

On non-arithmetic discontinuous groups

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In this talk, we will give a survey on arithmetic and non-arithmetic lattices in a semisimple algebraic group. After giving some basic results on the subject, we'll focus our attention to more recent results, mainly due to Mostow and Deligne, on non-arithmetic lattices in the (projective) unitary group $PU(n, 1)$ ($n \geq 2$). (For more details on these topics as well as the closely related rigidities of lattices, see [S 04]).

1. To begin with, we first fix our settings, giving basic definitions and notations. Let X denote a symmetric Riemannian space of non-compact type (with no flat or compact factors) and let $G = I(X)^o$ be the identity connected component of the isometry group of X . Then, as is well known, G is a connected semisimple Lie group of non-compact type, which is of adjoint type, i.e., with the center reduced to the identity 1. This implies that, denoting by $\mathfrak{g}$ the Lie algebra of G , one has $G = (\text{Aut } \mathfrak{g})^o$ (o denoting always the identity connected component). The group G acts transitively on X and for any $x_0 \in X$ the stabilizer $K = G_{x_0}$ is a maximal compact subgroup; thus one has $X \cong G/K$. In this manner, G and X determine one another uniquely (up to isomorphisms).

More generally, let G' denote a connected semisimple linear Lie group, which becomes automatically "real algebraic" in the sense that there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbf{R}$ (uniquely determined up to $\mathbf{R}$ -isomorphisms) such that $G' = \mathcal{G}(\mathbf{R})^o$. As typical examples, one has $G' = SL(n, \mathbf{R})$, $SO(p, q)^o$, etc. Let K' be a maximal compact subgroup of G' , and K'_0 the maximal compact normal subgroup of G' . Then one has

$$G' \supset K' \supset K'_0 \supset (\text{center of } G').$$

Therefore, setting

$$G = G'/K'_0, \quad K = K'/K'_0, \quad X = G/K = G'/K',$$

one obtains a pair (G, X) as described in the beginning; in particular, one has $G = G'$ if K'_0 reduces to the identity group $\{1\}$. We keep these notations throughout the paper.

When $G' = \mathcal{G}(\mathbf{R})^o$, the common dimension r of the maximal $\mathbf{R}$ -split tori in $\mathcal{G}$ is called the $\mathbf{R}$ -rank of G' and written as $r = \mathbf{R}\text{-rank } G'$. It is well known that, if $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ is a Cartan decomposition of $\mathfrak{g}' = \text{Lie } G'$, then

r coincides with the maximal dimension of the (abelian) subalgebras of $\mathfrak{g}'$ contained in $\mathfrak{p}'$. Thus one has $\mathbf{R}\text{-rank } G' = \mathbf{R}\text{-rank } G$.

When the algebraic group $\mathcal{G}$ is defined over $\mathbf{Q}$, G' is said to have a $\mathbf{Q}$ -structure and the $\mathbf{Q}$ -rank of G' (with this $\mathbf{Q}$ -structure) is the common dimension r_0 of the maximal $\mathbf{Q}$ -split tori in $\mathcal{G}$. G' is called $\mathbf{Q}$ -anisotropic when $r_0 = 0$.

2. A subgroup Γ of G' is called a *lattice* in G' if Γ is discrete and the covolume $\text{vol}(\Gamma \backslash G')$ (with respect to the Haar measure of G') is finite. A lattice Γ is called *uniform* if, in particular, the quotient space $\Gamma \backslash G'$ is compact.

Two subgroups Γ and Γ' of G' are said to be *commensurable* if the indices $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite, and one then writes $\Gamma \sim \Gamma'$. As is easily seen, this is an equivalence relation.

A lattice Γ in G is said to be *reducible* if there exists a non-trivial direct decomposition $G = G_1 \times G_2$ such that $\Gamma \sim (\Gamma \cap G_1) \times (\Gamma \cap G_2)$; otherwise, Γ is called *irreducible*. Every lattice in G is commensurable to the direct product of irreducible ones in the direct factors of G .

When $G' = \mathcal{G}(\mathbf{R})^\circ$ is given a $\mathbf{Q}$ -structure, a subgroup Γ of G' commensurable with $\mathcal{G}(\mathbf{Z})$ is called *arithmetic*; the projection of an arithmetic subgroup of G' in $G = G'/K'_0$ is called *arithmetic in a wider sense*. It is clear that arithmetic subgroups (in a wider sense) are discrete.

The following theorem is fundamental.

Theorem 1 (Borel–Harish-Chandra [BHC 62], Mostow–Tamagawa [MT 62])

If Γ is an arithmetic subgroup of G in a wider sense, then Γ is a lattice in G . Moreover, Γ is uniform (i.e., cocompact in G) if and only if G' is $\mathbf{Q}$ -anisotropic (i.e., $\mathbf{Q}\text{-rank } G' = 0$).

Note that, when Γ in G is arithmetic only in a wider sense, the $\mathbf{Q}$ -rank of G' being $= 0$, Γ is uniform. In the early 1960s it was conjectured by Selberg and others that the converse of Theorem 1 would also be true, if the $\mathbf{R}$ -rank of G is high. Actually, we now have

Theorem 2 (Margulis, 1973, [Ma 91]) *Suppose that the $\mathbf{R}$ -rank of G is ≥ 2 . Then any irreducible lattice Γ in G is arithmetic in a wider sense (for a certain choice of G' with a $\mathbf{Q}$ -structure).*

3. Thanks to the above result of Margulis, in order to study the arithmeticity of a lattice Γ , we may restrict ourselves to the case $\mathbf{R}\text{-rank } G = 1$, which

naturally implies that G is $\mathbf{R}$ -simple. According to the classification of $\mathbf{R}$ -simple Lie groups (due to E. Cartan), we have only the following possibilities for (G, X) :

$$G = PU(D; n, 1)^o = U(D; n, 1)^o / (\text{center}), \quad n \geq 2, \quad (n = 2 \text{ for } D = \mathbf{O}),$$

$$X = H_D^n \quad (\text{the hyperbolic } n\text{-space over } D),$$

D denoting a division composition algebra over $\mathbf{R}$, i.e.,

$$D = \mathbf{R}, \mathbf{C}, \mathbf{H} \text{ (Hamilton's quaternions), } \mathbf{O} \text{ (Cayley's octonions),}$$

and $U(D; n, 1)$ denoting the unitary group of the standard D -hermitian form of signature $(n, 1)$. In the case $D = \mathbf{O}$, which is non-associative, the projective unitary group is defined to be the automorphism group of the (split) exceptional Jordan algebra $\text{Her}_3(\mathbf{O}; 2, 1)$; hence G is of type $F_{4,1}$.

For $D = \mathbf{R}$, one has $G = SO(n, 1)^o$ (Lorentz group) and $X = H_{\mathbf{R}}^n$ is the "Lobachevsky space", i.e., the Riemannian n -space of constant curvature $\kappa = -1$, which can be realized by the hyperbolic hypersurface in $\mathbf{R}^{n+1}$ (with the Lorentz metric):

$$\{(x_i) \in \mathbf{R}^{n+1} \mid \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}.$$

In particular, $H_{\mathbf{R}}^2 (= H_{\mathbf{C}}^1)$ can be identified with the upper half-plane in $\mathbf{C}$ and the lattices in $G = SO(2, 1)^o (\cong SL(2, \mathbf{R}) / \{\pm 1\})$ are so-called Fuchsian groups. In this case, it is classical that there are continuous families of non-arithmetic lattices.

For $X = H_{\mathbf{R}}^n$, $n \geq 3$, non-arithmetic lattices, especially reflection groups, have been studied intensively by E. B. Vinberg and his school since 1965 (see e.g., [V 85], [V 90]). More recently, it was shown by Gromov and Piatetski-Shapiro [GPS 88] that for any $n \geq 2$ one can construct infinitely many non-arithmetic (uniform) lattices as the fundamental group of the "hybrid" of two quotient spaces $\Gamma_1 \backslash X$ and $\Gamma_2 \backslash X$ for non-commensurable arithmetic subgroups Γ_1 and Γ_2 of G .

On the other hand, for the case $D = \mathbf{H}$ and $\mathbf{O}$, Corlette [C 92] and Gromov and Schoen [GS 92] have shown that there exist no non-arithmetic lattices in G by a differential geometric method (harmonic maps), extending the idea of Margulis.

4. In the rest of the paper, we concentrate to the case $D = \mathbf{C}$, i.e., the case where $G = PU(n, 1)$ and $X = H_{\mathbf{C}}^n$, studied mainly by G. D. Mostow since the early 1970s.

The complex hyperbolic space $H_{\mathbb{C}}^n$ can be realized by the unit ball in $\mathbb{C}^n$ as follows. The unitary group $U(n, 1)$ acts on $\mathbb{C}^{n+1}$ and hence on the projective space $P^n(\mathbb{C}) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$ in a natural manner. The orbit of $e_{n+1} = (0, \dots, 0, 1) \pmod{\mathbb{C}^\times}$ in $P^n(\mathbb{C})$ is

$$\{z = (z_i) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^n |z_i|^2 - |z_{n+1}|^2 < 0\} / \mathbb{C}^\times,$$

which, in the inhomogeneous coordinates $z'_i = z_i/z_{n+1}$ ($1 \leq i \leq n$), is expressed by the unit ball

$$\{z' = (z'_i) \in \mathbb{C}^n \mid \sum_{i=1}^n |z'_i|^2 < 1\}.$$

The stabilizer of e_{n+1} in $U(n, 1)$ is $U(n) \times U(1)$. Hence $H_{\mathbb{C}}^n = U(n, 1)/U(n) \times U(1)$ is identified with the unit ball in $\mathbb{C}^n$, on which $G = PU(n, 1)$ acts as linear fractional transformations.

We denote by $\langle \rangle$ the standard hermitian inner product of signature $(n, 1)$ on $\mathbb{C}^{n+1}$. For $a \in \mathbb{C}^{n+1}$, $\langle a, a \rangle > 0$ and $\xi \in \mathbb{C}$, $|\xi| = 1$, we define (after Mostow) a "complex reflection" on $\mathbb{C}^{n+1}$ by

$$R'_{a, \xi} : z \mapsto z + (\xi - 1) \frac{\langle a, z \rangle}{\langle a, a \rangle} a \quad (z \in \mathbb{C}^{n+1}).$$

Then, for $\xi, \eta \in \mathbb{C}$, $|\xi| = |\eta| = 1$, one has

$$R'_{a, \xi} \circ R'_{a, \eta} = R'_{a, \xi\eta};$$

in particular, if ξ is a root of unity: $\xi^m = 1$, then one has $(R'_{a, \xi})^m = 1$. We denote the image of $R'_{a, \xi}$ in $G = PU(n, 1)$ by $R_{a, \xi}$.

In [M 80] Mostow studied the groups

$$\Gamma = \langle R_{e_i, \zeta_p} \ (i = 1, 2, 3) \rangle$$

generated by 3 reflections, where $\zeta_p = e^{2\pi i/p}$ with $p = 3$ or 4 or 5 and

$$e_i \in \mathbb{C}^{n+1}, \langle e_i, e_i \rangle = 1, \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = -\alpha\varphi,$$

$$\alpha = (2 \sin \frac{\pi}{p})^{-1}, \quad \varphi = e^{\pi i t/3}$$

with $t \in \mathbb{R}$. Mostow gave a criterion for Γ to be a lattice in G , and found 17 cases, showing that 7 among them are non-arithmetic (i.e., not arithmetic in a wider sense). The non-arithmetic cases are given by

$$[p, t] = [3, 1/12], [3, 1/30], [3, 5/42], [4, 1/12], [4, 3/20],$$

$$[5, 1/5], [5, 11/30].$$

(It has turned out that actually the Γ corresponding to $[5, 11/30]$ is arithmetic.)

5. Mostow then studied, in collaboration with Deligne, the analytic construction of lattices in $PU(n, 1)$. They consider a system of differential equations of Fuchsian type in n variables, studied for $n=2$ by Picard and in general by Lauricella (1893). The solution space of such equations is $\cong \mathbb{C}^{n+1}$, spanned by the period integrals generalizing the classical Euler integral:

$$F_{g,h}(x_1, \dots, x_n) = \int_g^h \prod_{i=1}^n (u - x_i)^{-\mu_i} \cdot u^{-\mu_{n+1}} (u - 1)^{-\mu_{n+2}} du,$$

where

$$\mu = (\mu_1, \dots, \mu_{n+3}) \in \mathbb{C}^{n+3}, \quad \mu_{n+3} = 2 - \sum_{i=1}^{n+2} \mu_i$$

is the parameter, which we will restrict to the so-called "disc $(n+3)$ -tuple" satisfying the condition $0 < \mu_i < 1$ ($1 \leq i \leq n+3$), and

$$g, h \in M = \{x = (x_1, \dots, x_n, 0, 1, \infty) \mid x_i \in \mathbb{C} - \{0, 1\}, x_i \neq x_j \text{ for } i \neq j\}.$$

Let $\hat{M}$ be the universal covering space of M . Then there exists a natural map from $\hat{M}$ to $P^n(\mathbb{C})$, the space of non-zero solutions modulo $\mathbb{C}^\times$, which is equivariant with respect to the actions of the fundamental group on $\hat{M}$ and the projective monodromy group, denoted by Γ_μ , on $P^n(\mathbb{C})$. It is also shown that there exists a hermitian inner product of signature $(n, 1)$ on the solution space such that Γ_μ is in $PU(n, 1)$.

In [DM 86] it was shown that the following condition (INT) is sufficient for Γ_μ to be a lattice in $G = PU(n, 1)$.

(INT) If $\mu_i + \mu_j < 1$ with $i \neq j$, then one has $(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z}$.

Actually, for $n=2$, this condition is equivalent to the one given by Picard in 1885, so that the 27 lattices obtained in this manner are called "Picard lattices". (In counting the lattices Γ_μ we disregard the order of μ_i 's because it is not essential.) There are 9 more μ 's satisfying the condition (INT) for $3 \leq n \leq 5$, the longest one being $\frac{1}{4}(1, 1, 1, 1, 1, 1, 1, 1)$.

In [M 86] Mostow showed that the following weaker condition (Σ INT) is sufficient to yield the same conclusion.

(Σ INT) One can choose a subset S_1 of $\{1, \dots, n+3\}$ such that $\mu_i = \mu_j$ for $i, j \in S_1$ and that, if $\mu_i + \mu_j < 1$ with $i \neq j$, one has $(1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2}\mathbb{Z}$ when $i, j \in S_1$ and $\in \mathbb{Z}$ otherwise.

In particular, taking S_1 with $|S_1| = 3$, one obtains Γ_μ commensurable to a lattice generated by 3 reflections, including all lattices constructed in [M 80].

In [M88] Mostow showed further that the converse of the above result is also true in the following sense. First, all Γ_μ which is discrete is a lattice in $PU(n, 1)$ (Prop. 5.3) and if $n > 3$ the condition (Σ INT) is necessarily satisfied (Th. 4.13). For $n = 2, 3$ there are 10 exceptional lattices Γ_μ with μ not satisfying (Σ INT). The list of all 94 μ 's satisfying the condition (Σ INT) is given in [M88], in which the longest one is $\frac{1}{6}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with $n = 9$.

6. As for the arithmeticity of Γ_μ , the following criterion was first given in [DM 86] under the assumption (INT):

(A) Let d be the least common denominator of the μ_i 's. Then, for all $A \in \mathbb{Z}$, $1 < A < d - 1$, $(A, d) = 1$, one has

$$\sum_{i=1}^{n+3} \langle A\mu_i \rangle = 1 \text{ or } n + 2,$$

where $\langle x \rangle = x - [x]$ for $x \in \mathbb{R}$, $[x]$ being the symbol of Gauss.

It was finally established in [M 88] (Prop. 5.4) that, without any additional assumption, the condition (A) is necessary and sufficient for Γ_μ to be an arithmetic lattice in $PU(n, 1)$.

Summing up the above results, we obtain the following

Theorem 3 (Mostow, 1988) *The projective monodromy group Γ_μ is a lattice in $PU(n, 1)$ if and only if the condition (Σ INT) is satisfied, except for the 10 exceptional lattices Γ_μ with $n = 2, 3$ not satisfying the condition (Σ INT). The group Γ_μ is an arithmetic lattice (in a wider sense) if and only if the condition (A) is satisfied.*

In the list of the μ 's satisfying (Σ INT) in [M 88], those giving non-arithmetic lattices are marked as NA. (However, this list still seems containing some misprints and erroneous markings.) We give below a (corrected) list of non-arithmetic lattices Γ_μ in $PU(n, 1)$, in which the numbering of the μ 's is the one given in [M 88].

List of non-arithmetic lattices Γ_μ in $PU(n, 1)$ $n = 3$ 39P $\frac{1}{12}(3, 3, 3, 3, 5, 7)$ $n = 2$

69P	$\frac{1}{12}(3, 3, 3, 7, 8)$	$[4, 1/12]$	NA1
71P	$\frac{1}{12}(3, 3, 5, 6, 7)$	(not uniform)	NA2
73P	$\frac{1}{12}(4, 4, 4, 5, 7)$	$[6, 1/6]$	NA3
74P	$\frac{1}{12}(4, 4, 5, 5, 6)$		NA1
78P	$\frac{1}{15}(4, 6, 6, 6, 8)$	$[10, 4/15]$	NA4
80	$\frac{1}{18}(2, 7, 7, 7, 13)$	$[9, 11/18]$	NA5
D7	$\frac{1}{18}(4, 5, 5, 11, 11)$		NA5
84	$\frac{1}{18}(7, 7, 7, 7, 8)$		NA5
85P	$\frac{1}{20}(5, 5, 5, 11, 14)$	$[4, 3/20]$	NA6
86	$\frac{1}{20}(6, 6, 6, 9, 13)$	$[5, 1/5]$	NA7
87	$\frac{1}{20}(6, 6, 9, 9, 10)$		NA6
D8	$\frac{1}{21}(4, 8, 10, 10, 10)$		NA9
88	$\frac{1}{24}(4, 4, 4, 17, 19)$	$[3, 1/12]$	NA8
D9	$\frac{1}{24}(5, 10, 11, 11, 11)$		NA8
89P	$\frac{1}{24}(7, 9, 9, 9, 14)$	$[8, 7/24]$	NA8
91	$\frac{1}{30}(5, 5, 5, 22, 23)$	$[3, 1/30]$	NA4
D10	$\frac{1}{30}(7, 13, 13, 13, 14)$		NA4
93	$\frac{1}{42}(7, 7, 7, 29, 34)$	$[3, 5/42]$	NA9
94	$\frac{1}{42}(13, 15, 15, 15, 26)$	$[7, 13/42]$	NA9

Remark 1. "P" indicates a *Picard lattice*, i.e. a lattice satisfying (INT). "D" indicates an exceptional lattice, i.e. a lattice not satisfying (Σ INT). For $n = 2$, there are 54 lattices (41–94) satisfying (Σ INT) (including 27 Picard lattices) and 9 exceptional lattices (D2–D10).

Remark 2. Γ_μ with $\mu = (\mu_1, \dots, \mu_5)$, $S_1 = \{\mu_1, \mu_2, \mu_3\}$, $\mu_4 \leq \mu_5$ is commensurable with a reflection group with $[p, t]$, where $p = 2(1 - 2\mu_1)^{-1}$, $t = \mu_5 - \mu_4$.

7. We say that two subgroups Γ and Γ' of G are *conjugate commensurable* if Γ is commensurable with a conjugate of Γ' . This kind of relations between the Γ_μ 's was studied in [M 88], [DM 93]. Some of their results are listed

below, where we write $\mu \approx \mu'$ if Γ_μ is conjugate commensurable with $\Gamma_{\mu'}$. It turns out that the 19 non-arithmetic lattices Γ_μ for $n = 2$ are divided into 9 conjugate commensurability classes (NA1–NA9).

It is still an open problem to decide whether or not there exist non-arithmetic lattices not conjugate commensurable to any of Γ_μ , especially such lattices for $n \geq 4$. It would also be interesting to study the *arithmetic* properties of the non-arithmetic lattices Γ_μ , e.g., the corresponding automorphic representations.

(A) ([DM 93], §10) For $a, b > 0$, $1/2 < a + b < 1$, one has

$$(a, a, b, b, 2 - 2a - 2b) \approx (1 - b, 1 - a, a + b - \frac{1}{2}, a + b - \frac{1}{2}, 1 - a - b).$$

In particular, for $a = b$,

$$\begin{aligned} (a, a, a, a, 2 - 4a) &\approx (1 - a, 1 - a, 2a - \frac{1}{2}, 2a - \frac{1}{2}, 1 - 2a) \\ &\approx (\frac{3}{2} - 2a, a, a, a, \frac{1}{2} - a). \end{aligned}$$

Example.

$$\begin{aligned} \frac{1}{18}(7, 7, 7, 7, 8) &\approx \frac{1}{18}(11, 11, 5, 5, 4) \approx \frac{1}{18}(13, 7, 7, 7, 2) \\ &\text{(i.e., } 84 \approx D7 \approx 80). \end{aligned}$$

For $a + b = 3/4$,

$$(a, a, b, b, \frac{1}{2}) \approx (1 - b, 1 - a, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).$$

Examples.

$$\begin{aligned} \frac{1}{12}(4, 4, 5, 5, 6) &\approx \frac{1}{12}(7, 8, 3, 3, 3) \quad \text{(i.e., } 74 \approx 69), \\ \frac{1}{20}(6, 6, 9, 9, 10) &\approx \frac{1}{20}(11, 14, 5, 5, 5) \quad \text{(i.e., } 87 \approx 85). \end{aligned}$$

(B) For π, ρ, σ with $1/\pi + 1/\rho + 1/\sigma = 1/2$, set

$$\mu(\pi, \rho, \sigma) = (\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\rho}, \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\sigma}).$$

Then ([M 88], Th. 5.6) for $1/\rho + 1/\sigma = 1/6$, one has

$$\mu(3, \rho, \sigma) \approx \mu(\rho, 3, \sigma) \approx \mu(\sigma, 3, \rho).$$

Examples.

$$\rho = 10, \sigma = 15 : \quad \frac{1}{30}(5, 5, 5, 22, 23) \approx \frac{1}{15}(6, 6, 6, 4, 8) \approx \frac{1}{30}(13, 13, 13, 7, 14)$$

$$(i.e., 91 \approx 78 \approx D10),$$

$$\rho = 8, \sigma = 24 : \quad \frac{1}{24}(4, 4, 4, 17, 19) \approx \frac{1}{24}(9, 9, 9, 7, 14) \approx \frac{1}{24}(11, 11, 11, 5, 10)$$

$$(i.e., 88 \approx 89 \approx D9),$$

$$\rho = 7, \sigma = 42 : \quad \frac{1}{42}(7, 7, 7, 29, 34) \approx \frac{1}{42}(15, 15, 15, 13, 26) \approx \frac{1}{21}(10, 10, 10, 4, 8)$$

$$(i.e., 93 \approx 94 \approx D8).$$

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