

ボロミアン普遍オービフォールドの分岐被覆のベッチ数と有限群の表現
(REPRESENTATION OF FINITE GROUPS AND THE
FIRST BETTI NUMBER OF BRANCHED COVERINGS
OF A UNIVERSAL BORROMEAN ORBIFOLD)

戸田正人 (MASAHITO TODA)

お茶の水女子大学理学部
Department of Mathematics,
Ochanomizu University

1. Motivation. The main object is the first homology of regular branched coverings of a hyperbolic 3-orbifold. We shall stick to a single, but universal, example of 3-orbifolds, which is called $B_{4,4,4}\backslash\mathbb{H}^3$ by Hilden, Lozano and Montesinos[HLM1]. The homology is given a structure of $\mathbb{C}[G]$ -module by the action of covering transformation group G . The main result is the structure of the $\mathbb{C}[G]$ -module. The investigation is motivated by the following problem in 3-dimensional topology:

Problem. *Does every aspherical 3-manifold have a finite-sheeted cover of positive first Betti number?*

This problem was raised by Thurston, which can be one of the crucial steps towards his hyperbolization conjecture of irreducible atoroidal 3-manifolds through his hyperbolization theorem for Haken 3-manifolds. The lemma below illustrate how irreducible components of the $\mathbb{C}[G]$ -module is related to the first Betti numbers of unbranched coverings of a given 3-manifold.

Lemma. *Suppose that Γ is an orientation-preserving cocompact Kleinian group and Γ_0 a normal subgroup of finite index in Γ . Then we have*

$$H_*(\Gamma\backslash\mathbb{H}^3, \mathbb{C}) \simeq H_*(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})^{\Gamma/\Gamma_0}$$

where superscript Γ/Γ_0 denotes the fixed point set by the action of Γ/Γ_0 .

The proof is a direct application of the basic homology theory, in particular the transfer map.

Now let us recall the definition of universal groups.

Definition. *Kleinian group Γ is universal if, for any given closed 3-manifold M , there is subgroup Γ_M of finite index in Γ such that $\Gamma_M\backslash\mathbb{H}^3$ is homeomorphic to M .*

See [HLM2] for the universality of Kleinian group $B_{4,4,4}$.

We denote by T_Γ the subgroup of Γ generated by all elements of finite order in Γ . The following assertion easily follows from above Lemma.

Proposition. *For given closed 3-manifold M , any subgroup Γ_M of universal group $B_{4,4,4}$ associated to M in the definition and each normal subgroup Γ_0 of finite index in $B_{4,4,4}$, we can find a finite-sheeted (unbranched) covering $\tilde{M}_{\Gamma_0}$ of M with*

$$b_1(\tilde{M}_{\Gamma_0}) \geq \dim(H_1(\Gamma_0 \backslash \mathbb{H}^3, \mathbb{C})^{T_{\Gamma_M} \Gamma_0 / \Gamma_0})$$

where $b_1(\cdot)$ denotes the first Betti number.

Hence the information of the irreducible component of G -module $H_1(\Gamma_0 \backslash \mathbb{H}^3, \mathbb{C})$ gives us the lower bound of the betti number of 3-manifolds which is covered by $\Gamma_0 \backslash \mathbb{H}^3$, possibly with branches. In view of the proposition Thurston's problem can be divided into two parts, the first is the investigation of the irreducible component of G -module $\dim(H_1(\Gamma_0 \backslash \mathbb{H}^3, \mathbb{C}))$ for various Γ_0 and the second is finding the nice Γ_0 in which the images of T_{Γ_M} is 'small'. We shall investigate the first part.

2. Results. $B_{4,4,4}$ is normalized by mutually orthogonal hyperbolic reflections r_1, r_2 and r_3 . r denotes orientation reversing element r_1 or $r_1 r_2 r_3$ of the normalizer.

Theorem A. *Let Γ_0 be the r -normal subgroup of $B_{4,4,4}$ with finite index. If the irreducible representation ρ of $G := B_{4,4,4}/\Gamma_0$ verifies*

$$(1) \quad \sum_i \alpha_i \chi_{\bar{\rho}}(\theta_i r) \neq 0$$

ρ appears as an irreducible component of $H_1(\Gamma_0 \backslash \mathbb{H}^3, \mathbb{C})$. Here, α_i 's are explicitly determined integers and $\bar{\rho}$ denotes the irreducible representation of semidirect product $G \rtimes \langle r \rangle$ which restricts to ρ , χ_* the character of the representation.

Since $B_{4,4,4}$ is known to be arithmetic lattice of $SO(3,1)$ over number field $K = \mathbb{Q}(\sqrt{5})$ (cf. [HLM1]) we can consider the congruence subgroups. For the case that Γ_0 is a principle congruence subgroup associated prime ideals of K we can compute the linear combination term on the left hand side of (1).

Theorem B. *Let $\Gamma_{\mathfrak{p}}$ be a principle congruence subgroup of $B_{4,4,4}$ associated to prime ideal $\mathfrak{p}$ in K . Set $G = B_{4,4,4}/\Gamma_{\mathfrak{p}}$.*

(i) *If $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 1 \pmod{8}$ every nontrivial irreducible representation of G appears in $H_1(\Gamma_{\mathfrak{p}} \backslash \mathbb{H}^3, \mathbb{C})$.*

(ii) *Let Γ be a congruence subgroup of $B_{4,4,4}$. If the image of Γ in G does not contain noncentral normal subgroup the first betti number of $\Gamma \backslash \mathbb{H}^3$ is positive.*

The method of the computation implies somewhat general type of result.

Theorem C. *Let Γ be a maximal $r_1 r_2 r_3$ -normal, but not maximal normal subgroup of finite index in $B_{4,4,4}$. Then any nontrivial $r_1 r_2 r_3$ -invariant irreducible representation of $G = B_{4,4,4}/\Gamma$ is an irreducible component of $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})$.*

3. Universal group $B_{4,4,4}$ and cell decomposition. The orbifold $B_{4,4,4} \backslash \mathbb{H}^3$ can be given by the pasting of hyperbolic polyhedron R according to the pattern in Fig. 1. Polyhedron R is a hyperbolic regular dodecahedron with right edge angle. We denote by θ_X the elliptic element of order 4 which pastes the side X to side X' . $B_{4,4,4} \backslash \mathbb{H}^3$ has the natural cell decomposition induced from faces of R .

For normal subgroup $\Gamma \subset B_{4,4,4}$ we can equivariantly lift the cell decomposition $\Gamma \backslash \mathbb{H}^3$. We denote by $(\mathfrak{F}_i)_{\Gamma}$ the set of i -cells in the decomposition. Labeling the cells according to Fig. 2 we can explicitly describe the action of G on $(\mathfrak{F}_i)_{\Gamma}$ as follows.

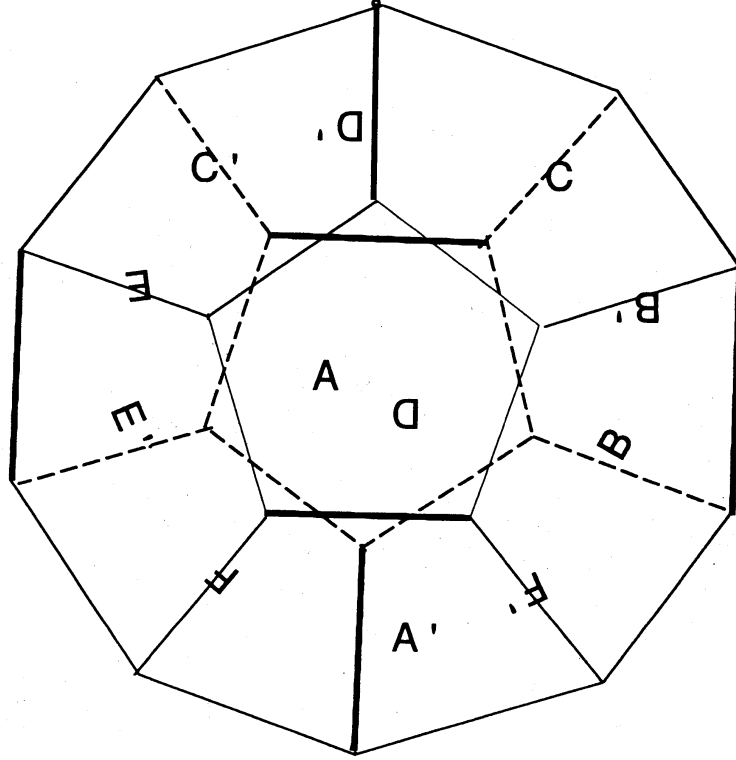


FIG. 1. Arrangement of Sides of Regular Dodecahedron R

Lemma 1. Let Γ be a normal subgroup of $B_{4,4,4}$ and $G = B_{4,4,4}/\Gamma$. ($\simeq$ denotes the isomorphism as G -set.)

- (0) $(\mathfrak{F}_0)_\Gamma = G(\Gamma Q) \cup \{G(\Gamma P_x); x = a, b, \dots, f\}$,
 $G(\Gamma Q) \simeq G$, $G(\Gamma P_x) \simeq G/\langle \theta_X \rangle$ ($x = a, b, \dots, f, X = A, B, \dots, F$).
- (1) $(\mathfrak{F}_1)_\Gamma = \{G(\Gamma xx'); x = a, b, \dots, f\} \cup \{G(\Gamma y); y = ab, bc, ca, de, ef, fd\}$,
 $G(\Gamma xx') \simeq G/\langle \theta_x \rangle$ ($x = a, b, \dots, f$), $G(\Gamma y) \simeq G$ ($y = ab, bc, ca, de, ef, fd$).
- (2) $(\mathfrak{F}_2)_\Gamma = \{G(\Gamma X); X = A, B, \dots, F\}$. $G(\Gamma X) \simeq G$ ($X = A, B, \dots, F$)
- (3) $(\mathfrak{F}_3)_\Gamma = G(\Gamma R) \simeq G$.

The lemma gives us the description of G -chain complex $\{C_*, \partial\}$ associated to cell decomposition $(\mathfrak{F}_*)_\Gamma$ as follows.

$$\begin{aligned}
 C_0 &\simeq \mathbb{C}[G] \cdot v_Q \oplus \bigoplus_x \mathbb{C}[G/\langle \theta_X \rangle] \cdot v_x := C'_0 \oplus C''_0 \\
 C_1 &= \bigoplus_x \mathbb{C}[G/\langle \theta_X \rangle] \cdot e_x \oplus \bigoplus_y \mathbb{C}[G] \cdot e_y := C'_1 \oplus C''_1 \\
 C_2 &= \bigoplus_x \mathbb{C}[G] \cdot s_X, \quad C_3 = \mathbb{C}[G] \cdot c_R.
 \end{aligned}$$

where the summation indexes varies according to the description in Lemma 1 and v_* , e_* , s_* , and c_* are the oriented cells of the corresponding 0-, 1-, 2- and 3-cells. We also decompose C_0 and C_1 into two summands.

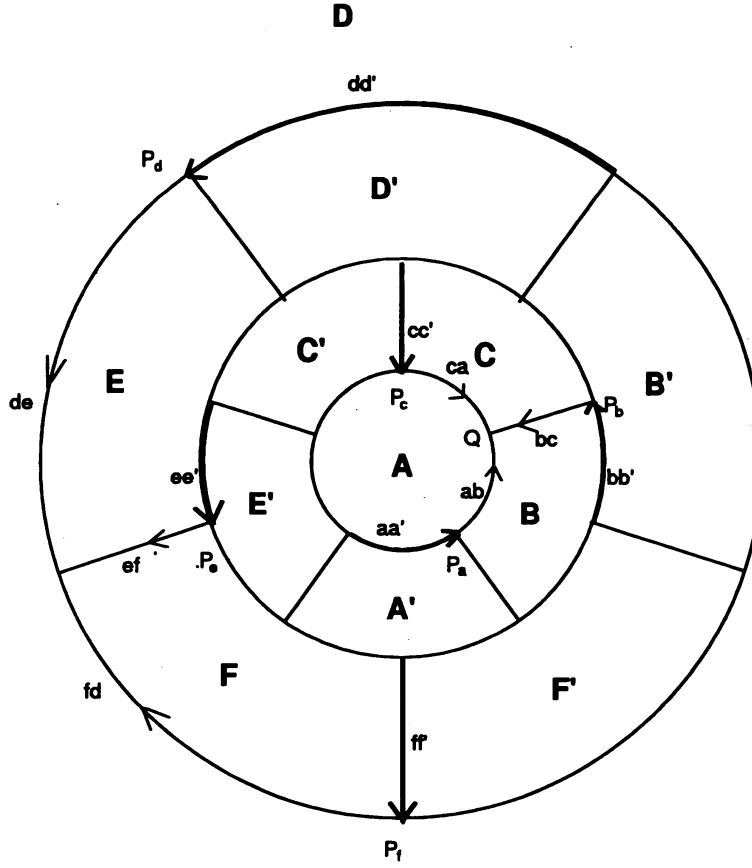


FIG. 2. Edges and Vertexes

In addition, if Γ is characterized by $r \{C_*, \partial\}$ has the action of $G \rtimes \langle r \rangle$. Observing the action of r in Fig. 1 we can explicitly describe action on the chain complex $\{C_*, \partial\}$ as follows.

Lemma 2. Suppose Γ is r_1 -normal. Then the action of r_1 on C_* is described as follows.

$$\begin{aligned}
 C'_0 \ni \alpha &\mapsto r_1 \alpha \theta_B \in C'_0, \\
 C''_0 \ni (\alpha_A, \alpha_B, \dots, \alpha_F) &\mapsto \\
 &\quad (r_1 \alpha_A \theta_B, r_1 \alpha_B, r_1 \alpha_F \theta_A, r_1 \alpha_D \theta_E, r_1 \alpha_E, r_1 \alpha_C \theta_D) \in C''_0, \\
 C'_1 \ni (\alpha_a, \alpha_b, \dots, \alpha_f) &\mapsto (-r_1 \alpha_d, r_1 \alpha_b, -r_1 \alpha_c, -r_1 \alpha_a, r_1 \alpha_e, -r_1 \alpha_f) \in C'_1, \\
 C''_1 \ni (\alpha_{ab}, \alpha_{bc}, \alpha_{ca}, \alpha_{de}, \alpha_{ef}, \alpha_{fd}) &\mapsto \\
 &\quad (r_1 \alpha_{ab} \theta_B, r_1 \alpha_{bc} \theta_B, r_1 \alpha_{fd} \theta_A \theta_C, r_1 \alpha_{de} \theta_E, r_1 \alpha_{ef} \theta_E, r_1 \alpha_{ca} \theta_D \theta_F) \in C''_1, \\
 C_2 \ni (\alpha_A, \alpha_B, \dots, \alpha_F) &\mapsto \\
 &\quad (r_1 \alpha_D \theta_A, r_1 \alpha_B \theta_B, -r_1 \alpha_C, r_1 \alpha_A \theta_D, r_1 \alpha_E \theta_E, -r_1 \alpha_F) \in C_2, \\
 C_3 \ni \alpha &\mapsto -r_1 \alpha \in C_3.
 \end{aligned}$$

Moreover if ρ is a r_1 -invariant irreducible representation of G these actions restrict to the homogeneous components of ρ .

Lemma 3. Suppose Γ is $r_1 r_2 r_3$ -normal. The actions of $r_1 r_2 r_3$ on six term modules C_0'', C_1', C_1'' and C_2 permute the components of pairs $A \leftrightarrow D, B \leftrightarrow E$ and $C \leftrightarrow F$. The actions on C_0' and C_3 are given by

$$C_0' \ni \alpha \mapsto r_1 r_2 r_3 \alpha \theta_E^{-1} \theta_F^{-1} \theta_A \in C_0', \quad C_3 \ni \alpha \mapsto -r_1 r_2 r_3 \alpha \in C_3.$$

If ρ is a $r_1 r_2 r_3$ -invariant irreducible representation of G , these actions restrict to the homogeneous components of ρ .

4. General principle. Let $\Gamma \subset B_{4,4}$ be a r -normal subgroup of finite index and (C_*, ∂_*) the chain complex described in section 3. Then the complex is $G \rtimes \langle r \rangle$ -module. The following two lemmas are direct consequences of elementary theory of representation of finite group and Poincare duality. Let $\text{Irr}(G)$ denote the set of irreducible representation of G . For G -module M and $\rho \in \text{Irr}(G)$ we denote by M_ρ the homogenous component of ρ .

Lemma 1. For any $\bar{\rho} \in \text{Irr}(G \rtimes S_0)$ chain complex (C_*, ∂_*) restricts to $G \rtimes S_0$ -subcomplex $(C_{*,\bar{\rho}}, \partial_*|_{C_{*,\bar{\rho}}})$ and $H_*(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_{\bar{\rho}} \simeq H_*(C_{*,\bar{\rho}}, \partial_*|_{C_{*,\bar{\rho}}})$.

Lemma 2. For any $\rho \in \text{Irr}(G)$, $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho \simeq H_2(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ as G -module.

For $\rho \in \text{Irr}(G)$ is r -invariant and M is a $G \rtimes \langle r \rangle$ -module, r stabilizes homogeneous component M_ρ of G -module $\text{Res}_G^{G \rtimes \langle r \rangle} M$. Hence M_ρ carries the action of $G \rtimes \langle r \rangle$ and we denote by $\bar{M}_\rho$ the associated character of $G \rtimes \langle r \rangle$.

Proposition 1. Suppose that Γ is r -normal and $\rho \in \text{Irr}(G)$ is nontrivial and r -invariant. Then ρ is an irreducible component of $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})$ if the generalized character

$$\bar{\mathcal{E}}_\rho := \sum_i (-1)^i \bar{\mathcal{C}}_{i,\rho}$$

of $G \rtimes \langle r \rangle$ is not trivial.

Proof. Since $\Gamma \backslash \mathbb{H}^3$ is a connected closed 3-manifold, the characters of homologies $\mathcal{H}_0(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ and $\mathcal{H}_3(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ are trivial for $\rho \neq 1_G$. Hence the alternated sum $\bar{\mathcal{E}}_\rho$ is equal to the generalized character $\bar{\mathcal{H}}_2(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho - \bar{\mathcal{H}}_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ by Lemma 1. If the action of $G \rtimes \langle r \rangle$ induces the nontrivial character, either of $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ or $H_2(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_\rho$ is at least nontrivial. The proposition follows from Lemma 2. Q.E.D.

5. Proof of Theorem A. In view of Proposition 1 $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})$ has ρ as irreducible component if $\bar{\mathcal{E}}_\rho$ is nonzero. Thus Theorem A reduces to the computation of $\bar{\mathcal{E}}_\rho$. Let $\rho \in \text{Irr}^r(G)$. For $\theta, g, h \in G$ with ${}^{hr}\theta \in \langle \theta \rangle$ we set

$$\varphi_\theta^r(g, h)_\rho : \mathbb{C}[G/\langle \theta \rangle]_\rho \ni \alpha \mapsto g {}^r \alpha h^{-1} \in \mathbb{C}[G/\langle \theta \rangle]_\rho, \quad T_\theta^r(g, h)_\rho := \text{Trace}(\varphi_\theta^r(g, h)_\rho).$$

We omit the upperscript r when it is obvious and the subscript θ if $\theta = 1$ (identity element).

Lemma 1. Let r be either r_1 or $r_1 r_2 r_3$. Suppose Γ is r -normal and $\rho \in \text{Irr}^r(G)$ is nontrivial. Then, for $r = r_1$

$$\begin{aligned} \bar{\mathcal{E}}_\rho(gr_1) = & -T^{r_1}(g, \theta_E^{-1})_\rho - T^{r_1}(g, 1)_\rho + T_{\theta_A}^{r_1}(g, \theta_B^{-1})_\rho \\ & + T_{\theta_D}^{r_1}(g, \theta_E^{-1})_\rho + T_{\theta_C}^{r_1}(g, 1)_\rho + T_{\theta_F}^{r_1}(g, 1)_\rho. \end{aligned}$$

and for $r = r_1 r_2 r_3$,

$$\bar{\mathcal{E}}_\rho(g r_1 r_2 r_3) = T^{r_1 r_2 r_3}(g, \theta_A^{-1} \theta_F \theta_E)_\rho - T^{r_1 r_2 r_3}(g, 1)_\rho.$$

Proof. The Lemma follows immediately from the definition of $\bar{\mathcal{E}}_\chi$ and the direct computation by the formulas in Lemma 2 in section 3. Q.E.D.

For simplicity we consider the case $r = r_1 r_2 r_3$, for which we only need the character formula for $T_\theta^r(*, *)$ with $\theta = 1$. The case $r = r_1$ is treated similarly but requires some more technical formula. First observe that the following is straightforward from the Clifford's theorem.

Lemma 2. *Let G be a finite group. Suppose that $r \in \text{Aut}(G)$ is of order two and $\rho \in \text{Irr}(G)$ is r -invariant. Then there exist exactly two irreducible representations $\bar{\rho}$ and $\sharp_r \bar{\rho}$ of $G \rtimes \langle r \rangle$ which restricts to ρ on G . The character of these satisfy $\chi_{\bar{\rho}}(x) + \chi_{\sharp_r \bar{\rho}}(x) = 0$ for $x \in G \rtimes \langle r \rangle \setminus G$.*

It is immediately verified that the bi-action of $G \times G$ and the action of r on $\mathbb{C}[G]_\rho$ induces the action of the semidirect product $(G \times G) \rtimes \langle r \rangle$ given by the r -action $(g, h) \mapsto ({}^r g, {}^r h)$. We denote by σ the representation on $\mathbb{C}[G]_\rho$. Considering $(G \times G) \rtimes \langle r \rangle$ as a normal subgroup of $(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)$ with index 2, we can define $\tau \in \text{Irr}((G \times G) \rtimes \langle r \rangle)$ with $\text{Res}_{(G \times G) \rtimes \langle r \rangle}^{(G \times G) \rtimes \langle r \rangle} \tau = \rho \times \rho^*$ by

$$\tau := \text{Res}_{(G \times G) \rtimes \langle r \rangle}^{(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)} (\bar{\rho} \times \bar{\rho}^*).$$

Since $\mathbb{C}[G]_\rho$ is equivalent to $\rho \times \rho^* \in \text{Irr}(G \times G)$, either $\sigma = \tau$ or $\sigma = \sharp \tau$ in view of Lemma 2. Thus $T^r(g, h) = \pm \chi_{\bar{\rho}}(gr) \chi_{\bar{\rho}^*}^*(hr)$. Therefor the computation of $T^r(g, h)$ reduces to the determination of the sign.

Lemma 3. $T^r(g, h) = \chi_{\bar{\rho}}(gr) \chi_{\bar{\rho}^*}^*(hr)$.

Proof. By the observations above we only have to prove that $\sigma \neq \sharp \tau$. Recall that $\mathbb{C}[G]_\rho$ is a simple component of $\mathbb{C}$ -algebra $\mathbb{C}[G]$. Since the action of r induces a $\mathbb{C}$ -algebra automorphism of $\mathbb{C}[G]$ together with the conjugation by elements of G , the idempotent associated to r -invariant representation ρ is fixed by these actions. Hence we have

$$(5.1) \quad \langle \text{Res}_H^{(G \times G) \rtimes \langle r \rangle} \sigma, 1_H \rangle_H \neq 0$$

where H is the diagonal subgroup in $(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)$, which is a subgroup of $(G \times G) \rtimes \langle r \rangle$. Since

$$\langle \bar{\rho}, \bar{\rho} \rangle = \frac{1}{2|G|} \left\{ \sum_{x \in G} |\chi_\rho(x)|^2 + \sum_{x \in G} |\chi_{\bar{\rho}}(xr)|^2 \right\} = \frac{1}{2} \left\{ \langle \rho, \rho \rangle + \frac{1}{|G|} \sum_{x \in G} |\chi_{\bar{\rho}}(xr)|^2 \right\},$$

we have

$$(5.2) \quad 1 = \frac{1}{|G|} \sum_{x \in G} |\chi_{\bar{\rho}}(xr)|^2.$$

By definition of $\sharp\tau$ and (5.2),

$$\begin{aligned} \langle \text{Res}_H^{(G \times G) \rtimes \langle r \rangle} \sharp\tau, 1_H \rangle_H &= \frac{1}{|H|} \left\{ \sum_{x \in G} |\chi_\rho(x)|^2 - \sum_{x \in G} |\chi_{\bar{\rho}}(xr)|^2 \right\} \\ &= \frac{1}{2} \left\{ \langle \rho, \rho \rangle_G - \frac{1}{|G|} \sum_{x \in G} |\chi_{\bar{\rho}}(xr)|^2 \right\} = 0. \end{aligned}$$

Hence by (5.1) $\sigma \neq \sharp\tau$.

Q.E.D.

As a consequence of Lemma 1 and Lemma 3, Theorem A follows for the case $r = r_1 r_2 r_3$. The explicit statement is as follows.

Theorem A. *Let Γ be a $r_1 r_2 r_3$ -normal subgroup of finite index in $B_{4,4,4}$ and ρ a nontrivial $r_1 r_2 r_3$ -invariant irreducible representation of G . If*

$$\chi_{\bar{\rho}}(\theta_A^{-1} \theta_F \theta_E r_1 r_2 r_3) + \chi_{\bar{\rho}}(r_1 r_2 r_3) \neq 0$$

for an irreducible representation $\bar{\rho}$ of $G \rtimes \langle r_1 r_2 r_3 \rangle$ which restricts to ρ on G , ρ is an irreducible component of G -module $H_1(\Gamma \backslash \mathbb{H}^3, \mathbb{C})$.

6. Remark on Theorem B and Theorem C. In view of Theorem A, Theorem B is the computation of character of $\bar{\rho}$ in the case that Γ is a congruence subgroup of arithmetic lattice. Since the character of irreducible representation of typical groups of Lie type is wellknown (cf. e.g. [Ca]) the problem reduces to the computation of the character of $\bar{\rho}$ from that of ρ . To clarify the points of the computations we briefly summarize the basic facts on arithmetic lattice and its congruence subgroups.

Let F be a field of characteristic $\neq 2$ and f a non-degenerate quadratic form on F^4 . Set

$$O_f(F) := \{g \in \text{GL}_4(F); g \cdot f = f\}$$

where g acts f by $g \cdot f(x, y) = f(g^{-1}x, g^{-1}y)$. For $\xi \in F^4$ with $f(\xi) \neq 0$ we denote by $r_\xi \in O_f(F)$ the orthogonal reflection with respect to plane $\xi^\perp$. r_ξ is obviously of determinant -1. Hence we have a normal subgroup of index two

$$SO_f(F) := \{g \in O_f(F); \det g = 1\}.$$

Spinorial norm Sp_f is the unique homomorphism of $O_f(F)$ to $F^*/(F^*)^2$ which takes reflection r_ξ to $f(\xi) \bmod (F^*)^2$. Let $\Omega_f(F) = SO_f(F) \cap \text{Ker } Sp_f$.

a. Arithmetic lattices Let k be a number field, $\mathfrak{o}$ the ring of integers in k and f a non-degenerate quadratic form on k^4 . Set

$$O_f(\mathfrak{o}) := \{g \in O_f(k); \text{all entries of } g \text{ are integers}\}.$$

Suppose that v is a real infinite place of k and f induces quadratic form f_v at v of type (p, q) . Then we have an associated embedding λ_v of $O_f(k)$ into $O(p, q; \mathbb{R})$. In particular, if $(p, q) = (3, 1)$, $\text{Ker } Sp_f$ and $\Omega_f(\mathfrak{o})$ embed into $\text{Ker } Sp_{3,1}(\mathbb{R}) = \text{Isom}(\mathbb{H}^3)$ and $\Omega(3, 1; \mathbb{R}) = \text{Isom}_0(\mathbb{H}^3)$ respectively. The following is derived from the classical theorem due to Siegel.

Theorem (Siegel). Suppose $k \neq \mathbb{Q}$ is a totally real number field and f is a non-degenerate anisotropic quadratic form on k^4 . If f is definite at all infinite places except for v_0 and of type $(3,1)$ at v_0 ,

$$\Gamma_{v_0} := \lambda_{v_0}(O_f(\mathfrak{o})) \cap SO(3,1 : \mathbb{R})$$

is a cocompact Kleinian group.

We say that Γ is an *arithmetic lattice* of $O(3,1 : \mathbb{R})$ if Γ is commensurable with Γ_{v_0} in Theorem above. In [HLM1] Hilden, Lozano and Montesinos proved that $B_{4,4,4}$ is arithmetic lattice over $\mathbb{Q}(\sqrt{5})$

b. congruence subgroups For ideal $\mathfrak{m}$ of $\mathfrak{o}$ we define congruence subgroup $O_f(\mathfrak{m})$ by

$$O_f(\mathfrak{m}) := \{g \in O_f(\mathfrak{o}); g \equiv 1 \pmod{\mathfrak{m}}\}.$$

Clearly $O_f(\mathfrak{m})$ is a normal subgroup of $O_f(\mathfrak{o})$. Set

$$\begin{aligned} \Gamma'_{v_0} &:= \lambda_{v_0}(O_f(\mathfrak{o}) \cap \Omega_f(k)), \quad \Gamma_{\mathfrak{m}} := \lambda_{v_0}(O_f(\mathfrak{m})) \cap \Omega_{f_0}(\mathbb{R}), \\ \Gamma'_{\mathfrak{m}} &:= \lambda_{v_0}(O_f(\mathfrak{m}) \cap \Omega_f(k)) := \Gamma'_{v_0} \cap \Gamma_{\mathfrak{m}}. \end{aligned}$$

Note that Γ'_{v_0} and $\Gamma'_{\mathfrak{m}}$ are of finite index in Γ_{v_0} and $\Gamma_{\mathfrak{m}}$, respectively since those groups are finitely generated by its cocompactness and the spinorial norm maps those groups to the abelian group any non-trivial element of which are of order two. Suppose that $\mathfrak{p}$ is prime. Reducing the entries modulo $\mathfrak{p}$ we have injections

$$\iota_{\mathfrak{p}} : \Gamma_{v_0}/\Gamma_{\mathfrak{p}} \longrightarrow SO_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p}) \quad \iota'_{\mathfrak{p}} : \Gamma'_{v_0}/\Gamma'_{\mathfrak{p}} \longrightarrow \Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$$

where $f_{\mathfrak{p}}$ denotes the quadratic form reduced from f modulo $\mathfrak{p}$. By Kneser's strong approximation $\iota'_{\mathfrak{p}}$ is surjective except for finite set $P_{B_{4,4,4}}$ of primes while $\iota_{\mathfrak{p}}$ may fail to be surjective by the lack of simply connectedness of SO_f . The following lemma is easily proved and describes when it fails.

Lemma. Suppose $\mathfrak{p} \notin P_{B_{4,4,4}}$.

- (1) $\iota_{\mathfrak{p}}(B_{4,4,4}) = \Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 1 \pmod{8}$.
- (2) $\iota_{\mathfrak{p}}(B_{4,4,4}) = SO_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 3 \pmod{8}$.

This dichotomy causes the restriction mod 8 in Theorem B (i). If $N_{k/\mathbb{Q}}(\mathfrak{p}) \equiv \pm 3 \pmod{8}$ we can prove that most of r -invariant characters appears in the first homology basing on Theorem A. Hence Theorem B (ii) follows from group theoretic technic and Proposition in Section 1.

We also have technically important dichotomy, which describes the two different types of group structures on $\Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$.

- Lemma.** (i) Let d be a non-square element of $F_{\mathfrak{p}} := \mathfrak{o}/\mathfrak{p}$. Quadratic form $f_{\mathfrak{p}}$ belongs to the (unique) cogridient class of isotropic quadratic forms or that of anisotropic ones according to $-a$ is a square in $F_{\mathfrak{p}}$ or not.
- (ii) If f is an isotropic quadratic form over $\mathbb{F}_q$, $\Omega_f(\mathbb{F}_q)$ is isomorphic to $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)/(\pm 1, \pm 1)$. If anisotropic, it is isomorphic to $SL_2(\mathbb{F}_{q^2})$.

For the isotropic case the computation of the character of $\bar{\rho}$ is relatively easy by the direct product structure. Under the assumption of Theorem C the direct product structure is always the case (by the validity of Schreier's conjecture). On the other hand for the anisotropic case we have to develop a general theory to compute the character of $G \rtimes \langle r \rangle$ from that of G .

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2-1-1, OHTSUKA, BUNKYO-KU, TOKYO 112-0012, JAPAN
 E-mail address: toda@math.ocha.ac.jp