

DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

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0. Introduction and Theorem.

In this paper we consider the following integro-differential equation with time delay in a real Hilbert space H :

$$(0.1) \quad \frac{d}{dt}u(t) + Au(t) + A_1u(t-h) + \int_{-h}^0 a(-s)A_2u(t+s)ds = f(t)$$

$$u(0) = x, \quad u(s) = y(s) \quad -h \leq s < 0.$$

Here, A is a positive definite self-adjoint operator and A_1 , A_2 are closed linear operators with domains containing that of A . The notations h and N denote a fixed positive number and a large natural number respectively. Let $a(\cdot)$ is a real valued function belonging to $C^3([0, h])$.

The equations of the type (0.1) were investigated by G.Di Blasio, K.Kunisch and E.Sinestrari [2], S.Nakagiri [4], H.Tanabe [6] and D.G.Park and S.Y.Kim [5]. Particular, G.Di Blasio, K.Kunisch and E.Sinestrari [2] showed the existence and uniqueness of a solution for $f \in L^2(0, T; H)$, $Ay \in L^2(-h, 0; H)$ and $x \in (D(A), H)_{1/2, 2}$ where $(D(A), H)_{1/2, 2}$ is a interpolation space.

Since the equation (0.1) is of parabolic type, we want x to be an arbitrary element of H . Then the integral in (0.1) exists only in the improper sense no

matter what nice functions f and Ay may be. Hence, it would be considered natural to investigate our problem under the following hypothesis:

$$f \in \cap_{\delta>0} L^2(\delta, T; H) \quad \text{and} \quad Ay \in \cap_{\delta>0} L^2(-h + \delta, 0; H),$$

$$f(t) \text{ and } Ay(t - h) \text{ are improperly integrable at } t = 0.$$

For the sake of simplicity we put

$$L_{loc}^2((0, T]; H) = \cap_{\delta>0} L^2(\delta, T; H).$$

We first shall state the definition of a weak solution of (0.1).

DEFINITION. We say that a function u defined on $[-h, T]$ is a weak solution of the equation (0.1) if the following four conditions satisfied: (see Definition 1.1 in [3])

- 1) $u \in L_{loc}^2((nh, (n+1)h]; D(A)) \cap W_{loc}^{1,2}((nh, (n+1)h]H) \cap C([0, Nh]; D(A^{-\alpha}))$
for $n = 0, 1, 2, \dots, N - 1$ and any $\alpha > 0$.
- 2) $\lim_{t \rightarrow 0} A^{-\alpha} u(t) = A^{-\alpha} x$
for any $\alpha > 0$ and $u(s) = y(s)$ for $-h \leq s < 0$.
- 3) $Au(\cdot + nh) \in L_{loc}^2((0, h]; H)$ and $A^{1-\alpha} u(\cdot + nh)$ is improper integrable at $t = 0$.
- 4) The function u satisfies the equation (0.1) for a.e t .

In Theorem 1 in [3] we showed the existence and uniqueness of a weak solution for which $A^{-\alpha} u$ is continuous in $[0, T]$ for an arbitray positive number α but this solution is not alway in $C([0, T]; H)$.

As the notations we put

$$F_{-1} = \{g \in L_{loc}^2((0, h]; H); \text{ there exists } \lim_{\epsilon \searrow 0} \int_{\epsilon} g(s) ds.\},$$

$$F_m = \{g \in F_{m-1}; \lim_{t \searrow 0} \int_{t/2}^t (t-s)^m A_1^m S(t-s)g(s)ds = 0\}$$

where $S(\cdot)$ is an analitic semigroup of the positive defined self-ajoint operator A and $m = 1, 2, \dots, N-1$.

In Proposition 6.9 of [3] we also showed the following resultant.

Let f belong to $F_{-1} \cap L_{loc}^2((0, Nh] : H)$ and m is a nonnegative integer such that $0 \leq m \leq N-1$. Then following two conditions are equivalent.

- 1) A weak solution of (0.1) is continuous on $[0, mh]$, but at $t = mh$ this solution is discontinuous.
- 2) $f - A_1 y(\cdot - h) \in F_{m-1}$, but $f - A_1 y(\cdot - h) \notin F_m$.

In [3] we could not show that F_m is a proper subset in F_{m-1} . The object in this paper is to show that F_m is a proper subset in F_{m-1} (i.e there exists a inhomogeneous function f and a initial data function y such that the solution of (0.1) is continuous on $[0, mh]$, but at $t = mh$ this solution is discontinuous on H .)

Throughout this paper we assume

$$A-1) \quad A = A_1 = A_2,$$

$$A-2) \quad \text{the operator } A \text{ holds eigenvalues } \{\lambda_q\}_{q=1}^{\infty} \text{ such that}$$

$$(0.2) \quad \lambda_q = Cq^\alpha + o(q^\alpha), \quad \lambda_q \leq \lambda_{q+1}$$

where α and C are some positive numnbers. We denote normal eigenfuctions of eigenvalues λ_q by φ_j .

THEOREM *Under the assumptions A-1) and A-2) there exist a inhomogeneous function f and the initial valued function y such that the weak solution of (0.1) is continuous on $[0, mh]$, but at $t = mh$ it is discontinuous.*

1. Properties of eigenvalues.

We denote 10^{-1} by ϵ_0 .

LEMMA 1. *Let ϵ_0 be a small positive number and t_0 be sufficiently small positive number. Then there exists a eigenvalue λ_q such that*

$$(1.1). \quad 1 - \epsilon_0 < t\lambda_q < 1 + \epsilon_0 \quad \text{for any } t : 0 < t < t_0.$$

Proof. We suppose that there exists a small positive number t_0 such that

$$t\lambda_q \leq 1 - \epsilon_0 \quad \text{or} \quad t\lambda_q \geq 1 + \epsilon_0 \quad \text{for any natural number } q.$$

We put $p = \max_q \{q : \lambda_q \leq (1 - \epsilon_0)/t\}$ and $r = \min_q \{q : \lambda_q \geq (1 + \epsilon_0)/t\}$. If t_0 is sufficiently small, p and r are sufficiently large natural number and $p + 1 = r$. From the assumption A-2) and (1.1) we get

$$Cp^\alpha + o(p^\alpha) \leq (1 - \epsilon_0)/t \quad \text{and} \quad C(p + 1)^\alpha + o((p + 1)^\alpha) \geq (1 + \epsilon_0)/t.$$

Then it follows

$$(1 + \epsilon_0)(C(p + 1)^\alpha + o((p + 1)^\alpha))^{-1} \leq t \leq (1 - \epsilon_0)(Cp^\alpha + o(p^\alpha))^{-1}.$$

Since p is sufficiently large natural number we obtain that the above inequalities are contadiction. Thus the proof is complte.

Let θ and N be $1/3 - 4/(3N)$ and 10^3 respectively.

We choose a sequence $\{t_n\}$ such that $t_1 = t_0/2$ and $0 < t_{n+1} < t_n\theta^n/2$ for any $n = 1, 2, 3, 4, \dots$.

where t_0 is of lemma 1

LEMMA 2. *Let j and n be natural number such that $0 < j \leq n$. Thus there exists a natural number $\ell(n, j)$ such that*

$$1 - \epsilon_0 < (\theta^j t_n)\lambda_{\ell(n, j)} < 1 + \epsilon_0,$$

and if $(n_1, j_1) \neq (n_2, j_2)$ then $\lambda_{\ell(n_1, j_1)} \neq \lambda_{\ell(n_2, j_2)}$.

where $\epsilon_0 = 10^{-1}$.

Proof. Since t_0 is sufficiently small positive number, from Lemma 1, we see that there exists λ_{ℓ} . Next we shall show the eigenvalue is unique. Suppose $(n_1, j_1) \neq (n_2, j_2)$ and $n_1 \geq n_2$. Then if $n_1 > n_2$ it follows $t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}$. If $n_1 = n_2$ and $j_1 > j_2$ it also follows $t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}$. From (1.1) and the above inequalities we have

$$\lambda_{\ell(n_2, j_2)} < (1+\epsilon_0)(t_{n_2}\theta^{j_2})^{-1} < (1+\epsilon_0)2^{-1}(t_{n_1}\theta^{j_1})^{-1} < (1+\epsilon_0)(1-\epsilon_0)^{-1}2^{-1}\lambda_{\ell(n_1, j_1)}.$$

Thus it follows $\lambda_{\ell(n_2, j_2)} < \lambda_{\ell(n_1, j_1)}$.

2. Constitution of functions.

We shall constitute our aim's function which satisfies the following conditions:

$$f \in F_{m-1} \cap L_{loc}^2((0, h]; H) \quad \text{but} \quad f \notin F_m.$$

For the sake of simplicity we suppose $h = 1$.

We first take a sequence $\{x_{n,j}\}$ such that

$$x_{n,0} = 2^{-1}t_n \quad \text{and} \quad x_{n,j} = x_{n,j-1} + (1 + 2/N)\theta^{j-1}t_n/3$$

where $n = 1, 2, \dots$ and $j = 1, 2, \dots \leq n$.

REMARK 1. Since $\sum_{j=1}^n (1 + 2/N)\theta^{j-1}/3 \leq 1/2$ it follows $t_n/2 \leq x_{n,j} < t_n$ where $j = 0, 1, 2, \dots, n$.

For the sake of the simplicity we put $\gamma_{n,j} = \theta^j t_n / (3N)$, and $\Gamma_{n,j} = (1 + 1/N)\theta^j t_n / 3$.

Let χ_1 and χ_2 be functions such that

- 1) $\chi_1, \chi_2 \in C^\infty([0, 1])$,
- 2) $\text{Supp } \chi_1 \subset [2^{-1}, 1]$ and $\text{Supp } \chi_2 \subset [0, 2^{-1}]$,
- 3) $\chi_1(\cdot) = 1$ on $[2/3, 1]$ and $\chi_2(\cdot) = 1$ on $[0, 1/3]$.

We denote $\chi_1((t - x_{n,j})/\gamma_{n,j})$ and $\chi_2((t - x_{n,j} - \Gamma_{n,j})/\gamma_{n,j})$ by $\chi_{1,n,j}(t)$ and $\chi_{2,n,j}(t)$ respectively.

Let p be an arbitrary natural number. We define a function $f_{n,j}^p(t) \in C([0, 1]; H)$ by

$$\begin{aligned} & 0 & \text{if } t \in [0, x_{n,j}] \cup [x_{n,j+1}, 1], \\ & \sum_{\alpha=0}^p (t - x_{n,j} - \gamma_{n,j})^\alpha A^{-p} a_\alpha \chi_{1,n,j}(t) & \text{if } t \in [x_{n,j}, x_{n,j} + \gamma_{n,j}], \\ & A^{-p} S(t - x_{n,j} - \gamma_{n,j} + \epsilon_0 \theta^j t_n / 3) \varphi_{\ell(n,j)} & \text{if } t \in [x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j}], \\ & \sum_{\alpha=0}^p (t - x_{n,j} - \Gamma_{n,j})^\alpha A^{-p} b_\alpha \chi_{2,n,j}(t) & \text{if } t \in [x_{n,j} + \Gamma_{n,j}, x_{n,j+1}] \end{aligned}$$

where

$$a_\alpha = (\alpha!)^{-1} (-A)^\alpha S(\epsilon_0 3^{-1} \theta^j t_n) \varphi_{\ell(n,j)} \text{ and } b_\alpha = (\alpha!)^{-1} (-A)^\alpha S((1 + \epsilon_0) 3^{-1} \theta^j t_n) \varphi_{n,j}.$$

REMARK 2. 1) a_α and b_α are α order's coefficients of Taylor expansion of the functions $S(s) \varphi_{n,j}$ at $s = \epsilon_0 \theta^j t_n / 3$ and $s = (1 + \epsilon_0) \theta^j t_n / 3$ respectively.

2) From the constructive method of the function $f_{n,j}^p$ we see

$$(\text{Supp } f_{n_1,j_1}^p) \cap (\text{Supp } f_{n_2,j_2}^p) = \emptyset \quad \text{if } (n_1, j_1) \neq (n_2, j_2).$$

3) $f_{n,j}^p \in C^p([0, 1]; D(A^\infty))$ and it is piecewise sufficiently smooth at $t \in [0, 1]$.

LEMMA 3. Let q and k be nonnegative integers such that $q \leq p$. Then we have

$$|(d/dt)^q A^k f_{n,j}^p(t)|_H \leq \text{Const} \lambda_{n,j}^{q+k-p}.$$

$$(d/dt)(d/dt)^q A^k f_{n,j}^p(t) \in L^2(0, 1; H).$$

Proof. We first shall show the former.

Let $t \in [x_{n,j}, x_{n,j} + \gamma_{n,j}]$. From the definition of $\chi_{1,n,j}$ and Lemma 1 it follows

$$(2.1) \quad |(d/ds)^\beta \chi_{1,n,j}| \leq \text{Const} / \gamma_{n,j}^\beta \leq C \lambda_{\ell(n,j)}^\beta.$$

If $\beta \leq \alpha$ we have

$$(2.2) \quad |(d/dt)^\beta (t - x_{n,j} - \gamma_{n,j})^\alpha| \leq \text{Const} \gamma_{n,j}^{\alpha-\beta} \leq C \lambda_{\ell(n,j)}^{\beta-\alpha}.$$

From the semigroup properties we see

$$(2.3) \quad |A^k S(s) \varphi_{n,j}|_H \leq \text{Const} \lambda_{\ell(n,j)}^k \exp(-s \lambda_{\ell(n,j)})$$

Combining (2.1), (2.2) and (2.3) we get

$$(2.4) \quad |(d/dt)^q A^k f_{n,j}^p|_H \leq \text{Const} \lambda_{\ell(n,j)}^{k-p} \exp(-\gamma_{n,j} \lambda_{\ell(n,j)}) \sum_{\alpha=0}^p \sum_{\beta=0}^{q \wedge \alpha} \lambda_{\ell(n,j)}^{\beta-\alpha} \lambda_{\ell(n,j)}^{q-\beta} \\ \leq \text{Const} \lambda_{\ell(n,j)}^{-p+q+k}.$$

Using the similar method to the above, for $t \in [x_{n,j} + \Gamma_{n,j}, x_{n,j+1}]$, we also get the same estimate as the above.

For $t \in [x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j}]$, from (2.3), we also get the same estimate as (2.4).

Then the former is proved.

Next we shall show the latter.

If $q+1$ is smaller than p , from the above, it is trivial. We suppose $q = p$. If $t \in (x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j})$ it follows

$$|(d/dt)(d/dt)^p A^k f_{n,j}^p(t)|_H \leq \text{Const} \lambda_{\ell(n,j)}^{k+1}.$$

If $t \in (x_{n,j}, x_{n,j} + \gamma_{n,j}) \cup (x_{n,j} + \Gamma_{n,j}, x_{n,j+1})$ it follows

$$(d/dt)(d/dt)^q A^k f_{n,j}^p(t) = 0.$$

Then the latter is proved.

Let b_n be a decreasing sequence such that

$$(2.5) \quad \lim_{n \rightarrow \infty} b_n = 0, \quad \inf_n n^{1/2} b_n \geq \delta_0 > 0.$$

From 2) of Remark 2 we know that there exists $\sum_{n=1}^{\infty} \sum_{j=1}^n f_{n,j}^p(t) b_n$. Thus we denote the above function by $f^p(t)$.

LEMMA 4. The function $f^p(\cdot)$ holds the following properties:

- 1) $f^p \in C^q([0, 1]; D(A^k)) \cap C^p((0, 1]; D(A^\infty))$ where $q + k \leq p$.
- 2) Let δ be any positive small number. This function is piecewise sufficiently smooth on $[\delta, 1]$.
- 3) $(d/dt + A)^k f^p \in C([0, 1]; H)$ and $\lim_{t \rightarrow 0} (d/dt + A)^k f^p(t) = 0$ where $k = 0, 1, \dots, p$.
- 4) $(d/dt)(d/dt + A)^p f^p \in L_{loc}^2((0, 1]; H)$.

Proof. Combining 2), 3) of Remark 2 and lemma 3 and noting (2.5) we get the proof of 1). Since the sum of f^p is finite on $[\delta, 1]$, from 3) of Remark 2, the proof of 2) is complete. From Lemma 3 and (2.5) the proof of 3) is complete. Noting the sum of f^p is finite on $[\delta, 1]$ and Lemma 3 we can prove 4).

LEMMA 5. Let t be any positive number such that $0 < t \leq 1$. Then there exists

$$\lim_{\epsilon \searrow 0} \int_{\epsilon}^t (d/ds)(d/ds + A)^k f^p(s) ds = 0$$

where $k = 0, 1, \dots, p$.

Proof. From 2) and 3) of Lemma 4 it is easy to prove this lemma.

LEMMA 6.

$$\left| A \int_{t_n/2}^{t_n} S(t_n - s) A^p f^p(s) ds \right|_H \geq \delta n^{1/2} b_n$$

where δ is a positive constant independent of n .

Proof. From the definition of f^p we have $f^p = \sum_{j=1}^n f_{n,j}^p b_n$ on $[t_n/2, t_n]$. We put

$$\begin{aligned} & \int_{x_{n,j}}^{x_{n,j+1}} AS(t_n - s) A^p f_{n,j}^p ds = \\ & \left(\int_{x_{n,j}}^{x_{n,j} + \gamma_{n,j}} + \int_{x_{n,j} + \gamma_{n,j}}^{x_{n,j} + \Gamma_{n,j}} + \int_{x_{n,j} + \Gamma_{n,j}}^{x_{n,j+1}} \right) \{ AS(t_n - s) A^p f_{n,j}^p(s) \} ds \end{aligned}$$

$$= I_1 + I_2 + I_3.$$

We first shall estimate I_1 . From the definition of $f_{n,j}^p$ on $[x_{n,j}, x_{n,j} + \gamma_{n,j}]$ and semigroup properties we have

$$\begin{aligned} & |AS(t_n - s)A^p f_{n,j}^p|_H \\ & \leq \sum_{\alpha=0}^p 1/(\alpha!) |s - x_{n,j} - \gamma_{n,j}|^\alpha \lambda_{n,j}^{\alpha+1} \exp(-(t_n - s + \epsilon_0 \theta^j t_n/3)\lambda_{n,j}). \end{aligned}$$

Since

$$s - x_{n,j} \geq \lambda_{n,j} \quad \text{and} \quad \gamma_{n,j} \lambda_{\ell(n,j)} \leq 1/N$$

we see

$$(2.6) \quad |I_1|_H \leq \sum_{\alpha=0}^p \text{Const}(\gamma_{n,j})^{\alpha+1} \lambda_{\ell(n,j)}^{\alpha+1} \leq \text{Const}/N.$$

where Const is a constant independent of n, j and N . Using the similar method to the above we get

$$(2.7) \quad |I_3|_H \leq \text{Const}/N.$$

Let us estimate I_2 . Using the semigroup properties we get

$$AS(t_n - s)A^p f_{n,j}^p = \exp(-(t_n - x_{n,j} + (\epsilon_0 - 1/N)\theta^j t_n/3)\lambda_{n,j}))\lambda_{n,j} \varphi_{n,j}.$$

Since $t_n - x_{n,j} = (1 + 2/N)(1 - \theta)^{-1} \theta^j t_n/3$, from lemma 2 and the above equality we have

$$|I_2|_H \geq (1 - \epsilon_0) \exp(-\delta_1)/3$$

where $\delta_1 = (1 - \epsilon_0)\{1/3(1 + 2/N)(1 - \theta)^{-1} + (\epsilon_0 - 1/N)\}$. Then combining (2.6), (2.7) and the above inequality and noting N is a sufficiently large number there exists a constant δ_0 such that

$$|I_1 + I_2 + I_3|_H^2 \geq (|I_2|_H - |I_1|_H - |I_3|_H)^2 \geq ((1 - \epsilon_0) \exp(-\delta_1) - 2\text{Const}/N)^2 = \delta_0^2.$$

Thus we complete the proof of this lemma.

LEMMA 7. *Let k be a nonnegative integer such that $k \leq p$. Then we get the following equality:*

$$\begin{aligned} & \int_{t/2}^t (t-s)^k A^{k+1} S(t-s) (d/dt + A)^p f^p(s) ds \\ &= - \sum_{q=0}^{k-1} (t/2)^{k-q} A^{k-q} S(t/2) (d/ds + A)^{p-q-1} A^{j+1} f^p(t/2) C_q \\ & \quad + C_k \int_{t/2}^t S(t-s) (d/ds + A)^{p-k} A^{k+1} f^p(s) ds \end{aligned}$$

where $C_q = k!/(k-q)!$.

Proof. Using the integration by parts we get the following recurrence formula for q .

$$\begin{aligned} & \int_{t/2}^t (t-s)^{k-q} A^{k+1} S(t-s) (d/ds + A)^{p-q} f^p(s) ds \\ &= -(t/2)^{k-q} A^{k+1} S(t/2) (d/ds + A)^{p-q-1} f^p(t/2) \\ & \quad + (k-q) \int_{t/2}^t (t-s)^{k-q-1} A^{k+1} S(t-s) (d/ds + A)^{p-q-1} f^p(s) ds. \end{aligned}$$

Solving the above recurrence formula we get the proof of this lemma.

LEMMA 8. *We get the following inequality:*

$$\limsup_{t \searrow 0} \left| \int_{t/2}^t (t-s)^p A^p S(t-s) d/ds (d/ds + A)^p f^p(s) ds \right|_H > 0.$$

Proof. From the definition of f^p it follows, for any nonnegative integer α ,

$$(2.8) \quad ((d/dt)^\alpha f^p)(t_n/2) = 0 \quad \text{and} \quad ((d/dt)^\alpha f^p)(t_n) = 0.$$

Let p be 0. Using the integration by parts and (2.8) we see

$$\left| \int_{t_n/2}^{t_n} S(t_n-s) d/ds f^0(s) ds \right|_H = \left| -A \int_{t_n/2}^{t_n} S(t_n-s) f^0(s) ds \right|_H.$$

From Lemma 6 it follows the right term of the above equation is uniformly positive about n .

Let p be larger than 1. Then from the integration by parts and (2.8) we have

$$\begin{aligned} & \int_{t_n/2}^{t_n} (t_n - s)^p A^p S(t_n - s) d/ds (d/ds + A)^p f^p(s) ds \\ &= p \int_{t_n/2}^{t_n} (t_n - s)^{p-1} A^p S(t_n - s) (d/ds + A)^p f^p(s) ds \\ & - \int_{t_n/2}^{t_n} (t_n - s)^p A^{p+1} S(t_n - s) (d/ds + A)^p f^p(s) ds = I_1 + I_2. \end{aligned}$$

From Lemma 7 and (2.8) we get

$$I_1 = Const \int_{t_n/2}^{t_n} S(t_n - s) (d/ds + A)^p f^p(s) ds.$$

On the other hand from the integration by parts it follows

$$\int_{t_n/2}^{t_n} S(t_n - s) (d/ds + A)^p f^p(s) ds = 0.$$

Then $I_1 = 0$.

Combining Lemma 6 we obtain $|I_2|_H \geq \delta_0$. The proof is complete.

LEMMA 9. *Let k be a nonnegative integer smaller than $p - 1$. Then it follows*

$$\lim_{t \searrow 0} \left| \int_{t/2}^t (t - s)^k A^k S(t - s) d/ds (d/ds + A)^p f^p(s) ds \right|_H = 0.$$

Proof. From the integration by parts we get

$$\begin{aligned} & \int_{t/2}^t (t - s)^k A^k S(t - s) d/ds (d/ds + A)^p f^p(s) ds = -(t/2)^k A^k S(t/2) (d/ds + A)^p f^p(t/2) \\ & + k \int_{t/2}^t (t - s)^{k-1} A^k S(t - s) (d/ds + A)^p f^p(s) ds = I_1 + I_2. \end{aligned}$$

On the other hand we have the operator norm: $|s^k A^k S(s)|_{H \rightarrow H} \geq Const$. Combining 3) of Lemma 4 and the above result we obtain $\lim_{t \searrow 0} I_1 = 0$. From Lemma 7 and 3) of Lemma 4 we get $\lim_{t \searrow 0} I_2 = 0$. Thus the proof is complete.

3. Proof of Theorem.

We take a function f defined on $[0, 1]$ such that

$$f(t) = (d/dt)(d/dt + A)^p f^p(t).$$

From then 4) of Lemma 4, Lemma 5, Lemma 8 and Lemma 9 we get

$$f \in F_{p-1} \quad \text{and} \quad f \notin F_p.$$

Combining Proposition 6.9 in [3] and the above result we obtain the proof of Theorem is complete.

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