

The generalized least-square estimate of autoregressive coefficients

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The purpose of this note is to derive the asymptotic distribution of the generalized least-square estimate of autoregressive coefficients. Let $\{\varepsilon_t : t = 0, \pm 1, \pm 2, \dots\}$ be a real-valued stationary Gaussian process with mean 0 and suppose that the spectrum of the process is absolutely continuous with respect to the Lebesgue measure and has the spectral density $f_\varepsilon(\omega)$ ($-\pi \leq \omega < \pi$). Let $\{X_t\}$ be a stationary Gaussian process generated by the equation $X_t - \sum_{j=1}^p \alpha_j X_{t-j} = \varepsilon_t$, where the coefficients α_j are such that the zeroes of $z^p - \sum_{j=1}^p \alpha_j z^{p-j}$ are all inside the unit circle. Denote by f_X the spectral density of the process $\{X_t\}$ and denote by $f_{X\varepsilon}$ the cross-spectral density of the bivariate process $\{(X_t, \varepsilon_t)\}$; then it evidently holds that $f_X(\omega) = f_\varepsilon(\omega) / |1 - \sum \alpha_j e^{i\omega j}|^2$ a.e., and $f_{X\varepsilon}(\omega) = (1 - \sum \alpha_j e^{i\omega j}) f_X(\omega)$. For later use, write the covariances $E(X_t X_{t-s})$, $E(X_t \varepsilon_{t-s})$ and $E(\varepsilon_t \varepsilon_{t-s})$ respect

ively as $r_X(s)$, $r_{XE}(s)$ and $r_E(s)$.

If the spectral density f_E is known, an estimate of the α_j can be obtained, based on observations $X_{1-p}, \dots, X_1, \dots, X_N$, as the value which minimizes the weighted square integral given as

$$\int_{-\pi}^{\pi} \left| \sum_{t=1}^N (X_t - \sum_{j=1}^p \alpha_j X_{t-j}) e^{i\omega t} \right|^2 / f_E(\omega) d\omega.$$

Call the estimate $\hat{\alpha}_{j,N}$ thus obtained the generalized least-square estimate of the α_j . Suppose that the inverse of $f_E(\omega)$ is integrable and let $D(k)$ be the k -th Fourier coefficient of $1/f_E(\omega)$; namely, $D(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1/f_E(\omega) d\omega$. Let Γ_N be the p by p matrix whose (k, j) element is $\sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s)$ and let γ_N be the p -vector whose k -th component is $\sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_s D(t-s)$. Denote by α without suffix the p -vector whose k -th component is α_k . Define $\hat{\alpha}_N$ in the same way; that is, the j -th element of $\hat{\alpha}_N$ is $\hat{\alpha}_{j,N}$. Then it holds that $\Gamma_N \hat{\alpha}_N = \gamma_N$ and, if α^0 is the true value of α , $\Gamma_N (\hat{\alpha}_N - \alpha^0) = \zeta_N$ where ζ_N is the p -vector with $\sum \sum X_{t-j} \varepsilon_s D(t-s)$ in the j th element. Let Ω be the p by p matrix whose (k, j) element is $\frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - \sum \alpha_k e^{i\omega k}|^{-2} e^{i\omega(k-j)} d\omega$; then the following theorem establishes that the generalized least-square estimate $\hat{\alpha}_N$ is asymptotically normally distributed with covariance matrix Ω^{-1} ; namely, the asymptotic covariance of the generalized least-square estimate

is the same with that of the least-square estimate applied to the ordinary autoregressive process (that is, $\{\varepsilon_t\}$ is an independent process).

Theorem. Assume that

- i) f_ε and $1/f_\varepsilon$ are square-integrable and $1/f_\varepsilon$ has a bounded derivative with respect to ω ,
- ii) $\sum_{j=0}^{\infty} |r_\varepsilon(j)| < \infty$, and $\sum_{j=0}^{\infty} |D(j)| < \infty$
- iii) The process $\{\varepsilon_t\}$ is uniform mixing; namely, if $B(t \leq p)$ and $B(t \geq q)$ are the Borel fields determined by $\{\varepsilon_t; t \leq p\}$ and $\{\varepsilon_t; t \geq q\}$ respectively, there exists a sequence of positive numbers g_n such that $g_n \rightarrow 0$ as $|n| \rightarrow \infty$ and $|\Pr(E \cap F) - \Pr(E)\Pr(F)| < g_{q-p}$, where $E \in B(t \leq p)$ and $F \in B(t \geq q)$. Then under assumptions i), ii), iii), $\sqrt{N}(\hat{\alpha}_N - \alpha^0)$ is asymptotically normally distributed with zero mean vector and with covariance matrix Ω^{-1} .

Proof.

Define $F_{h,N}(\ell)$ as $F_{h,N}(\ell) = \sum_{m=1}^{N-\ell} X_{m+\ell-h} \varepsilon_m$ for $\ell \geq 0$;

$F_{h,N}(\ell) = \sum_{m=|\ell|}^N X_{m+\ell-h} \varepsilon_m$ for $\ell < 0$. Moreover let

$$\xi_N(h) = \sum_{\ell=-N+1}^{N-1} \{F_{h,N}(\ell) - E(F_{h,N}(\ell))\} D(\ell) / \sqrt{N},$$

$$\xi_{N,L}(h) = \sum_{\ell=-L}^L \{F_{h,N}(\ell) - E(F_{h,N}(\ell))\} D(\ell) / \sqrt{N} \text{ and } \xi_{N,L}^*(h) = \xi_N(h)$$

$-\xi_{N,L}(h)$. Observe first of all for a fixed positive integer

that by assumptions i) and iii) the statistics $\{F_{h,N}(\ell) - E(F_{h,N}(\ell))\} / \sqrt{N}$

($l = 0, \pm 1, \dots, \pm L$; $k = 1, \dots, p$) are asymptotically jointly normally distributed with covariances

$$\begin{aligned} C_{l,m}(k,j) &= \lim_{N \rightarrow \infty} E \{ (F_{k,N}(l) - E(F_{k,N}(l))) (F_{j,N}(m) - E(F_{j,N}(m))) \} / N \\ &= \sum_{u=-\infty}^{\infty} \{ r_X(u) r_E(u+l+m-k-j) + r_{XE}(u+l-k) r_{EX}(u-m+j) \} \end{aligned}$$

where the last expression above is finite by assumption ii [cf.

Hannan (1970), p 209 and 228]. Accordingly, $\xi_{N,L}(k)$, $k = 1, 2, \dots, p$, are asymptotically jointly normally distributed with mean 0 and with covariance matrix whose (k,j) element $C_L(k,j)$ is given as $C_L(k,j) = \sum_{l=-L}^L \sum_{m=-L}^L D(l) D(m) C_{l,m}(k,j)$. Now,

$$\begin{aligned} \lim_{L \rightarrow \infty} C_L(k,j) &= \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} D(l) D(m) \left\{ \sum_{u=-\infty}^{\infty} r_X(u) r_E(u+l-m-k+j) \right. \\ &\quad \left. + r_{XE}(u+m-j) r_{EX}(u-l+k) \right\} \end{aligned}$$

where the right-hand side converges absolutely. Repeated applications of the Parseval equality lead to the equation

$$\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} D(l) D(m) \sum_{u=-\infty}^{\infty} r_X(u) r_E(u+l-m-k+j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_X(\omega)}{f_E(\omega)} e^{i(k-j)\omega} d\omega,$$

whereas, since $\int_{-\pi}^{\pi} e^{i\ell\omega} f_{XE} f_E^{-1} d\omega = 0$ for $\ell < 0$,

$$\begin{aligned} &\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} D(l) D(m) r_{XE}(u+m-j) r_{EX}(u-l+k) \\ &= \sum_{u=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(u+k)\omega} \frac{f_{XE}(\omega)}{f_E(\omega)} d\omega \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(u-j)\omega} \frac{f_{XE}(\omega)}{f_E(\omega)} d\omega = 0. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\lim_{L \rightarrow \infty} C_L(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\omega) f_X^{-1}(\omega) e^{i(k-j)\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - \sum_{l=1}^p \alpha_l e^{i\omega l}|^{-2} e^{i(k-j)\omega} d\omega.\end{aligned}$$

Secondly, an upper bound of the absolute mean of $\xi_{N,L}^*(k)$ is evaluated as follows:

$$E|\xi_{N,L}^*(k)| \leq \sum_{L+1 \leq |l| \leq N-1} |D(l)| [E\{F_{k,N}(l) - E(F_{k,N}(l))\}^2 / N]^{1/2},$$

whereas, for a certain positive number S , it holds uniformly that

$$E\{F_{k,N}(l) - E(F_{k,N}(l))\}^2 / N < S.$$

Therefore $E|\xi_{N,L}^*(k)| \leq 2S \sum_{l=L+1}^{\infty} |D(l)|$ uniformly in N and k .

By use of Chebychev's inequality for the first-order absolute moment, it follows from Assumption ii) that there exists a L_0 such that for $L > L_0$ $\Pr\{|\xi_{L,N}^*| > S\} < \varepsilon$ for all $N (\geq L)$. Then the limit theorem given by T.W. Anderson (1971) says that the asymptotic distribution of $\xi_N = \xi_{L,N} + \xi_{L,N}^*$ is multivariate normal distribution with zero mean vector and with covariance matrix Ω , where

ξ_N , $\xi_{L,N}$ and $\xi_{L,N}^*$ are p -vectors whose k -th elements are $\xi_N(k)$, $\xi_{L,N}(k)$ and $\xi_{L,N}^*(k)$ respectively.

The convergence of $\frac{1}{N}\Gamma_N$ to Ω is shown as this. Observe that $E \frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s) = \sum_{l=-N+1}^{N-1} (1 - \frac{|l|}{N}) \gamma(l+k-j) D(l)$. Then,

$$\lim_{N \rightarrow \infty} E \frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s) = \sum_{l=-\infty}^{\infty} \gamma_X(l+k-j) D(l)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\omega) f_E(\omega)^{-1} e^{i(k-j)\omega} d\omega.$$

On the other hands, it is straightforward to see that

$$\frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N \{X_{t-k} X_{s-j} - E(X_{t-k} X_{s-j})\} D(t-s)$$

converges in mean-square to 0. Thus $\frac{1}{N} \Gamma_N$ converges in probability to Ω .

Finally the convergence of $\frac{1}{\sqrt{N}} E\left\{\sum_{l=-N+1}^{N-1} F_{k,N}(l) D(l)\right\}$ to 0 is demonstrated as follows. By the application of the Grenander - Rosenblatt theorem (1953, pp 543-544) after its slight modification

$$\frac{1}{\sqrt{N}} \left| E\left(\sum F_{k,N}(l) D(l)\right) - \sum_{-\infty}^{\infty} \gamma_{X_E}(k-l) D(l) \right| = O\left(\frac{\log N}{\sqrt{N}}\right).$$

On the other hand,

$$\sum_{-\infty}^{\infty} \gamma_{X_E}(k-l) D(l) = 0$$

$$\text{since } \sum_{-\infty}^{\infty} \gamma_{X_E}(l-k) D(l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\omega} \frac{1 - \sum \alpha_j e^{ij\omega}}{|1 - \sum \alpha_j e^{ij\omega}|^2} d\omega = 0.$$

Consequently, $\frac{1}{\sqrt{N}} \sum (F_{k,N}(l) - E(F_{k,N}(l))) D(l)$ is asymptotically distributed in the same way as $\frac{1}{\sqrt{N}} \sum F_{k,N}(l) D(l)$. To summarize, $\sqrt{N}(\hat{\alpha}_N - \alpha_0)$ is asymptotically distributed as $(\frac{1}{N} \Gamma_N)^{-1} \xi_N$, while $\frac{1}{N} \Gamma_N \rightarrow \Omega$ in probability and ξ_N is asymptotically normal with zero mean vector and with covariance Ω . Thus the proof is complete.

References

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