curves. We remark that the domain of applicability of the first method briefly described here is (R1), while the domain of applicability of the second method (using some of the Gruson-Peskine ideas) is (R2), where:

$$(3.1.4) \quad (R1) : n \geq 9, n/3 + 2 \leq n - 4 \text{ (so } \#\Sigma_p^n \geq 12)$$

$$(3.1.5) \quad (R2) : n \geq 8, n/3 \leq p, n/3 + 2 \text{ (so } \#\Sigma_p^n \leq 11)$$

We end this section remarking that the constructions from the subsection 3.4 and 3.5 have *no degree of freedom*, but they cover all the ranges. Let's remark that the method used in (R2) cannot be used in general, because the condition (C5) from Prop. 3.2.3 gives (smooth, irreducible) curves only for $d > a$ quadratic function in n if used for $p = n - 4, n - 5, \dots$

3.2 Two smoothing criteria; the surfaces X_p^n

Let there be $\Sigma \subset \mathbf{P}^2$, $\Sigma = \{P_0, P_1, \dots, P_s\}$ a set of general points and $S := Bl_\Sigma(\mathbf{P}^2) \xrightarrow{\pi} \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ in Σ .

The first smoothing criterion is ii) from the next Proposition 3.2.1 (reformulated in Prop. 3.2.3).

Proposition 3.2.1. *If $\mathcal{D} = (a; b_0, b_1, \dots, b_s) \in \text{Pic}(S)$ is so that $a \geq b_0 \geq b_1 \geq \dots \geq b_s > 0$, then:*

i) each one of the conditions (i.1), (i.2), (i.2)', (i.3), (i.3)' implies $h^1(\mathcal{D}) = 0$, where

$$(i.1): a - \sum_{l=0}^s b_l \geq -1;$$

$$(i.2): 2 \leq s \leq 7, a \geq \sum_{l=0}^2 b_l;$$

$$(i.2)': 2 \leq s \leq 7, a - \sum_{l=0}^s b_l \geq -1, b_0 > b_1;$$

$$(i.3): s \geq 8, a \geq \sum_{l=0}^2 b_l, 3a - \sum_{l=0}^s b_l \geq 1;$$

$$(i.3)': s \geq 8, a - \sum_{l=0}^2 b_l \geq -1, 3a - \sum_{l=0}^s b_l \geq 1, b_0 > b_1;$$

ii) each one of the conditions (ii.1), (ii.2), (ii.2)', (ii.3), (ii.3)' implies $h^0(\mathcal{D}) \neq 0$, $[\mathcal{D}]$ has no base point and contains a (smooth, irreducible) curve, where

$$(ii.1): a \geq \sum_{l=0}^s b_l;$$

$$(ii.2): 2 \leq s \leq 6, a \geq \sum_{l=0}^2 b_l;$$

$$(ii.2)': 2 \leq s \leq 7, a - \sum_{l=0}^2 b_l \geq -1, b_0 \geq b_1 + 2;$$

$$(ii.3): s \geq 7, a \geq \sum_{l=0}^2 b_l, 3a - \sum_{l=0}^s b_l \geq 2;$$

$$(ii.3)': s \geq 8, a - \sum_{l=0}^2 b_l \geq -1, 3a - \sum_{l=0}^s b_l \geq 2, b_0 \geq b_1 + 2.$$

Remark. This is a Harbourne type result ([Hb1]).

Proof. A direct proof can be obtained specializing the points from Σ on a smooth plane cubic $\Gamma_0 \subset \mathbf{P}^2$ (general on Γ_0), using induction on s and standard exact sequences. The details are left to the reader (or see [P4]).

q.e.d.

Proposition 3.2.2. *Let there be $\mathcal{D} = (p+2; p, 1^s) \in \text{Pic}(S)$, $p \in \mathbf{Z}$, $p \geq 1$. Then: i) if $s \leq 3p+3$, then $h^0(\mathcal{D}) \neq 0$, $[\mathcal{D}]$ has no base point and contains a (smooth, irreducible) curve;*

ii) if $s \leq 3p$, then $\mathcal{D}$ is very ample.

Remark. A proof of ii) can be found in [Gi].

Proof. A direct proof can be obtained specializing the points from Σ (except for the last one) on a rational irreducible plane curve $\Delta_0 \subset \mathbf{P}^2$ of degree $p+1$ having only one ordinary singularity of multiplicity p : P_0 will be the singular point, $P_1, \dots, P_{s-1} \in \Delta_0$, $P_s \notin \Delta_0$. Then use standard exact sequences and standard techniques in order to separate points and tangent vectors. The details are left to the reader (or see [P4]).

q.e.d.

Now, we are ready to *define* the surfaces X_p^n . Precisely, let there be $p \in \mathbf{Z}$, $k \leq p \leq n-4$, $k = [n/3]_*$, $n \in \mathbf{Z}$, $n \geq 5$ and $\Sigma_p = \Sigma_p^n := \{P_0, P_1, \dots, P_{s_p^n}\} \subset \mathbf{P}^2$ ($s_p^n := 3p - n + 5$) be a set of general points. From Prop. 3.2.2 ii) follows that $\mathcal{H}_p^n := (p+2; p, 1^{s_p^n}) \in \text{Pic}(S_p^n)$ ($S_p^n := \text{Bl}_{\Sigma_p^n}(\mathbf{P}^2)$) is very ample on S_p^n . Then:

$$X_p^n := \text{Im} \left(\varphi_{[\mathcal{H}_p^n]} \right).$$

The surfaces X_p^n are rational having hyperelliptic hyperplane sections and one can easily check (doing computations) that $X_p^n \subset \mathbf{P}^n$ and $\deg(X_p^n) = n + p - 1$. If $p = k$ ($= [n/3]_*$) one obtains the surface used in [Ci].

Now we need to reformulate Prop. 3.2.3 ii) in the Gruson-Pesquine coordinates $(d, r : \theta_1, \dots, \theta_{s_p^n})$ ([GP2, step 2]); this is possible. So, for $\mathcal{D} \in \text{Pic}(X_p^n) \cong \mathbf{Z} \oplus \mathbf{Z}_{s_p^n}^{n+1}$, $\mathcal{D} = \mathcal{O}_X(D) = (a; b_0, b_1, \dots, b_{s_p^n})$, satisfying $a \geq b_0 \geq b_1 \geq \dots \geq b_{s_p^n} \geq 0$, ($D \in \text{Div}(X_p^n)$), we consider the change of coordinates (in $\text{Pic}(X_p^n) \otimes \mathbf{Q}$) given by:

$$(3.2.1) \quad r := a - b_0, \quad \theta_i := \frac{1}{2}r - b_i, \quad i = \overline{1, s_p^n}.$$

If $d = \deg(D) := (\mathcal{D} \cdot H_p^n)$ ($H_p^n \in [\mathcal{H}_p^n]$) and $g = p_a(D)$, using the genus

formula one obtains:

$$(3.2.2) \quad d = \frac{p-n+5}{2}r + 2a + \sum_{i=1}^{s_p^n} \theta_i; \quad g = F_d(r) - \frac{1}{2} \sum_{i=1}^{s_p^n} \theta_i^2.$$

$$(3.2.3) \quad F_d(r) = F_d^{p,n}(r) := \frac{1}{2} \left[d(r-1) + (p-1)r - \frac{n+p-1}{4}r^2 \right] + 1.$$

Prop. 3.2.1 ii) can be reformulated now as:

Proposition 3.2.3. *Let there be*

$$\mathcal{D} = \mathcal{O}_{X_p^n}(D) \in \text{Pic}(X_p^n), \quad \mathcal{D} = (a; b_0, b_1, \dots, b_{s_p^n}).$$

If the following conditions (C1), (C2), (C3), (C4), (C5) are simultaneously satisfied by $\mathcal{D}$ in the coordinates $(d, r; \theta_1, \dots, \theta_{s_p^n})$, then $[\mathcal{D}] = |D| \neq \emptyset$, has no base point and contains a (smooth, irreducible) curve, where:

$$(C1) \quad \theta_i \equiv \frac{1}{2}r \pmod{1}, \quad i = \overline{1, s_p^n};$$

$$(C2) \quad d + \frac{1}{2}(p-n+5)r - \sum_{i=1}^{s_p^n} \theta_i \equiv 0 \pmod{2};$$

$$(C3) \quad |\theta_1| \leq \theta_2 \leq \dots \leq \theta_{s_p^n} \leq \frac{r}{2};$$

$$(C4) \quad -\theta_1 + \sum_{i=2}^{s_p^n} \theta_i \leq d - \frac{1}{2}(n-p+1)r;$$

$$(C5) \quad d \geq (p-1)r + 2.$$

Proof. Left to the reader (similar to the case $n = 3$, see [Ha2], [GP2]).
q.e.d.

We end this section with a technical lemma (which represents the second smoothing criterion), essential in section 3.5.

Lemma 3.2.4. *If $p, n \in \mathbf{Z}$, $n \geq 3$, $p \geq \frac{n}{3} + 2$, let's consider the following invertible sheaves from $\text{Pic}(X_p^n)$:*

$$\begin{aligned}\mathcal{H}_1 &= \mathcal{H}_p^n := (p+2; p, 1^{s_p^n}), \\ \mathcal{H}_2 &= \tilde{\mathcal{H}}_p^n := (p+2; p, 1^{s_p^n-1}, 0), \\ \mathcal{H}'_3 &= (\tilde{\mathcal{H}}_{p-1,1}^{n+1})' := (p+1; p-1, 1^{s_p^n-5}, 0^2, 1^2, 0), \\ \mathcal{H}'_4 &= (\tilde{\mathcal{H}}_{p-1,2}^{n+1})' := (p+1; p-1, 1^{s_p^n-7}, 0^2, 1^4, 0), \\ \mathcal{H}'_5 &= (\tilde{\mathcal{H}}_{p-1,3}^{n+1})' := (p+1; p-1, 1^{s_p^n-9}, 0^2, 1^6, 0), \\ \mathcal{H}'_6 &= (\tilde{\mathcal{H}}_{p-1,4}^{n+1})' := (p+1; p-1, 1^{s_p^n-11}, 0^2, 1^8, 0).\end{aligned}$$

If $t_1, t_2, \dots, t_6 \in \mathbf{Z}$, $t_1, \dots, t_6 \geq 0$, let there be

$$(3.2.4) \quad \mathcal{D} := \mathcal{D}_0 + t_1 \mathcal{H}_1 + t_2 \mathcal{H}_2 + t_3 \mathcal{H}'_3 + t_4 \mathcal{H}'_4 + t_5 \mathcal{H}'_5 + t_6 \mathcal{H}'_6 \in \text{Pic}(X_p^n)$$

where

$$\mathcal{D}_0 := (a+2; a, \nu_1, \nu_2, \dots, \nu_{s_p^n}), \quad \nu_j \in \{0, 1\}, \quad j = \overline{1, s_p^n}$$

and $\nu_j = 1$ for u values of the index j ($0 \leq u \leq s_p^n$). Let's denote by $t := t_1 + t_2 + \dots + t_6$. Then, the condition

$$(3.2.5) \quad 3a \geq u + n - 7 - (t+1)(n-4)$$

implies $[\mathcal{D}] \neq \emptyset$, $[\mathcal{D}]$ without base points and contains a (smooth, irreducible) curve.

Proof. Write

$$\mathcal{D} = t_1 \mathcal{D}_{x_1} + t_2 \mathcal{D}_{x_2} + t_3 \mathcal{D}_{x_3} + t_4 \mathcal{D}_{x_4} + t_5 \mathcal{D}_{x_5} + t_6 \mathcal{D}_{x_6} + \mathcal{R}$$

$x_1, \dots, x_6 \in \mathbf{Z}$, where

$$\begin{aligned}\mathcal{D}_{x_1} &:= (x_1+2; x_1, 1^{s_p^n}), \\ \mathcal{D}_{x_2} &:= (x_2+2; x_2, 1^{s_p^n-1}, 0), \\ \mathcal{D}_{x_3} &:= (x_3+2; x_3, 1^{s_p^n-5}, 0^2, 1^2, 0), \\ \mathcal{D}_{x_4} &:= (x_4+2; x_4, 1^{s_p^n-7}, 0^2, 1^4, 0), \\ \mathcal{D}_{x_5} &:= (x_5+2; x_5, 1^{s_p^n-9}, 0^2, 1^6, 0), \\ \mathcal{D}_{x_6} &:= (x_6+2; x_6, 1^{s_p^n-11}, 0^2, 1^8, 0),\end{aligned}$$

$\mathcal{R} := (b + 2; b, \nu_1, \nu_2, \dots, \nu_{s_p^n})$, where $b := a + t(p - 1) + (t_1 + t_2) - \sum_{i=1}^6 t_i x_i$.

Take then $x_1, \dots, x_6 \geq 0$ *minimal* so that Prop. 3.2.2 i) applies to $\mathcal{D}_{x_1}, \dots, \mathcal{D}_{x_6}$. We deduce that:

$$(3.2.6) \quad 3p - n + 2 \leq 3x_1 \leq 3p - n + 4$$

$$(3.2.7) \quad 3p - n + 1 \leq 3x_2 \leq 3p - n + 3$$

$$(3.2.8) \quad 3p - n - 1 \leq 3x_i \leq 3p - n + 1, \quad i = \overline{3, 6}$$

In order to obtain the conclusion from the lemma for $[\mathcal{D}]$, it's enough to have the same for $[\mathcal{R}]$ so, by Prop. 3.2.2 i) again, we need

$$(3.2.9) \quad u \leq 3b + 3, \quad b = a + t(p - 1) + (t_1 + t_2) - \sum_{i=1}^6 t_i x_i$$

Replacing b , (3.2.9) becomes

$$u \leq 3a + 3t(p - 1) + 3(t_1 + t_2) - \sum_{i=1}^6 t_i(3x_i) + 3, \quad \text{or}$$

$$\sum_{i=1}^6 t_i(3x_i) \leq 3a + 3t(p - 1) + 3(t_1 + t_2) - u + 3.$$

Using (3.2.6), (3.2.7), (3.2.8) we can see that this last inequality is implied by

$$\begin{aligned} t(3p - n + 1) + 3t_1 + 2t_2 &\leq 3a + 3t(p - 1) + 3(t_1 + t_2) - u + 3 \Leftrightarrow \\ &\Leftrightarrow 3a \geq u + n - 7 - (t + 1)(n - 4) - t_2. \end{aligned}$$

Because $t_2 \geq 0$, this last inequality follows from (3.2.5).

q.e.d.

Corollary 3.2.5. *In the same hypotheses as in Lemma 3.2.4, the same conclusion holds for $[\mathcal{D}]$ if*

$$(3.2.10) \quad 3a \geq 3p - (t + 1)(n - 4) - 2.$$

Proof. Give to u the biggest possible value $u = s_p^n = 3p - n + 5$ in (3.2.5).
q.e.d.

3.3 Numerical properties of the functions $\alpha_p(d, n)$

We recall that the functions $\alpha_p = \alpha_p(d, n)$ were defined in section 2 ((2.1)-(2.5)). Let it be now

$$(3.3.1) \quad \alpha'_p(d, n) := \alpha_{p-1}(d+1, n+1).$$

We prove now the following *key lemma*:

Lemma 3.3.1. *a) Let's denote by $d' := d + (n + p - 1)$. Then:*

$$\begin{aligned} x_p(d', n) &= x_p(d, n) + 1; \quad t_p(d', n) = t_p(d, n); \quad u_p(d', n) = u_p(d, n), \\ \alpha_p(d', n) &= \alpha_p(d, n) + (d + p - 1); \quad (\Leftrightarrow \alpha_p(d - (n + p - 1), n) = \alpha_p(d, n) - d + n); \\ \text{the same for } \alpha'_p. \end{aligned}$$

$$\text{b) } \alpha_{p+1}(d-1, n) \leq \alpha_p(d, n), \quad (\forall) d \geq a_p^n + 1, \quad d \in \mathbf{Z}.$$

$$\alpha_{p+1}(d, n) \leq \alpha_p(d, n), \quad (\forall) d \geq a_p^n + n + p, \quad d \in \mathbf{Z}.$$

$$\begin{aligned} \text{c) } x_{p+1}(d, n) &= x_p(d+1, n+1), \quad t_{p+1}(d, n) = t_p(d+1, n+1) - 1; \\ \alpha_p(d+1, n+1) &= \alpha'_{p+1}(d, n) \in \{\alpha_{p+1}(d, n) - 1, \alpha_{p+1}(d, n)\}. \end{aligned}$$

Proof. b) We'll show first that

$$(3.3.2) \quad \alpha_{p+1}(d-1, n) \leq \alpha_{p+1}(d, n), \quad (\forall) d \geq a_{p+1}^n + 1, \quad d \in \mathbf{Z}.$$

We use *induction* on $x_{p+1}(d-1, n)$, using a).

$$(3.3.3) \quad \begin{cases} x_{p+1}(d-1, n) = 0 \Rightarrow \alpha_{p+1}(d-1, n) = [(d-n+p)/2]_* \\ x_{p+1}(d-1, n) = 0 \Rightarrow \alpha_{p+1}(d, n) = \begin{cases} [(d-n+p+1)/2]_*, & d \neq a_{p+1}^n + (n+p) \\ [(3d-4n+1)/2]_*, & d = a_{p+1}^n + (n+p). \end{cases} \end{cases}$$

So $x_{p+1}(d-1, n) = 0 \Rightarrow \alpha_{p+1}(d-1, n) \leq \alpha_{p+1}(d, n)$ (doing some computations for $d = a_{p+1}^n + (n+p)$).

Suppose now that we have proved (3.3.2) for $x_{p+1}(d-1, n) = x \geq 0$. Let d_1 so that $x_{p+1}(d_1-1, n) = x+1$. Put $d := d_1 - (n+p)$. Using a) we deduce that $x_{p+1}(d-1, n) = x$. Then $\alpha_{p+1}(d_1-1, n) - \alpha_{p+1}(d_1-1, n) = \alpha_{p+1}(d+(n+p), n) - \alpha_{p+1}(d-1+(n+p), n) =$ (use a) again) $(\alpha_{p+1}(d, n) - \alpha_{p+1}(d-1, n)) + 1 \geq 1 > 0$ (we used the induction hypothesis). Now (3.3.2) is proved.

We'll now prove the first inequality from b) using induction on $x_p(d-1, n)$. We have

$$(3.3.4) \quad x_p(d-1, n) = 0 \Rightarrow \alpha_p(d, n) = \begin{cases} [(d-n+p)/2]_*, & d \neq a_p^n + (n+p-1) \\ [(3d-4n+1)/2]_*, & d = a_p^n + (n+p-1). \end{cases}$$

It can be seen that

$$(3.3.5) \quad x_p(d-1, n) = 0 \Rightarrow [(d-n+p)/2]_* \leq \alpha_p(d, n) \leq [(d-n+p+1)/2]_*$$

Because $x_p(d-1, n) = 0 \Rightarrow x_{p+1}(d-1, n) = 0$, from (3.3.3) we now deduce the first inequality from b) if $x_p(d-1, n) = 0$.

Suppose now that the inequality holds for $x_p(d-1, n) = x \geq 0$ and let d_2 be so that $x_p(d_2-1, n) = x+1$. Put $d := d_2 - (n+p-1)$. Then $x_p(d-1, n) = x$ and we have: $\alpha_p(d_2, n) - \alpha_{p+1}(d_2, n) = \alpha_p(d + (n+p-1), n) - \alpha_{p+1}(d-1 + (n+p), n) = (\text{use a)}) \alpha_p(d, n) - \alpha_{p+1}(d-1, n) \geq 0$ (induction hypothesis). So $\alpha_p(d_2, n) \geq \alpha_{p+1}(d_2, n)$. Using now (3.3.2) we get the first inequality from b).

The second inequality from b) comes from the first one, as before.

q.e.d.

Lemma 3.3.2. (for $F_d^{p,n}(r)$ see (3.2.3))

$$\text{a) } F_{d+(n+p-1)}^{p,n}(r+2) = F_d^{p,n}(r) + (d+p-1);$$

$$F_d^{p,n+1}(r) = F_{d-1}^{p+1,n}(r) - \frac{1}{2};$$

$$\text{b) i) } \alpha_p(d, n) = \left[F_d^{p,n}(2(x_p(d, n) + 1)) - \frac{1}{2} \right]_*;$$

$$\text{ii) } \alpha_{p+1}(d-1, n) = \left[F_d^{p,n+1}(2(x_{p+1}(d-1, n) + 1)) \right]_*.$$

Proof. b) Use a), Lemma 3.3.1 a) and induction on $x_p(d, n)$.

q.e.d.

Remark. The previous lemma shows that the functions $\alpha_p(d, n)$ are related in a "good" way to the genus formula (see (3.2.2)) and this will be very important in subsection 3.4. Moreover, we already used this property of the functions α_p when proving the Theorem A (section 2) in [P2].

3.4 Invertible sheaves $\mathcal{L} \in \text{Pic}(X_p^n)$ whose

$(\deg(\mathcal{L}), P_a(\mathcal{L}))_{\mathcal{L}}$ cover the domain

$$A_p^n \quad (n \geq 5, p \geq n/3)$$

We recall that the domains A_p^n , $p \geq n/3$, $p \in \mathbf{Z}$ from the (d, g) -plane were defined in the section 2, (2.11). Here we consider, for $p \geq n/3$, $p \in \mathbf{Z}$, the extended domains $\tilde{A}_p^n$:

$$(3.4.1) \quad \tilde{A}_p^n : \alpha_{p+1}(d-1, n) \leq g \leq \alpha_p(d, n), \quad d \geq a_p^n + 1, \quad d, g \in \mathbf{Z}$$

(for the definition of α_p^n see (2.3)). Obviously

$$(3.4.2) \quad \tilde{A}_p^n \supset A_p^n, \quad (\forall) p \geq n/3, \quad (\forall) n \geq 3, \quad p, n \in \mathbf{Z}$$

In this section we prove the following

Proposition 3.4.1. *Let there be $n \geq 5$, $n \in \mathbf{Z}$ and $(d, g) \in \tilde{A}_p^n$, $p \geq n/3$, $p \in \mathbf{Z}$. Then there is $\mathcal{D} \in \text{Pic}(X_p^n)$ such that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$.*

We recall that $\deg(\mathcal{D}) := (\mathcal{D} \cdot \mathcal{H}_p^n)$, $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in \text{Pic}(X_p^n)$.

Proof. It is divided in 4 steps.

Step 1. Let there be $d, g \in \mathbf{Z}$, $d \geq a_{p+1}^n + 1$ so that

$$\alpha'_{p+1}(d-1, n) \leq g \leq \beta_{p+1}(d, n) - x_{p+1}(d, n),$$

where α'_p is defined in (3.3.1) and

$$\beta_p(d, n) = \begin{cases} \alpha'_p(d, n), & d \not\equiv a_p^n \pmod{(n+p-1)} \\ \alpha_p(d, n), & d \equiv a_p^n \pmod{(n+p-1)} \end{cases}$$

Then there is $\mathcal{D} \in \text{Pic}(X_p^n)$ so that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$.

Step 2. Let there be $d, g \in \mathbf{Z}$, $d \geq a_p^{n+1}$ so that

$$\alpha'_p(d, n+1) - x_p(d, n+1) \leq g \leq \alpha_p(d, n+1).$$

Then there is $\mathcal{D} \in \text{Pic}(X_p^{n+1})$ so that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$.

Step 3. Let there be $d, g \in \mathbf{Z}$, $d \geq a_p^{n+1} - 1 = a_{p+1}^n$ so that

$$\alpha'_{p+1}(d, n) - x_{p+1}(d, n) \leq g \leq \alpha'_{p+1}(d, n).$$

Then there is $\mathcal{D} \in \text{Pic}(X_p^n)$ so that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$.

Step 4. The statement of Prop. 3.4.1.

Proof of Step 1. We use induction on $x_{p+1}(d-1, n)$.

If $x_{p+1}(d-1, n) = 0$, we have:

$$\begin{aligned} [(d-n+p-1)/2]_* &= \alpha'_{p+1}(d-1, n) \leq \beta_{p+1}(d, n) - x_{p+1}(d, n) = \\ (3.4.3) \quad &= \begin{cases} [(d-n+p)/2]_*, & d \neq a_{p+1}^n + (n+p) \\ [(3d-4n-1)/2]_*, & d = a_{p+1}^n + (n+p). \end{cases} \end{aligned}$$

We consider the following invertible sheaves $\mathcal{D}_0 \in \text{Pic}(X_p^n)$:

$$(3.4.4) \quad \mathcal{D}_0 = (g+2; g, 1^u, 0^{s_p^n-u}), \quad 0 \leq u \leq s_p^n = 3p-n+5.$$

Then $g = p_a(\mathcal{D}_0)$ and, if $d := \deg(\mathcal{D}_0) = (\mathcal{D}_0 \cdot \mathcal{H}_p^n)$ it result that $g = (d-2p+u-4)/2$. Taking now

$$(3.4.5) \quad u = 3p-n+4, \quad 3p-n+3, \quad 3p-n+2 \geq 0$$

and using (3.4.3), one obtains invertible sheaves $\mathcal{D}_0 \in \text{Pic}(X_p^n)$ *having the last component zero* and degrees and arithmetical genera in the range from Step 1, for $x_{p+1}(d-1, n) = 0$. Because

$$(3.4.6) \quad \begin{cases} \alpha'_{p+1}(d-1+(n+p), n) = \alpha'_{p+1}(d-1, n) + (d+p-1) \\ (\beta_{p+1} - x_{p+1})(d+(n+p), n) = (\beta_{p+1} - x_{p+1})(d, n) + (d+p-1) \end{cases}$$

(see Lemma 3.3.1 a)) we obtain from (3.4.3), by induction on $x_{p+1}(d-1, n)$ the inequality $\alpha'_{p+1}(d-1, n) \leq \beta_{p+1}(d, n) - x_{p+1}(d, n)$, $(\forall) d \geq a_{p+1}^n + 1$.

Let there be $\tilde{H}_p^n = (p+2; p, 1^{s_p^n-1}, 0) \in \text{Pic}(X_p^n)$. It is easy to check that, if $\mathcal{D} \in \text{Pic}(X_p^n)$ *has the last component zero*, then

$$(3.4.7) \quad \begin{cases} \deg(\mathcal{D} + \tilde{\mathcal{H}}_p^n) = d + (n+p), \quad d = \deg(\mathcal{D}) \\ p_a(\mathcal{D} + \tilde{\mathcal{H}}_p^n) = g + (d+p-1), \quad g = p_a(\mathcal{D}) \end{cases}$$

Now, using (3.4.6) and (3.4.7) it follows that we cover the range from Step 1, adding to the sheaves $\mathcal{D}_0$ used for $x_{p+1}(d-1, n) = 0$, successively, the sheaf $\tilde{\mathcal{H}}_p^n$ (at each addition, $x_{p+1}(d-1, n)$ increase by 1, see Lemma 3.3.1 a)).

Remark 3.4.2. *The invertible sheaves used in order to cover the domain from Step 1 are of the form: $\mathcal{D} = \mathcal{D}'_0 + t_2 \mathcal{H}_2$ (see Lemma 3.2.4), where $\mathcal{H}_2 = \tilde{\mathcal{H}}_p^n = (p+2; p, 1^{s_p^n-1}, 0)$, $t_2 \in \mathbf{Z}$, $t_2 \geq 0$,*

$$\mathcal{D}'_0 = (a+2; a, 1^u, 0^{s_p^n-u}), \quad u \in \{3p-n+2, 3p-n+3, 3p-n+4\},$$

$$a \in \{(d'_0 - n + p - 2)/2, (d'_0 - n + p - 1)/2, (d'_0 - n + p)/2\} \cap \mathbf{Z},$$

$$d'_0 = (\mathcal{D}'_0 \cdot \mathcal{H}_p^n) \geq a_{p+1}^n + 1, d'_0 \leq a_{p+1}^n + (n+p).$$

(see (3.4.4), (3.4.5) and the construction).

Proof of Step 2. This step is the most difficult. We'll give two constructions, both of them necessary in a complementary way in the subsection 3.5.

Construction A (works for $\#\Sigma_p^n \geq 12$, hence $p \geq \frac{n}{3} + 2$).

1) We'll prove here the existence of $\mathcal{D} \in \text{Pic}(X_p^{n+1})$ of degree d and arithmetical genus $g = \alpha_p(d, n+1)$ for any $d \geq a_p^{n+1}$. We recall that $X_p^{n+1} \subset \mathbf{P}^{n+1}$.

Let there be $\mathcal{H}_p^{n+1} = (p+2; p, 1^{s_p^{n+1}}, 0) \in \text{Pic}(X_p^{n+1})$. Because $\deg(X_p^{n+1}) = n+p$, it can be seen that, for any $\mathcal{D} \in \text{Pic}(X_p^{n+1})$ we have

$$(3.4.8) \quad \begin{cases} \deg(\mathcal{D} + \mathcal{H}_p^{n+1}) = d + (n+p), \quad d = \deg(\mathcal{D}) = (\mathcal{D} \cdot \mathcal{H}_p^{n+1}) \\ p_a(\mathcal{D} + \mathcal{H}_p^{n+1}) = g + (d+p-1), \quad g = p_a(\mathcal{D}). \end{cases}$$

Using (3.4.8) and Lemma 3.3.1 a), it follows that we can use the same argument as in the proof of Step 1, doing induction on $x_p(d, n+1)$. So, we check the existence for $x_p(d, n+1) = 0$ and then we add successively the class of the hyperplane section of X_p^{n+1} , namely $\mathcal{H}_p^{n+1}$. If $x_p(d, n+1) = 0$, then $\alpha_p(d, n+1) = [(d-n+p-1)/2]_*$. So, we cover this initial range with sheaves $\mathcal{D}_0 \in \text{Pic}(X_p^{n+1})$ of the form

$$(3.4.9) \quad \mathcal{D}_0 = (g+2; g, 1^u, 0^{s_p^n - u}), \quad g = [(d-n+p-1)/2]_* \text{ with}$$

$$(3.4.10) \quad u = 3p - n + 2 = s_p^n - 2 \quad \text{or} \quad u = 3p - n + 3 = s_p^{n+1} - 1.$$

Let's *remark* that the sheaves used here are of the form

$$(3.4.11) \quad (t+2l; t, l^{s_p^{n+1}-2}, l-1 \text{ or } l, l-1), \text{ where}$$

$$l = x_p(d, n+1) + 1 \in \mathbf{Z}.$$

2) Now, we consider $\mathcal{H}_3 := \tilde{H}_{p-1,1}^{n+1} = (p+1; p-1, 1^{3p-n}, 0^2, 1^2) \in \text{Pic}(X_p^{n+1})$. Then

$$(3.4.12) \quad p_a(\tilde{H}_{p-1,1}^{n+1}) = p-1, \quad \deg(\tilde{H}_{p-1,1}^{n+1}) = n+p.$$

We start by proving

$$(3.4.13) \quad \begin{cases} \text{If } g = \alpha_p(d, n+1) - (x_p(d, n+1) - x)^2, \quad x_p(d, n+1) \geq x, \\ x \in \mathbf{Z}, \quad x \geq 0, \text{ then } (\exists) \mathcal{D} \in \text{Pic}(X_p^{n+1}) \text{ so that} \\ (\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g). \end{cases}$$

We'll use induction on $x_p(d, n+1) \geq x$. During the induction process we'll need the following $\varepsilon(\mathcal{D})$:

$$(3.4.14) \quad \begin{cases} \text{If } \mathcal{D} = (a; b_0, b_1, \dots, b_{3p-n+4}) \in \text{Pic}(X_p^{n+1}) \\ \text{then } \varepsilon(\mathcal{D}) := a - b_0 - b_{3p-n+1} - b_{3p-n+2} \end{cases}$$

(the indices $3p-n+1$ and $3p-n+2$ correspond to the two consecutive zeros from $\mathcal{H}_3$, so that $\varepsilon(\mathcal{H}_3) = 2$).

If $x_p(d, n+1) = x$, (3.4.13) is just 1). If $x_p(d, n+1) = x$, let's denote by $\mathcal{D}_x$ an invertible sheaf satisfying (3.4.13), from 1). Then $\varepsilon(\mathcal{D}_x) = 0$

($= 2(x_p(d, n+1) - x)$), by (3.4.11). Actually, we'll prove by induction on $x_p(d, n+1) \geq x$ the following statement (stronger than (3.4.13)):

$$(3.4.15) \quad \begin{cases} \text{If } g = \alpha_p(d, n+1) - (x_p(d, n+1) - x)^2, \ x_p(d, n+1) \geq x \\ x \in \mathbf{Z}, \ x \geq 0, \text{ then } (\exists) \mathcal{D} \in \text{Pic}(X_p^{n+1}) \text{ so that} \\ (\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g) \text{ and } \varepsilon(\mathcal{D}) = 2(x_p(d, n+1) - x). \end{cases}$$

We have just verified (3.4.15) for $x_p(d, n+1) = x$. In order to finish the inductive process, let's suppose that we have constructed invertible sheaves $\mathcal{D}_{x+t}$ having $(\deg(\mathcal{D}_{x+t}), p_a(\mathcal{D}_{x+t})) = (d, g)$ in the situation ($t \in \mathbf{Z}, t \geq 0$).

$$(3.4.16) \quad \begin{cases} x_p(d, n+1) = x + t \\ g = \alpha_p(d, n+1) - t^2 = \alpha_p(d, n+1) - (x_p(d, n+1) - x)^2 \\ \text{so that } \varepsilon(\mathcal{D}_{x+t}) = 2t = 2(x_p(d, n+1) - x) \end{cases}$$

We need $\mathcal{D}_{x+t+1} \in \text{Pic}(X_p^{n+1})$ to do the same job for the next range in d (namely $(x_p(d', n+1) = x + t + 1)$). Or, using the induction hypothesis and Lemma 3.3.1 a) we can see that the sheaves $\mathcal{D}_{x+t+1} := \mathcal{D}_{x+t} + \tilde{H}_{p-1,1}^{n+1}$ are the good ones. Indeed: $\deg(\mathcal{D}_{x+t+1}) = d + (n+p) := d'; x_p(d', n+1) = x + t + 1; p_a(\mathcal{D}_{x+t+1}) = p_a(\mathcal{D}_{x+t}) + (p-1) + (\mathcal{D}_{x+t} \cdot \tilde{H}_{p-1,1}^{n+1}) - 1 = p_a(\mathcal{D}_{x+t}) + (p-1) + \deg(\mathcal{D}_{x+t}) - \varepsilon(\mathcal{D}_{x+t}) - 1 =$ (use the induction hypothesis) $\alpha_p(d, n+1) - (x_p(d, n+1) - x)^2 - 2(x_p(d, n+1) - x) + d + p - 2$.

We replace now d by $d' - (n+p)$ and we use Lemma 3.3.1 a). We obtain

$$\begin{aligned} p_a(\mathcal{D}_{x+t+1}) &= \alpha_p(d' - (n+p), n+1) - (x_p(d' - (n+p), n+1) - x)^2 - \\ &\quad - 2(x_p(d' - (n+p), n+1) - x) + d' - (n+p) + p - 2 = \\ &= \alpha_p(d', n+1) - d' + n + 1 - (x_p(d', n+1) - 1 - x)^2 - \\ &\quad - 2(x_p(d', n+1) - 1 - x) + d' - n - 2 = \\ &= \alpha_p(d', n+1) - (x_p(d', n+1) - x)^2 + 2(x_p(d', n+1) - x) - 1 - \\ &\quad - 2(x_p(d', n+1) - x) + 2 - 1 = \alpha_p(d', n+1) - (x_p(d', n+1) - x)^2. \end{aligned}$$

Moreover, $\varepsilon(\mathcal{D}_{x+t+1}) = \varepsilon(\mathcal{D}_{x+t}) + 2 = 2(x_p(d', n+1) - x)$. Now, the inductive process is finished. So (3.4.15) (hence (3.4.13)) is proved.

Now, let a be arbitrary, $a \in \mathbf{Z}, a \geq 0$ and put $x := a^2 - a \geq 0$. We obtain by (3.4.13) elements from $\text{Pic}(X_p^{n+1})$ of degree d and arithmetical genus

$$p_a = \alpha_p(d, n+1) - (x_p(d, n+1) - a^2 + a)^2$$

for any d so that $x_p(d, n+1) \geq a^2 - a$. Take now d so that $x_p(d, n+1) = a^2$ ($\geq x = a^2 - a$). One obtains invertible sheaves of degree d and $p_a = \alpha_p(d, n+1) - a^2$ for $x_p(d, n+1) = a^2$. Now, adding succesively $\mathcal{H}_p^{n+1}$ and using Lemma 3.3.1 a), we obtain invertible sheaves of degrees d and arithmetical genera g covering by (d, g) the domain

$$(a) \quad g = \alpha_p(d, n+1) = a^2, \quad x_p(d, n+1) \geq a^2.$$

3) Now, start again with the invertible sheaves covering the domain (a) (from 2)). Using the invertible sheaf

$$\mathcal{H}_4 = \tilde{\mathcal{H}}_{p-1,2}^{n+1} = (p+1; p-1, 1^{3p-n-2}, 0^2, 1^4) \in \text{Pic}(X_p^{n+1})$$

(instead of $\tilde{H}_{p-1,1}^{n+1}$) and using (3.4.11), we obtain, as before, invertible sheaves covering by degrees and arithmetical genera the domains from the (d, g) -plane defined by

$$g = \alpha_p(d, n+1) - a^2 - (x_p(d, n+1) - y)^2, \quad x_p(d, n+1) \geq y$$

(for any $y \geq a^2, y \in \mathbf{Z}$).

Take now $b \in \mathbf{Z}, b \geq 0$ arbitray and put $y = a^2 + b^2 - b \geq a^2$. It follows that we obtained invertible sheaves from $\text{Pic}(X_p^{n+1})$ covering by degrees and arithmetical genera the domain

$$\begin{cases} g = \alpha_p(d, n+1) - a^2 - (x_p(d, n+1) - a^2 - b^2 + b)^2 \\ x_p(d, n+1) \geq a^2 + b^2 - b. \end{cases}$$

Take now d such that $x_p(d, n+1) = a^2 + b^2$ ($\geq y = a^2 + b^2 - b$). We obtain invertible sheaves of degrees d and arithmetical genera $g = \alpha_p(d, n+1) - (a^2 + b^2)$. Adding now succesively $\mathcal{H}_p^{n+1}$ we get invertible sheaves covering by dedrees and arithmetical genera the domain

$$(b) \quad g = \alpha_p(d, n+1) - (a^2 + b^2), \quad x_p(d, n+1) \geq a^2 + b^2.$$

4), 5). Continuity for another two times as for (a) and (b), using (3.4.11) (and $s_p^{n+1} \geq 10$, because $p \geq \frac{n}{3} + 2$) and the invertible sheaves $\mathcal{H}_5 = \tilde{\mathcal{H}}_{p-1,3}^{n+1} = (p+1; p-1, 1^{3p-n-4}, 0^2, 1^6)$ and $\mathcal{H}_6 = \tilde{\mathcal{H}}_{p-1,4}^{n+1} = (p+1; p-1, 1^{3p-n-6}, 0^2, 1^8)$, $\mathcal{H}_5, \mathcal{H}_6 \in \text{Pic}(X_p^{n+1})$ one obtains sheaves from $\text{Pic}(X_p^{n+1})$ covering by degrees and arithmetical genera the domain from the (d, g) -plane defined by

$$(e) \quad g = \alpha_p(d, n+1) - (a^2 + b^2 + c^2 + e^2), \quad x_p(d, n+1) \geq a^2 + b^2 + c^2 + e^2$$

where $a, b, c, e \in \mathbf{Z}$, $a, b, c, e \geq 0$ are *arbitrary*.

6) The domain (e) is exactly the domain from Step 2, because any positive integer can be written as a sum of 4 squares of positive integers. Indeed, let there be $g \in \mathbf{Z}$ so that

$$\alpha_p(d, n+1) - x_p(d, n+1) \leq g \leq \alpha_p(d, n+1)$$

and

$$f := \alpha_p(d, n+1) - g \geq 0, \quad f \in \mathbf{Z}.$$

Write $f = a^2 + b^2 + c^2 + e^2$, $a, b, c, e \in \mathbf{Z}$, $a, b, c, e \geq 0$. Certainly $0 \leq f \leq \alpha_p(d, n+1)$. Hence the domain (e) from (5) is

$$\begin{cases} g = \alpha_p(d, n+1) - (a^2 + b^2 + c^2 + e^2) (= \alpha_p(d, n+1) - f) \\ x_p(d, n+1) \geq a^2 + b^2 + c^2 + e^2 (= f = \alpha_p(d, n+1) - g) \end{cases}$$

$$0 \leq f \leq x_p(d, n+1).$$

Replacing f we get $\alpha_p(d, n+1) - x_p(d, n+1) \leq g \leq \alpha_p(d, n+1)$; but this is exactly the domain from Step 2, so it is covered by the $(\deg(\mathcal{L}), p_a(\mathcal{L}))$ for various $\mathcal{L} \in \text{Pic}(X_p^{n+1})$.

Construction B (works for $\#\Sigma_p^n \geq 6$, hence $p \geq \frac{n}{3}$).

From Lemma 3.3.2 b)i) we have

$$(3.4.17) \quad \alpha_p(d, n+1) = \left[F_d^{p, n+1} (2(x_p(d, n+1) + 1) - \frac{1}{2}) \right]_*.$$

Due to this equality we can use the Gruson-Pesquine coordinates $(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^{n+1}})$, where $r = a - b_0$, $\theta_i = \frac{1}{2}r - b_i$, $= \overline{1, s_p^{n+1}}$ for $\mathcal{D} = (a; b_0, b_1, \dots, b_{s_p^{n+1}}) \in \text{Pic}(X_p^{n+1})$ (see (3.2.1)).

Step 2 will be proved as for as for any (d, g) from the domain of the Step 2 we'll prove the existence of a sheaf $\mathcal{D} \in \text{Pic}(X_p^{n+1})$ with $\deg \mathcal{D} = d$, such that, in the coordinates $(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^{n+1}})$, $\mathcal{D}$ satisfies the conditions (C1) and (C2) from the Prop. 3.2.3 (see [GP2] also), and

$$(3.4.18) \quad F_p^{p, n+1}(r) - \frac{1}{2} \sum_{i=1}^{s_p^{n+1}} \theta_i^2 = g.$$

Indeed, the left member of (3.4.18) is just $p_a(\mathcal{D})$ (see (3.2.2)), hence $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$. Moreover, it is necessary, performing the transformation of coordinates of $\mathcal{D}$ from

$$(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^{n+1}}) \text{ to } (a; b_0, b_1, \dots, b_{s_p^{n+1}})$$

that these last coordinates be integers. But (C1) means $b_i \in \mathbf{Z}$, $i = \overline{1, s_p^{n+1}}$ (see (3.2.2)). Now, (C2) means $a \in \mathbf{Z}$, so $b_0 \in \mathbf{Z}$, because $r \in \mathbf{Z}$.

Now we'll prove the existence of such $\mathcal{D}$ covering the domain from Step 2 by (degree, arithmerical genus). Using (3.4.17), take

$$(3.4.19) \quad r := 2(x_p(d, n+1) + 1).$$

Let there be $b \in \mathbf{Z}$ so that

$$(3.4.20) \quad 1 \leq b \leq r = 2(x_p(d, n+1) + 1)$$

$$(3.4.21) \quad \begin{cases} \text{Write } b = \sum_{i=1}^4 c_i^2, \ c_i \in \mathbf{Z}, \ c_1 \geq c_2 \geq c_3 \geq c_4 \geq 0 \\ \text{Then take } \theta_{s_p^{n+1}-j+1} := c_j, \ j = \overline{1, 4}, \ \theta_i = 0, \ i = \overline{s_p^{n+1}-4} \end{cases}$$

(recall that $s_p^{n+1} \geq 4$, because $p \geq n/3$).

Given r as in (3.4.19) and θ_i as in (3.4.21), $\mathcal{D}$ is determined by these numbers and d , performing the inverse transformation (3.2.1), with $d = \deg(\mathcal{D})$.

We have $p_a(\mathcal{D}) = F_d^{p, n+1}(r) - \frac{1}{2} \sum_{i=1}^{s_p^{n+1}} \theta_i^2 = F_d^{p, n+1}(r) - \frac{1}{2}b$ (from 3.2.2) and (3.4.21)). Because b moves in the range (3.4.20), it follows from (3.4.17) that we get the necessary invertible sheaves for Step 2 as far as we check (C1) and (C2). Now, (C1) is clear, because $r \in 2\mathbf{Z}$ and $\theta_i \in \mathbf{Z}$. As for (C2), $g = p_a(\mathcal{D}) \in \mathbf{Z}$ and $p_a(\mathcal{D}) = F_d^{p, n+1}(r) - \frac{1}{2} \sum_{i=1}^{s_p^{n+1}} \theta_i^2$; so

$$2F_d^{p, n+1}(r) - \sum_{i=1}^{s_p^{n+1}} \theta_i^2 \equiv 0 \pmod{2}.$$

Because $l^2 \equiv l \pmod{2}$ if $l \in \mathbf{Z}$ and because $r \in 2\mathbf{Z}$, we can see that the left member of the above congruence is congruent $\pmod{2}$ with the left member of (C2).

Remark 3.4.3. *The Construction B is, obviously shorter than the Construction A, but in the subsection 3.5 we'll need both of them.*

Proof of Step 3. The existence of the invertible sheaves in the domain from Step 3 in a consequence of the construction from Step 2, using Lemma 3.3.1 c).

Recall that $X_p^n = \varphi_{[\mathcal{H}_p^n]}(S_p^n) \subset \mathbf{P}^n$, where $\varphi_{[\mathcal{H}_p^n]}$ is the embedding defined by the very ample sheaf $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in \text{Pic}(S_p^n)$ and $S_p^n = \text{Bl}_{\Sigma_p^n}(\mathbf{P}^2)$ ($\Sigma_p^n \subset \mathbf{P}^2$ a set of $s_p^n + 1$ general points). The sheaves from Step 2 were constructed using the surfaces $X_p^{n+1} \subset \mathbf{P}^{n+1}$. If we consider them on the abstract surface S_p^{n+1} , there invertible sheaves can be considered on S_p^n also, S_p^n being obtained from S_p^{n+1} blowing up a new general point $P_{s_p^n}$ (considered in $\mathbf{P}^2$). Put $\Sigma_p^n := \Sigma_p^{n+1} \cup \{P_{s_p^n}\}$. The sheaves from $\text{Pic}(X_p^{n+1})$ used in Step 2, considered now in $\text{Pic}(S_p^n)$ have the last component equal to zero. Using Lemma 3.3.1c), we obtain invertible sheaves $\mathcal{D} \in \text{Pic}(S_p^n)$ of degree d and $p_a = g$ in the domain

$$\alpha'_{p+1}(d-1, n) - x_{p+1}(d-1, n) \leq g \leq \alpha'_{p+1}(d-1, n), \quad d \geq a_{p+1}^{n+1} = a_{p+1}^n + 1.$$

Now, putting 1 (instead of 0) on the last component of the previous sheaves, the arithmetical genus doesn't change and the degree is translated by 1. We get now exactly the necessary domain for the proof of Step 3.

Remark 3.4.4. *The invertible sheaves $\mathcal{D} \in \text{Pic}(X_p^n)$ used in order to cover the domain from Step 3 coming from the Construction A in Step 2, are of the following form: $\mathcal{D} = \mathcal{D}_0'' + t_2\mathcal{H}_2 + t_3\mathcal{H}_3' + t_4\mathcal{H}_4' + t_5\mathcal{H}_5' + t_6\mathcal{H}_6'$, where $\mathcal{H}_2, \mathcal{H}_3', \mathcal{H}_4', \mathcal{H}_5', \mathcal{H}_6'$, are as in Lemma 3.2.4, $t_2, \dots, t_6 \in \mathbf{Z}$, $t_2, \dots, t_6 \geq 0$*

$$\mathcal{D}_0'' = (a+2; a, 1^u, 0^{s_p^{n+1}-u}, 1), \quad u \in \{3p-n+2, 3p-n+3\}.$$

$$a \in \{(d_0'' - n + p - 1)/2, (d_0'' - n + p)/2\} \cap \mathbf{Z},$$

$$d_0'' = (\mathcal{D}_0'' \cdot \mathcal{H}_p^n) \geq a_{p+1}^n = [(n-p-1)/2]_* + 1,$$

$$d_0'' \leq a_{p+1}^n + n + p - 1.$$

This follows from (3.4.9), (3.4.10), (3.4.11), the inductive processes, used in proving (a), (b), (c), (e) and the transformation from Step 3 (1 instead of 0 on the last component, giving 1 on the last component of $\mathcal{D}_0''$ and translating d_0 by 1, giving a as in the Remark.)

Proof of Step 4. Let's remark at the beginning that, putting together the Steps 1 and 3 and using Lemma 3.3.1 c), it follows that we covered with invertible sheaves from $\text{Pic}(X_p^n)$, by degrees and arithmetical genera the domain

$$(3.4.22) \quad \alpha'_{p+1}(d-1, n) \leq g \leq \alpha'_{p+1}(d, n), \quad d \geq a_{p+1}^n + 1.$$

We need sheaves from $\text{Pic}(X_p^n)$ covering by degrees and arithmetical genera the domain

$$\alpha_{p+1}(d-1, n) \leq g \leq \alpha_p(d, n), \quad d \geq a_p^n + 1.$$

We'll do this by induction on $x_p(d-1, n)$, using (3.4.22). If $x_p(d-1, n) = 0$, then $x_{p+1}(d-1, n) = 0$. Hence $\alpha_{p+1}(d-1, n) = [(d-n+p)/2]_*$. Using (3.3.5), it follows that we need invertible sheaves which cover by degrees and arithmetical genera the domain

$$[(d-n+p)/2]_* \leq g \leq [(d-n+p+1)/2]_*$$

$$a_p^n + 1 \leq d \leq a_p^n + (n+p-1).$$

But then the necessary degrees and genera are realized by

$$(3.4.23) \quad \mathcal{D}_0 = (g+2; g, 1^u, 0^{s_p^n-u}) \in \text{Pic}(X_p^n), \text{ with}$$

$$(3.4.24) \quad u = 3p-n+3, 3p-n+4 \text{ or } 3p-n+5.$$

Then, if we succeeded to construct invertible sheaves in the domain $\alpha_{p+1}(d-1, n) \leq g \leq \alpha_p(d, n)$. $x_p(d-1, n) = x$, adding them the hyperplane class $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n})$ we obtain, as many times before, using Lemma 3.3.1 a), invertible sheaves from $\text{Pic}(X_p^n)$ covering by degrees and arithmetical genera the domain $\alpha_{p+1}(d, n) \leq g \leq \alpha_p(d, n)$ for $x_p(d-1, n) = x+1$. Filling on the left from (3.4.22) (using Lemma 3.3.1 c), last part) we finish the inductive process. Now the Prop. 3.4.1 is proved.

q.e.d.

Before ending this subsection, some remarks, necessary in the next subsection 3.5. The next Remark 3.4.5 is a consequence of the Remarks 3.4.2, 3.4.4 and of the construction given in Step 4.

Remark 3.4.5. *The invertible sheaves $\mathcal{D} \in \text{Pic}(X_p^n)$ used in order to cover by degrees and arithmetical genera the domain from prop. 3.4.1, using the Construction A in Step 2, are of one of the following forms:*

a) $\mathcal{D} = \mathcal{D}'_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are as in the Lemma 3.2.4, $t_1, t_2 \in \mathbf{Z}$, $t_1, t_2 \geq 0$, $\mathcal{D}'_0 = (a+2; a, 1^u, 0^{s_p^n-u})$, $s_p^n = 3p-n+5$, $u \in \{3p-n+2, 3p-n+3, 3p-n+4, 3p-n+5\}$, $a \in \{(d'_0-n+p-2)/2, (d'_0-n+p-1)/2, (d'_0-n+p)/2, (d'_0-n+p+1)/2\} \cap \mathbf{Z}$, $d'_0 = (\mathcal{D}'_0 \cdot \mathcal{H}_p^n) \geq a_{p+1}^n + 1$, $d'_0 \leq a_p^n + n + p$;

b) $\mathcal{D} = \mathcal{D}''_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}'_3 + t_4\mathcal{H}'_4 + t_5\mathcal{H}'_5 + t_6\mathcal{H}'_6$ with $\mathcal{H}_1, \dots, \mathcal{H}'_6$ as in the Lemma 3.2.4, $t_1, \dots, t_6 \in \mathbf{Z}$, $t_1, \dots, t_6 \geq 0$, $\mathcal{D}''_0 = (a+2; a, 1^u, 0^{s_p^n-u}, 1)$, $u \in \{3p-n+2, 3p-n+3\}$, $a \in \{(d''_0-n+p-1)/2, (d''_0-n+p)/2\} \cap \mathbf{Z}$, $d''_0 = (\mathcal{D}''_0 \cdot \mathcal{H}_p^n) \geq a_{p+1}^n$ and $d''_0 \leq a_{p+1}^n + n + p - 1$.

From Remark 3.4.5 we deduce the following remark, which we'll use in the next subsection.

Remark 3.4.6. If $n \geq 9$ and $\frac{n}{3} \leq p \leq n-4$, $n, p \in \mathbf{Z}$, then the invertible sheaves $\mathcal{D} \in \text{Pic}(X_p^n)$ used in order to cover by degrees and arithmetical genera the domain from Prop. 3.4.1 (using Construction A in Step 2) are of the following form: $\mathcal{D} = \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}'_3 + t_4\mathcal{H}'_4 + t_5\mathcal{H}'_5 + t_6\mathcal{H}'_6$ with $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}'_4, \mathcal{H}'_5, \mathcal{H}'_6$ as in Lemma 3.2.4 and $\mathcal{D}_0 = (a+2; a, 1^u, 0^{s_p^n-u-1}, \varepsilon)$, $\varepsilon \in \{0, 1\}$, $0 \leq u \leq s_p^n - 1 = 3p - n + 4$, $a \geq (d_0 - n + p - 1)/2$, $a \in \mathbf{Z}$, $d_0 \in [a_{p+1}^n, a_p^n + n + p - 1] \cap \mathbf{Z}$ ($a_p^n = [(n-p)/2]_* + 1$).

Put $d_0 = d'_0 - 1$ if $\mathcal{D}$ is of the category a) in Remark 3.4.5 and $d_0 = d''_0$ if $\mathcal{D}$ is of the category b).

Remark 3.4.7. Let there be $\mathcal{D} = (a; b_0, b_1, \dots, b_{s_p^n}) \in \text{Pic}(X_p^n)$ one of the invertible sheaves used in the proof of Proposition 3.4.1. Let there be $d = \deg \mathcal{D} = (\mathcal{D} \cdot \mathcal{H}_p^n)$. Then $r \leq 2(x_p(d, n) + 1)$.

Indeed, let's denote by $r(d, n) := 2(x_p(d, n) + 1)$. If $\mathcal{D}$ has been used in Step 1, then $r = 2(x_{p+1}(d-1, n) + 1) \leq r(d, n)$. If $\mathcal{D}$ has been in Step 3 (with any construction in Step 2), then $r = 2(x_{p+1}(d, n) + 1) \leq r(d, n)$. If $\mathcal{D}$ has been used in Step 4, then $r = 2(x_{p+1}(d, n) + 1) \leq r(d, n)$ or $\mathcal{D}$ is obtained from some $\mathcal{D}_1 = (a^1; b_0^1, b_1^1, \dots, b_{s_p^n}^1) \in \text{Pic}(X_p^n)$ adding a number, let's say t , of $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in \text{Pic}(X_p^n)$; then $r_1 := a^1 - b_0^1 \leq r(d_1, n)$, where $d_1 = \deg(\mathcal{D}_1)$; now, adding t times $\mathcal{H}_p^n$, r_1 increases with at most $2t$ in order to become r and $r(d_1, n)$ increases with exactly $2t$ in order to become $r(d, n)$ (due to the form of the three previous r); so the inequality $r \leq r(d_1, n)$ is transferred to $r \leq r(d, n)$.

3.5 Curves $C \subset X_p^n$ whose $(\deg(C), g(C))_C$ cover the domain A_p^n ($n \geq 8, n/3 \leq p \leq n-4$)

We recall that the domain A_p^n were defined in Section 2, (2.11). In this subsection we prove the following

Proposition 3.5.1. Let there be $(d, g) \in A_p^n$, where $n, p \in \mathbf{Z}$, $n \geq 8$, $n/3 \leq p \leq n-4$, and put $k := [n/3]_*$. Then:

i) if $d \geq \frac{2}{3}(3p + n + 9)$ there is a (smooth, irreducible) curve $C \subset X_p^n$, non-degenerate in $\mathbf{P}^n$, so that $(\deg(C), g(C)) = (d, g)$;

ii) if $d < \frac{2}{3}(3p + n + 9)$ there is a (smooth, irreducible) curve $C \subset \mathbf{P}^n$, non-degenerate in $\mathbf{P}^n$, so that $(\deg(C), g(C)) = (d, g)$. If $n \equiv 0 \pmod{3}$ then C can be found on X_k^n and if $n \equiv 1, 2 \pmod{3}$ then C can be found on X_{k+1}^n .

Proof. The proof is a consequence of the analysis which we'll do on the linear systems associated to the invertible sheaves appearing in the proof of

Prop. 3.4.1. Precisely, we'll test when these linear systems are nonempty containing (smooth, irreducible) curves, using the smoothing criteria from subsection 3.2. Namely, for the situation (R1) (see (3.1.4)) we'll apply essentially the criterion given by the Corollary 3.2.5 (considering Construction A in Step 2 of the proof of Prop. 3.4.1) and for the situation (R2) (see (3.1.5)) we'll apply the criterion given by the Prop. 3.2.3 (considering Construction B in Step 2 of the proof of Prop. 3.4.1). The situations (R1) and (R2) are complementary and their union cover the hypothesis.

1) **For the situation (R1):**

Lemma 3.5.2. *If (p, n) is in the situation (R1), $(d, g) \in A_p^n$ and $\mathcal{D} \in \text{Pic}(X_p^n)$ is one of the invertible sheaves used in the proof of Prop. 3.4.1 (considering Construction A in Step 2) so that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$, then we know that $\mathcal{D} = \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}'_3 + t_4\mathcal{H}'_4 + t_5\mathcal{H}'_5 + t_6\mathcal{H}'_6$ as in Remark 3.4.6.; let's denote by $t := \sum_{i=1}^6 t_i$ and suppose $t \geq 3$. Then $[\mathcal{D}] \neq \emptyset$ and contains a (smooth, irreducible) curve C , non-degenerate in $\mathbf{P}^n$ (recall $\deg(\mathcal{D}) = (\mathcal{D} \cdot \mathcal{H}_p^n)$).*

Proof. We use Corollary 3.2.5, i.e. we check that (3.2.10) is verified if $t \geq 3$. Indeed, for $t = 3$ (it's enough to consider this case) (3.2.10) becomes

$$(3.5.1) \quad 3a \geq 3p - 4n + 14.$$

Because $a \geq (d_0 - n + p - 1)/2$, $d_0 \geq (n - p)/2$ (see Remark 3.4.6), minorating a and then d_0 , it follows that (3.5.1) holds if

$$(3.5.2) \quad 13n \geq 9p + 62.$$

Now, it is clear that (3.5.2) is satisfied in (R1) ($p \leq n - 4$, $n \geq 9$). Because $\deg \mathcal{H} \geq n + p - 1$, $(\forall) \mathcal{H} \in \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}'_4, \mathcal{H}'_5, \mathcal{H}'_6\}$ and $\deg(\mathcal{D}_0) \geq 0$ (with the definition of the degree as in the statement of the Lemma), the curve C which we just obtained has $\deg(C) \geq 3(n + p - 1)$ (because $t = 3$) so $\deg(C) > n + p - 1 = \deg(X_p^n)$, and $C \subset X_p^n$. So C is non-degenerate in $\mathbf{P}^n$.
q.e.d.

Lemma 3.5.3. *If $(p, n), (d, g), \mathcal{D}, t$ are as in the previous lemma and $t = 2$, then the same conclusion holds for $[\mathcal{D}]$.*

Proof. We apply again Corollary 3.2.5 for $t = 2$ and we use that $a \in \mathbf{Z}$. Doing computations (as before), it follows that the only case when (3.2.10) for $t = 2$ is not satisfied is for $p = n - 4$ and $a = -1$. We'll consider separately this case.

It's easy to see that the invertible sheaves $\mathcal{D}$ from our Lemma are of the form $\mathcal{D} = (b + 6; b, \eta_1, \eta_2, \dots, \eta_{sp})$, $\eta_i \in \{0, 1, 2, 3\}$, $(\forall)i$, $b \in \{a + 2p -$

$2, a + 2p - 1, a + 2p\}$, $a \in \mathbf{Z}$ as in the Remark 3.4.6. Let E be one of the exceptional divisors lying on X_p^n and corresponding to a $P_i \in \Sigma_p^n$. From the exact sequence

$$0 \rightarrow \mathcal{D} \rightarrow \mathcal{D}(E) \rightarrow \mathcal{D}(E)|_E \rightarrow 0$$

we deduce that the conclusion of the Lemma holds for $[\mathcal{D}]$ iff it holds for $[\mathcal{D}(E)]$, as far as $b_i = (\mathcal{D} \cdot \mathcal{O}_{X_p^n}(E)) \geq 1$. Moreover, it is easy to see that all $\eta_i = 3$ only if $b = a + 2p$. So, to conclude the Lemma it remains to prove the conclusion for $\mathcal{D} \in \{\mathcal{D}', \mathcal{D}''\}$ where $\mathcal{D}' = (a + 2p + 4; a + 2p - 2, 3^{s_p^n - 1}, 2)$, $\mathcal{D}'' = (a + 2p + 6; a + 2p, 3^{s_p^n})$, of course for $p = n - 4$ and $a = -1$. We'll study the case $\mathcal{D} = \mathcal{D}'$ the other one being similar. So,

$$\mathcal{D}' = (2p + 3; 2p - 3, 3^{2p}, 2), \quad p = n - 4 \geq 5.$$

We specialize the points from Σ_p^n on a smooth cubic curve $\Gamma_0 \subset \mathbf{P}^2$; we denote this specialization by $\tilde{\Sigma}_p^n = \{\tilde{P}_0, \tilde{P}_1, \dots, \tilde{P}_{2p+1}\}$; we suppose, moreover that the points from $\tilde{\Sigma}_p^n$ are general on Γ_0 (see the sketch of the proof of Prop. 3.2.1), in particular every 3 of them are not collinear. Let there be by $\tilde{S}_p^n = Bl_{\tilde{\Sigma}_p^n}(\mathbf{P}^2)$. Then $(p + 2; p, 1^{2p+1})$ is very ample on $\tilde{S}_p^n$ (actually, if $s \leq 2q + 3, q \geq 1, s \geq 0, s, q \in \mathbf{Z}$ and $\Sigma = \{R_0, R_1, \dots, R_s\} \subset \Gamma_0$ general on Γ_0 , $S := BL_{\Sigma}(\mathbf{P}^2)$, then $(q + 2; q, 1^s)$, is very ample on S , see [Hb2]; for a direct proof see [P4]). Let's denote by $\tilde{X}_p^n := \varphi_{[p+2; p, 1^{2p+1}]}(\tilde{S}_p^n) \subset \mathbf{P}^n$. Using Prop. 3.2.1 (i3)' and (ii3)' it results that, if $\tilde{\mathcal{D}}_1 := (2p; 2p - 3, 2^{2p}, 1)$, then

$$(3.5.3) \quad \begin{cases} h^1(\tilde{\mathcal{D}}_1) = 0 \text{ and } [\mathcal{D}_1] \neq \emptyset, \text{ base point free,} \\ \text{containing a (smooth, irreducible) curve.} \end{cases}$$

By semicontinuity (see [CS], Remark 1, p. 324) we deduce that (3.5.3) holds again if we replace $\tilde{P}_0$ with another point (denoted $\tilde{P}'_0$) from a small neighbourhood of $\tilde{P}_0$, not belonging to Γ_0 and not collinear with any others $\tilde{P}_i$ and $\tilde{P}_j$. Then, the proper transformation of Γ_0 in $\tilde{X}_p^n$ is $\Gamma \in [3; 0, 1^{2p+1}]$.

Now, we denote by δ_{ij} the quadratic transformation based on $\{\tilde{P}'_0, \tilde{P}_i, \tilde{P}_j\}$. Performing the successive quadratic transformations $\delta_{12}, \delta_{34}, \dots, \delta_{2p-1, 2p}$, the curve Γ_0 becomes a curve $\Delta_0 \subset \mathbf{P}^2$ of degree $p + 3$ and having $2p + 1$ singular points (with distinct tangents) with multiplicities $p, 2, \dots, 2$ respectively; $\tilde{X}_p^n$ becomes $\bar{X}_p^n := \varphi_{[p+2; p, 1^{2p+1}]}(\bar{S}_p^n) \subset \mathbf{P}^n$, where $\bar{S}_p^n = Bl_{\bar{\Sigma}_p^n}(\mathbf{P}^2)$, $\bar{\Sigma}_p^n = \{\bar{P}_0, \bar{P}_1, \dots, \bar{P}_{2p}, \bar{P}_{2p+1}\}$, $\bar{P}_0, \bar{P}_1, \dots, \bar{P}_{2p}$ being the singularities of Δ_0 with multiplicities $p, \dots, 2$ respectively and $\bar{P}_{2p+1} \in \Delta_0$ a smooth point so that every 3 points in $\bar{X}_p^n$ are non-collinear; of course $\tilde{X}_p^n \cong \bar{X}_p^n (\cong \tilde{S}_p^n = \bar{S}_p^n)$; moreover

$\tilde{\mathcal{D}}_1$ becomes, in the new coordinates, $\bar{\mathcal{D}}_1 = (p; p-3, 1^{2p+3}) \in \text{Pic}(\bar{X}_p^n)$. If we consider $\tilde{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_1$ on $\tilde{S}_p = \bar{S}_p$, then $\tilde{\mathcal{D}}_1 = \bar{\mathcal{D}}_1$.

Then, (3.5.3) can be written as

$$(3.5.4) \quad \begin{cases} h^1(\bar{\mathcal{D}}_1) = 0 \text{ and } [\bar{\mathcal{D}}_1] \neq \emptyset, \text{ without base} \\ \text{points and containing a (smooth, irreducible) curve.} \end{cases}$$

Let's denote by $\Delta \in [p+3; p, 2^{2p}, 1]$ the proper transformation of Δ_0 in $\bar{X}_p^n$. We can see that $\bar{\mathcal{D}}_1 = \mathcal{D}'(-\Delta)$, where $\mathcal{D}'$ is our initial invertible sheaf (considered on $\bar{X}_p^n$). We obtain then the exact sequence of sheaves on $\bar{X}_p^n$

$$0 \rightarrow \bar{\mathcal{D}}_1 \rightarrow \mathcal{D}' \rightarrow \mathcal{D}'|_{\Delta} \rightarrow 0.$$

Since $(\mathcal{D}' \cdot \mathcal{O}_{\bar{X}_p^n}(\Delta)) = 7 \geq 2g(\Delta) = 2(= 2g(\Gamma))$ using (3.5.4) it results that $[\bar{\mathcal{D}}_1] \neq \emptyset$, base point free containing a (smooth, irreducible) curve, if the points of Σ_p^n are specialized in $\bar{\Sigma}_p^n$. So, the same fact remains true on X_p^n , by semicontinuity. q.e.d.

Lemma 3.5.4. *If $(p, n), (d, g), \mathcal{D}$ are as in the Lemma 3.5.2 and $d = \deg(\mathcal{D}) \geq \max(2n+1, \frac{2}{3}(3p+n+9))$, then $[\mathcal{D}] \neq \emptyset$, without base points and contains a (smooth, irreducible) curve C , non-degenerate in $\mathbf{P}^n$.*

Proof. Apply Corollary 3.2.5 for $t = 1$.

Preliminary Conclusion 3.5.5. *If $(d, g) \in A_p^n$ with (p, n) in situation (R1) and $d \geq \max(2n+1, \frac{2}{3}(3p+n+9))$, then there is a (smooth, irreducible) curve $C \subset X_p^n$, non-degenerate in $\mathbf{P}^n$, with $(\deg(C), g(C)) = (d, g)$.*

2) For the situation (R2).

Lemma 3.5.6. *If (p, n) is in the situation (R2), $(d, g) \in A_p^n$ and $\mathcal{D} \in \text{Pic}(X_p^n)$ is one of the invertible sheaves used in the proof of Prop. 3.4.1 (considering Construction B in Step 2) so that $(\deg(\mathcal{D}), p_a(\mathcal{D})) = (d, g)$, then $\mathcal{D}$ satisfied the condition (C3) from the Prop 3.2.3 in the coordinates $(d, r; \theta_1, \dots, \theta_{s_p^n})$ (here $\deg(\mathcal{D}) = (\mathcal{D} \cdot \mathcal{H}_p^n)$).*

Proof. The sheaves used in Step 1 satisfy (C3). The invertible sheaves used in Step 3 come from the sheaves used in Step 2 putting 1 (instead 0) on the last component. So, it's enough to verify that the sheaves $\mathcal{D}$ used in Step 2 apply to

$$(3.5.5) \quad |\theta_1| \leq \theta_2 \leq \dots \leq \theta_{s_p^{n+1}} < \frac{r}{2} \text{ for } d = \deg(\mathcal{D}) \geq 2n+2$$

The inequalities $|\theta_1| \leq \theta_2 \leq \dots \leq \theta_{s_p^{n+1}}$ come from (3.4.21). From $d \geq 2n + 2$ we deduce that $r = 2(x_p(d, n + 1) + 1) \geq 4$ (cf. (3.4.19)). Then $c_i > \frac{r}{2} \Rightarrow b \geq c_i^2 > \frac{r^2}{4}$ (see (3.4.21)); but $b \leq r$ (see (3.4.20)) and $r \leq \frac{r^2}{4}$, that is a contradiction ! So, $c_i \leq \frac{r}{2}$, hence $\theta_i \leq \frac{r}{2}$, $i = \overline{1, s_p^{n+1}}$. Moreover, $c_1 = \frac{r}{2} \Leftrightarrow r = 4$. Then we replace (if necessary) $(c_1, c_2, c_3, c_4) = (2, 0, 0, 0)$ by $(c_1, c_2, c_3, c_4) = (1, 1, 1, 1)$ and we get $\theta_{s_p^{n+1}} < \frac{r}{2}$. Now, the invertible sheaves used in Step 4 satisfy (C3), because $\mathcal{H}_p^n$ satisfies this condition.

q.e.d.

Lemma 3.5.7. *If $(p, n), (d, g)$ and $\mathcal{D}$ are as in the previous lemma, then $\mathcal{D}$ satisfies the condition (C4) from Prop. 3.2.3 in the coordinates $(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^n})$.*

Proof. (C4) means $b_0 \geq b_1$ if $\mathcal{D} = (a; b_0, b_1, \dots, b_{s_p^n})$ is written in the usual coordinates on $\text{Pic}(X_p^n)$. We will check this (in i)) for the sheaves used in Step 1 from the proof of Prop. 3.4.1 obtained by adding a (finite) number of $\tilde{\mathcal{H}}_p^n$ to the initial family $\mathcal{D}_0$, for the sheaves used in Step 4 of Prop. 3.4.1 obtained adding a (finite) number of $\mathcal{H}_p^n$ to the sheaves used in Step 1 (all these are sheaves like the invertible sheaves from Remark 3.4.5 a)). We'll also check (C4) for the sheaves used in Step 2, Construction B (in ii)). It's clear that from these two verifications (i) and (ii), we get the conclusion of the Lemma.

i) We consider sheaves $\mathcal{D} = \mathcal{D}_0 + t_1 \mathcal{H}_1 + t_2 \mathcal{H}_2$, $\mathcal{H}_1 = \mathcal{H}_p^n$, $\mathcal{H}_2 = \tilde{\mathcal{H}}_p^n$, as in Remark 3.4.5. a). Put $t := t_1 + t_2$. If $t = 1$, because $d = \deg(\mathcal{D}) \geq 2n + 1$, it follows that $d_0 := \deg \mathcal{D}_0 \geq n - p + 1$. But, then $\mathcal{D} = (a + p + 4; a + p, 2^u, 1^{s_p^n - u - 1}, \varepsilon)$, $\varepsilon \in \{0, 1\}$. We need $a + p \geq 2$. Minorating a as in Remark (3.4.5 a)) and d_0 from before, it remains to have $p \geq 3$, which is true in (R2). If $t \geq 2$, then $b_0 \geq b_1$ (i.e. (C4)) becomes $a + tp \geq t + 1$. It's enough to consider the case $t = 2$ ($a + tp - t - 1$) is an increasing function of t). So, we need $a + 2p \geq 3$. Minorating a and d_0 from Remark 3.4.5 a), it results that we need $p \geq (n + 14)/9$, this last inequality being satisfied in (R2).

ii) Now we are going to study the sheaves used in the proof of Step 2 of Prop. 3.4.1 (Construction B). Their degrees are $d \geq 2n + 2$, so $x_p(d, n + 1) \geq 1$.

1) If $x_p(d, n + 1) \geq 3$, we have $-\theta_1 + \sum_{i=2}^{s_p^{n+1}} |\theta_i| = \sum_{i=1}^4 c_i$ (see (3.4.21))
 $\leq 2 \sqrt{\sum_{i=1}^4 c_i^2} \leq 2\sqrt{b} \leq 2\sqrt{2(x_p(d, n + 1) + 1)}$ (see 3.4.20). Now, it's enough to

prove

$$(3.5.6) \quad \begin{aligned} x_p(d, n+1) \geq 3 &\Rightarrow \\ \Leftrightarrow d - (n - p + 2)(x_p(d, n+1) + 3) &\geq 2(x_p(d, n+1) + 1) \end{aligned}$$

(indeed, then $\sqrt{2(x_p(d, n+1) + 1)} \leq x_p(d, n+1) + 1$, so combining (3.5.6) with the previous inequality, we get exactly (C4) for $r = 2(x_p(d, n+1) + 1)$, cf. (3.4.19)). As for (3.5.6), we have

$$(3.5.7) \quad x_p(d, n+3) \geq 3 \Rightarrow d \geq (7n + 5p + 2)/2.$$

a) If $p \geq 4$, then

$$\begin{aligned} d - (n - p + 2)(x_p(d, n+1) + 1) - 2(x_p(d, n+1) + 1) &\geq \\ \geq d - n(x_p(d, n) + 1) &\geq d - \frac{n}{2(n+p)}(2d + n + 3p - 2). \end{aligned}$$

This last number is ≥ 0 iff

$$(3.5.8) \quad d \geq n(n + 3p - 2)/2p.$$

But in (R2), (3.5.8) follows from (3.5.7). We get then (3.5.6).

b) If $p = 3$, then $n \in \{8, 9\}$. If $n = 8$, (3.5.6) becomes $d \geq 9(x_3(d, 9) + 1)$, true for $d \geq 27$. But from (3.5.7) it results that $d \geq 37$, so we are ready. Similarly for $n = 9$ ((3.5.6) becomes $d \geq 10(x_3(d, 10) + 1)$, true for $d \geq 30$; but (3.5.7) $\Rightarrow d \geq 40$).

2) If $x_p(d, n+1) = 2$. Then

$$(3.5.9) \quad d \geq (5n + 3p + 2)/2.$$

In this case (C4) becomes

$$(3.5.10) \quad -\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq d - 3(n - p + 2).$$

From (3.5.9) it follows $d - 3(n - p + 2) \geq (9p - n - 10)/2 \geq 4$ (since $p \geq n/3$, $n \geq 8$, and $p = (n + 1)/3 = 3$ if $n = 8$). Then in order to obtain (3.5.10)

it's enough to choose $\theta_1, \dots, \theta_{s_p^{n+1}}$ such that $-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq 4$. By (3.4.20)

and (3.4.21) it's enough to express each $b \in \{1, 2, \dots, 6\}$ as $b = \sum_{i=1}^4 c_i^2$ so that

$\sum_{i=1}^4 c_i \leq 4$ which is, obviously possible.

3) If $x_p(d, n+1) = 1$ (C4) becomes

$$(3.5.11) \quad -\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq d - 2(n - p + 2).$$

But $p \geq 3$ and $d \geq 2n + 2$, so $d - 2(n - p + 2) \geq 4$. By (3.4.20) and (3.4.21) it's enough to express each $b \in \{1, 2, 3, 4\}$ as $b = \sum_{i=1}^4 c_i^2$ so that $c_i \in \{0, 1\}$ for

all i (hence $\sum_{i=1}^4 c_i \leq 4$), which is possible.

q.e.d.

Lemma 3.5.8. *If $(p, n), (d, g), \mathcal{D}$ are as in Lemma 3.5.6 and $d = \deg(\mathcal{D}) \geq \max\left(2n + 1, \frac{2}{3}(3p + n + 9)\right)$, then $\mathcal{D}$ satisfies the condition (C5) in the coordinates $(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^n})$.*

Proof. From Remark 3.4.7 it follows that it's enough to check (C5) for $r = 2(x_p(d, n) + 1)$. This condition becomes:

$$(3.5.12) \quad d \geq 2(p - 1)(x_p(d, n) + 1) + 2.$$

If we denote by $E_p(d, n) := d - 2(p - 1)(x_p(d, n) + 1) - 2$, then (3.5.12) can be written as

$$(3.5.13) \quad E_p(d, n) \geq 0.$$

We can see that, if (3.5.13) is true for integers d so that $x_p(d, n) = x$, then it is true for any integer d' so that $x_p(d', n) \geq x$. Indeed, if $E_p(d, n) \geq 0$ for all d so that $x_p(d, n) = y$, let d' be so that $x_p(d', n) = y + 1$ and $d := d' - (n + p - 1)$. Then $E_p(d', n) = E_p(d, n) + (n - p - 1) \geq E_p(d, n) \geq 0$ (use Lemma 3.3.1 a)). The previous statement follows then by induction on $x_p(d, n)$.

Now, if $x_p(d, n) = 2$, (3.5.12) becomes

$$(3.5.14) \quad d \geq 6p - 4.$$

Since $x_p(d, n) = 2$, we have $d \geq (5n + 3p - 3)/2$. But then (3.5.14) holds in (R2) (use $3p \leq n + 5$).

So (3.5.12) is true for any d so that $x_p(d, n) \geq 2$. As for $x_p(d, n) = 1$, then (3.5.12) becomes $d \geq 4p - 2$. This follows now from $d \geq \frac{2}{3}(3p + n + 9)$ in (R2) (use again $3p \leq n + 5$).

q.e.d.

Because (C1) and (C2) were verified during the Construction B in the proof of Step 2 of Prop. 3.4.1, and (C3), (C4), (C5) were verified in Lemmas 3.5.6, 3.5.7, 3.5.8, we deduce from Prop. 3.2.3 the following

Preliminary Conclusion 3.5.9. *If $(d, g) \in A_p^n$ with (p, n) in the situation (R2) and $d \geq \max\left(2n + 1, \frac{2}{3}(3p + n + 9)\right)$, then there is a (smooth, irreducible) curve $C \subset X_p^n$, non-degenerate in $\mathbf{P}^n$, with $(\deg(C), g(C)) = (d, g)$.*

Lemma 3.5.10. *If $n \geq 8$ and $n/3 \leq p \leq n - 4$, $n, p \in \mathbf{Z}$, $(d, g) \in A_p^n$ and $2n + 1 \leq d \leq \frac{2}{3}(3p + n + 9)$, there is a (smooth, irreducible) curve $C \subset X_k^n$ if $n \equiv 0 \pmod{3}$ and $C \subset X_{k+1}^n$ if $n \equiv 1, 2 \pmod{3}$, non-degenerate in $\mathbf{P}^n$, with $(\deg(C), g(C)) = (d, g)$ (here $k = [n/3]_*$, as usual).*

Proof. We use Prop. 3.2.3. We need curves in the range

$$\alpha_{p+1}(d - 1, n) \leq g \leq \alpha_p(d, n), 2n + 1 \leq d < \frac{2}{3}(3p + n + 9).$$

Then $\alpha_{p+1}(d - 1, n) = [(3d - 4n - 2)/2]_*$, $\alpha_p(d, n) = [(3d - 4n + 1)/2]_*$. Because the set $[\alpha_{p+1}(d - 1, n), \alpha_p(d, n)] \cap \mathbf{Z}$ does not depend on p we construct the necessary curves on X_k^n if $n = 3k$ and on X_{k+1}^n if $n = 3k + 1, 3k + 2$. We remark that $F_d^{p,n}(4) = (3d - 4n + 2)/2$ (see (3.2.3)).

Precisely, we consider linear systems associated to invertible sheaves $\mathcal{L} \in \text{Pic}(X_p^n)$, $p \in \{k, k + 1\}$, $k = [n/3]_*$, which in the Gruson-Peskine coordinates are $(d, 4; 0^{s_p^n - t}, 1^t)$, with $2n + 1 \leq d < \frac{2}{3}(3p + n + 9)$, $t \in \{1, 2, 3, 4, 5\}$ (see (3.2.1) and (3.2.2)). Hence, $r = 4$, $\theta_i = 0$, $i = \overline{0, s_p^n - t}$, $\theta_{s_p^n - j + 1} \in \{0, 1\}$, $j = \overline{2, 5}$, $\theta_{s_p^n} = 1$, $s_p^n \in \{5, 6, 7\}$. We can see immediately that the conditions (C1)-(C5) are satisfied (use $d \geq 2n + 1$ and (3.2.2)).

q.e.d.

Now the **proof** of the Prop. 3.5.1 follows immediately from and Lemma 3.5.10.

q.e.d.

3.6 Conclusion

Now, the **Proof** of the Theorem C (Section 2) follows, putting together the domains A_p^n from Prop. 3.5.1. Then our Main Theorem follows from (2.15), Theorem A, Theorem B and Theorem C (Section 2).

q.e.d.

4 Comments and further developments; D_2^n - the expected lacunary domain

1) One can see that our Main Theorem remains true over an algebraically closed field of *arbitrary characteristic*. This follows replacing the Bertini Theorem (which we used several times during our proof) - true in characteristic zero - by the Hartshorne's Bertini-type theorem ([Ha2], théorème 5.1) - true in arbitrary characteristic. The verifications are similar to Rathmann's verifications for curves in $\mathbf{P}^4$ and $\mathbf{P}^5$ ([Ra]).

2) We can consider other topics from Halphen-Castelnuovo theory (see §1) also. One example is when $\mathcal{P}$ = linear normality. In [P5] and [P6] we extended the results of Dolcetti and Pareschi ([DP]) from $\mathbf{P}^3$ to higher dimensional projective spaces.

3) Moreover, concerning the study of families of non-degenerate (smooth, irreducible) curves from $\mathbf{P}^n$, we can consider the Hilbert scheme $H_{d,g}^n$ (see §1). As Kleppe pointed out ([Kl], §6) our results can be used in order to stand out "good" components of $H_{d,g}^n$ in a big range on (d, g, n) .

4) Finally, few words concerning the domain D_2^n (see (2.10)). We proved that there is no gap for $HC(n)$ in D_1^n (see (2.9)) in our Main Theorem. We recall from §1 that for $n = 3, 4, 5, 6, 7$ is known that the (d, g) -plane is divided in a lacunary and a non-lacunary domain in each case. We *conjecture* that D_1^n and D_2^n from this article represents indeed the good definitions for the *non-lacunary* (D_1^n) and the *lacunary* (D_2^n) domains for any $n \geq 7$. Although we do not deal in the present article with D_2^n , let's do some hints whose aim is to suggest that D_2^n should be a good definition for the lacunary domain.

If $n \geq 7$ and $k : [n/3]_*$ we consider the subdomains of D_2^n given by:

$$D_2'^n : A(d, n) < g \leq \pi_0(d, n), \quad d \geq 2n + 1$$

$$D_2''^n : \begin{cases} \alpha_{k+1}(d, n), & \text{if } n \equiv 1, 2 \pmod{3} \\ \alpha_k(d, n), & \text{if } n \equiv 0 \pmod{3} \end{cases} < g \leq A(d, n),$$

$$2n + 1 \leq d < d_1(n)$$

(see (1.1), (2.1), (2.6), (2.7)). Then $D_2^n = D_2'^n \cup D_2''^n$. If $(d, g) \in D_2'^n$, inspired from the Harris-Eisenbud Conjecture, true for $d > 2^{n+1}$ (see [H], ch. III) and using the Horowitz results from [Ho], §1 and the Example from [Ci], we expect that any such a (d, g) which *is not* a gap for $HC(n)$ be the $(deg(C), g(C))$ for some smooth, non-degenerate curve $C \subset \mathbf{P}^n$ lying either on a surface X_p^n , $\frac{n-5}{3} \leq p < \frac{n}{3}$ or on a (possibly singular) scroll. In both cases we get gaps (these X_p^n are obtained blowing up too few points from $\mathbf{P}^2$, at most 3, and on scrolls there is a formula relating d and g , see [Ci], section 2, g). If $(d, g) \in D_2''^n$, by the same argument ([H], [Ho]), the necessary curves are expected to lie either on X_p^n ($\frac{n}{2} \leq p < \frac{n}{3}$) or on scrolls. We also expect gaps on X_p^n for $d < a$ functions of degree $3/2$ in n , whose existence could be proved using an argument similar to the case of cubic surfaces from [GP2]. We do not insist more on D_2^n , the classification of gaps from it being a rather delicate problem.

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Address of author:
 Institute of Mathematics
 Romanian Academy of Sciences
 P.O. Box 1-764
 014700 Bucharest, Romania
 e-mail: Ovidiu.Pasarescu@imar.ro