

Note on Proizvolov's example

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Let f be a map (= continuous map) of a topological space X onto a topological space Y . We say that f is a compact-covering map if every compact subset of Y is the image of some compact subset of X under f . In the following cases, every open map is compact-covering :

- (1) (E. Michael [4]) X is a metric space and Y is a T_2 space and, for some metric on X , $f^{-1}(y)$ is complete for each $y \in Y$.
- (2) (A. Arhangel'skii [2]) X is a Čech-complete space and Y is a T_2 space.
- (3) (K. Alster [1]) X is a metric space and Y is a countable T_2 space.
- (4) (K. Nagami [5]) X is a p -space and Y is a T_3 space, and $f^{-1}(y)$ is compact for each $y \in Y$.

On the other hand, V.V.Proizvolov [6] constructed an

example such that there exists an open, at most two-to-one map from a Lindelöf, first countable T_3 space onto a compact metric space, which is not compact-covering.

In this note, we would like to give an adding explanation for his example, using the following lemma :

Lemma. Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ be compact T_2 spaces with $\mathcal{T}_1 \subset \mathcal{T}_2$. Then $\mathcal{T}_1 = \mathcal{T}_2$ holds.

This is an immediate consequence from the fact that the identity map from $(X, \mathcal{T}_2)$ to $(X, \mathcal{T}_1)$ is homeomorphic.

Proizvolov's example. Let P be the set $[0,1] \times [0,1]$ and $P_0 = [0,1] \times \{\frac{1}{2}\}$, and define the topology of P as below :
If $p \in P - P_0$, p has a neighborhood base in the usual sense of Euclidean plane. For any $p = (p_1, \frac{1}{2}) \in P_0$ and for any natural numbers l, m and n , let $U_{lmn}(p)$ be the subset of P which consists of p and of all points satisfying one of the following three conditions : (1) $p_2' \leq \frac{1}{2}$, and $(p_1 + \frac{1}{n} - p_1')^2 + (\frac{1}{2} - p_2')^2 < \frac{1}{n^2}$ or $(p_1 - \frac{1}{n} - p_1')^2 + (\frac{1}{2} - p_2')^2 < \frac{1}{n^2}$; (2) $p_1 - \frac{2}{n} < p_1' < p_1 + \frac{2}{n}$, $\frac{1}{2} \leq p_2'$ and $p_2' - \frac{1}{2} < \frac{1}{m} |p_1' - p_1|$; (3) $\frac{1}{2} \leq p_2' < \frac{1}{2} + \frac{1}{l}$ and $p_2' - \frac{1}{2} > m |p_1' - p_1|$, , and let $\{U_{lmn}(p)\}_{l,m,n=1}^{\infty}$ be the neighborhood base at p . Then it is easily seen that P is a Lindelöf, first countable T_3 space. Let $Y = [0,1] \times [0, \frac{1}{2}]$ be the subspace of

the Euclidean plane and f the map from P onto Y such that it identifies the points which are symmetric with respect to F_0 . Then f is clearly an open, at most two-to-one map. It remains to show that f is not compact-covering. On the contrary, suppose f is compact-covering. Then there exists a compact subset K of P , whose image by f covers Y . Let $\mathcal{T}_1$ be the topology of K as the subspace of the Euclidean plane, and let $\mathcal{T}_2$ be the topology of K as the subspace of P . Then $\mathcal{T}_1 < \mathcal{T}_2$ holds. Hence, by Lemma $\mathcal{T}_1 = \mathcal{T}_2$ holds. On the other hand, since K covers F_0 , by the definition of $\mathcal{T}_2$ it contains no countable base; however, $\mathcal{T}_1$ contains a countable base. This contradiction shows that f is not compact-covering.

Supplement. A space X is called a space of countable (resp. point-countable) type if every compact subset (resp. point) of X is contained in some compact subset of X with a countable neighborhood base (cf. [3]).

As for the research of K. Nagami [5], there was a question whether every open compact map defined on a T_3 space of countable type is compact-covering, and it was informed that V. V. Proizvolov [6] solved it in the negative. However, in his example mentioned above, P could not be of countable type.

Because, P_0 is a compact G_δ -subset of P which has no countable neighborhood base in P . Hence, it seems that such question remains still open.

References

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