

Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity

Kenji Yajima¹ and Guoping Zhang²

Department of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

1 Introduction

In this paper we study the local smoothing property and Strichartz inequality for n -dimensional Schrödinger equations with potentials which grow super-quadratically at infinity:

$$i \frac{\partial u}{\partial t} = -(1/2) \Delta u + V(x)u, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}; \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^n. \quad (1.1)$$

Assumption 1.1. $V(x)$ is real valued and is of C^∞ -class. There exist $m > 2$ and $R > 0$ such that:

- (1) For $|x| \geq R$, $D_1 \langle x \rangle^m \leq V(x) \leq D_2 \langle x \rangle^m$, where $D_1 \leq D_2$ are positive constants.
- (2) For any α , $|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|}$.

Under the assumption, the operator $L : u \mapsto -(1/2) \Delta u + V(x)u$ defined on $C_0^\infty(\mathbb{R}^n)$ is essentially selfadjoint in $L^2(\mathbb{R}^n)$ and the solution in $L^2(\mathbb{R}^n)$ of the initial value problem (1.1) is given by $u(t, \cdot) = U(t)u_0$ via the unitary group $U(t) = e^{-itH}$ generated by the unique selfadjoint extension H of L . We shall show that the solution $u(t, \cdot)$, nonetheless, is much smoother than u_0 and $1/m$ times differentiable at almost all time $t \neq 0$. More precisely, we prove the following theorem. We write $\langle A \rangle = (1 + |A|^2)^{\frac{1}{2}}$ for a self-adjoint operator A and $D = (D_1, \dots, D_n)$, $D_j = -i\partial/\partial x_j$. $\|\cdot\|_p$ is the norm of Lebesgue space $L^p(\mathbb{R}^n)$ and $\|\cdot\| = \|\cdot\|_2$, $1 \leq p \leq \infty$.

¹Partly supported by the Grant-in-Aid for Scientific Research, The Ministry of Education, Science, Sports and Culture, Japan Grant Nr. 11304006

²Partly supported by the TonenGeneral International Scholarship Foundation

Theorem 1.2. *Let V satisfy Assumption 1.1 and $\Psi \in C_0^\infty(\mathbb{R}^n)$. Then, for any $T > 0$, there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^T \|\Psi(x) \langle D \rangle^{\frac{1}{m}} e^{-itH} u_0\|^2 dt \right)^{\frac{1}{2}} \leq C \|u_0\|, \quad u_0 \in L^2(\mathbb{R}^n). \quad (1.2)$$

Theorem 1.2 is an extension of the one dimensional result by [YZ] to multi-dimensional cases and it is sharp in the sense that the exponent $1/m$ in (1.2) cannot in general be replaced by any larger number. This can be seen by taking the potential $V(x) = \langle x_1 \rangle^m + \dots + \langle x_n \rangle^m$ and the initial state $u_0(x) = e_{i_1}(x_1) \cdots e_{i_n}(x_n)$, where $e_j(x)$ is the j -th eigenfunction of the one dimensional Schrödinger operator $-(1/2)(d^2/dx^2) + \langle x \rangle^m$, and by using the well known result on the asymptotic behavior as $j \rightarrow \infty$ of $e_j(x)$ for x in a compact set (see e.g. [YZ]). However, a slightly stronger result $\sup_{x \in \mathbb{R}^1} \left(\int_{-T}^T |\Psi(x) \langle D \rangle^{\frac{1}{m}} e^{-itH} u_0(x)|^2 dt \right)^{\frac{1}{2}} \leq C \|u_0\|$ is known in one dimension (see [YZ]).

On the way to the proof of Theorem 1.2 we prove the following Strichartz type inequality with “derivative loss”.

Theorem 1.3. *Let V satisfy Assumption 1.1. Let $2 \leq p, \theta \leq \infty$ be such that $\frac{2}{\theta} = n \left(\frac{1}{2} - \frac{1}{p} \right)$ and $p \neq \infty$ if $n = 2$. Then, for any $T > 0$ and $\gamma > \frac{1}{\theta} \left(\frac{1}{2} - \frac{1}{m} \right)$ there exists a constant $C > 0$ such that*

$$\left(\int_{-T}^T \|e^{-itH} u_0\|_p^\theta dt \right)^{\frac{1}{\theta}} \leq C \|\langle H \rangle^\gamma u_0\|, \quad u_0 \in L^2(\mathbb{R}^n). \quad (1.3)$$

Note that $\|\langle H \rangle^\gamma u_0\| < \infty$ requires u_0 also to decay at infinity: $\langle x \rangle^{m\gamma} u_0 \in L^2(\mathbb{R}^n)$. In one dimension a related result $\|\langle H \rangle^{\theta(m,p)} e^{-itH} u_0(x)\|_{L^p(\mathbb{R}_x, L^2(-T, T))} \leq C \|u_0\|$ is known for a certain $\theta(m, p)$ which is *positive* for any $2 \leq p \leq \infty$ if $m < 4$ and for $\frac{1}{p} > \frac{m-4}{4(m-1)}$ if $m \geq 4$ (see [YZ]). This suggests that Theorem 1.3 is far from best possible. For Schrödinger equations on compact Riemannian manifolds, Strichartz’ inequality with sharp derivative loss $\gamma = \frac{1}{2\theta}$ has recently been obtained by [Bu]. See also [Bo1], [Bo2] for related results. Applications of Theorem 1.2 and Theorem 1.3 to the initial value problem for nonlinear Schrödinger equations will be discussed elsewhere.

The estimates of the forms (1.2) and (1.3) have been long known for the free Schrödinger equation in the following stronger forms (see e.g. [Sj], [KY] for (1.4) and [St], [GV], [Y1] for (1.5); the “end-point” case of (1.5), however, has been proved by [KT] only recently) and they have been widely applied, in particular, to nonlinear Schrödinger equations ([K3],

[KPV]) or to the convergence problem ([V]). We write H_0 for $-(1/2)\Delta$ with the domain $D(H_0) = H^2(\mathbb{R}^n)$, where $H^\sigma(\mathbb{R}^n)$ is Sobolov space of order σ .

(1) Local smoothing property: For any $T > 0$ and $\Psi \in C_0^\infty(\mathbb{R}^n)$, there exists $C > 0$ such that

$$\left(\int_0^T \|\Psi(x) \langle D \rangle^{\frac{1}{2}} e^{-itH_0} u_0\|_2^2 dt \right)^{\frac{1}{2}} \leq C \|u_0\|, \quad u_0 \in L^2(\mathbb{R}^n), \quad (1.4)$$

where T can be set $T = \infty$ if $n \geq 3$.

(2) Strichartz inequality: Let $2 \leq p, \theta \leq \infty$ be such that $\frac{2}{\theta} = n \left(\frac{1}{2} - \frac{1}{p} \right)$ and $p \neq \infty$ if $n = 2$. Then, there exists $C > 0$ such that

$$\left(\int_0^\infty \|e^{-itH_0} u_0\|_p^\theta dt \right)^{\frac{1}{\theta}} \leq C \|u_0\|_2, \quad u_0 \in L^2(\mathbb{R}^n). \quad (1.5)$$

For generalizations of these inequalities to the case with decaying potentials, see e.g. [CS], [BAD] and [Y1].

Before proceeding further, we present here the outlines of the proofs of (1.4) (for $T < \infty$) and (1.5) which explain their “physical contents” because they will guide our proofs of Theorem 1.2 and Theorem 1.3 and “physically explain” why $1/m$ in (1.2) is sharp. We consider along with the equation (1.1) corresponding Newton’s equations:

$$\begin{aligned} \dot{q}(t) &= p(t), & \dot{p}(t) &= -\nabla_q V(q), \\ q(0) &= y, & p(0) &= k, \end{aligned} \quad (1.6)$$

and denote their solutions by $(q(t, y, k), p(t, y, k))$. If $V = 0$, $q(t, y, k) = y + tk$ and $p(t, y, k) = k$.

For proving (1.4) for $T < \infty$, we use the formula $e^{itH_0} x e^{-itH_0} = x + tD$ and write

$$\begin{aligned} \int_0^T \|\Psi(x) \langle D \rangle^{1/2} e^{-itH_0} u\|_2^2 dt &= \int_0^T (\langle D \rangle^{1/2} e^{itH_0} \Psi^2(x) e^{-itH_0} \langle D \rangle^{1/2} u, u) dt \\ &= \left(\langle D \rangle^{1/2} \cdot \left\{ \int_0^T \Psi^2(x + tD) dt \right\} \cdot \langle D \rangle^{1/2} u, u \right). \end{aligned} \quad (1.7)$$

Here we have $\left| \partial_\xi^\alpha \partial_x^\beta \int_0^T \Psi^2(x + t\xi) dt \right| \leq C_{\alpha\beta} \langle \xi \rangle^{-1}$ for any α, β and $\int_0^T \Psi^2(x + tD) dt$ is a pseudo-differential operator (Φ DO for short) of order -1 . Hence, the right hand side of (1.7) is bounded by $C \|u\|^2$ and (1.4) follows. Notice that the identity $e^{itH_0} \Psi^2(x) e^{-itH_0} = \Psi^2(x + tD)$ is nothing but the so called Egorov formula which “quantizes” the map $y \mapsto y + tk$ and the relation $\int_0^T \Psi^2(x + t\xi) dt \sim |\xi|^{-1}$ is a result of the obvious fact that the free particles $y + tk$ with velocity k stay in a compact set for the time $\leq C|k|^{-1}$. Thus, we may consider that the local smoothing inequality (1.4) is nothing but the “quantization” of this obvious fact.

We now turn to the proof of (1.5). For $1 \leq p \leq \infty$, p' denotes its dual exponent: $1/p + 1/p' = 1$. Because $U_0(t) = e^{-itH_0}$ is unitary and because the integral kernel of $U_0(t)$ is bounded in modulus by a constant times $|t|^{-n/2}$, we have

$$\|U_0(t)u\|_2 = \|u\|_2, \quad \text{and} \quad \|U_0(t)u\|_\infty \leq C|t|^{-n/2}\|u\|_1. \quad (1.8)$$

(1.5) then follows by applying the the following result of Keel and Tao [KT]: Let (X, dx) be a measure space and $\{U(t) : t \in \mathbb{R}\}$ a one parameter family of operators acting on complex-value functions on X . Suppose that $\{U(t)\}$ satisfies

$$\|U(t)f\|_2 \leq C\|f\|_2, \quad \|U(t)U(s)^*f\|_\infty \leq C|t-s|^{-\sigma}\|f\|_1. \quad (1.9)$$

Then, for $2 \leq p, \theta \leq \infty$ such that $2/\theta = \sigma(1/2 - 1/p)$ and $(p, \theta, \sigma) \neq (\infty, 2, 1)$, there exists a constant $C > 0$ such that $\left(\int_{\mathbb{R}} \|U(t)f\|_p^\theta dt\right) \leq C\|f\|_2$ for any $f \in L^2(X)$. Thus, (1.5) is a result of the unitarity and the dissipative property (1.8) of e^{-itH_0} .

If $V(x)$ grows at most quadratically at infinity in the sense

$$|\partial_x^\alpha V(x)| \leq C_\alpha, \quad 2 \leq |\alpha| \leq 2(n+2), \quad (1.10)$$

it is shown (cf. [F]) that the fundamental solution (FDS for short) $E(t, x, y)$ for (1.1), viz. the integral kernel of e^{-itH} , can be written for short $0 < |t| < \delta$ in the form

$$E(t, x, y) = \frac{1}{(2\pi it)^{n/2}} e^{iS(t, x, y)} a(t, x, y), \quad (1.11)$$

where $S(t, x, y)$ is real smooth and $a(t, x, y)$ is smooth and bounded. It follows that $U(t) = \exp(-itH)$ satisfies (1.8) for $|t| < \delta$ and, hence, (1.3) with finite $T > 0$ (note that the time global estimates do not hold in general because eigenfunctions exist for H). Moreover, $e^{itH}\Psi(x)^2e^{-itH}$ is a Φ DO with principal symbol $\Psi(q(t, x, k))^2$ and, if k is large and $y \in \text{supp } \Psi$, $q(t, y, k) \in \text{supp } \Psi$ for the time $|t| \leq C|k|^{-1}$ (see [Y2]). Thus, the local smoothing property (1.2) holds with $m = 2$ as in the case $V = 0$.

When V is superquadratic at infinity, $q(t, y, k)$ as well as $E(t, x, y)$ behave very differently from the case that V grows at most quadratically at infinity. To see this, we consider $V(x) = \langle x \rangle^m$ in one dimension, $m > 0$. Then, classical particles are subject to periodic motion and, when energy $\sim k^2$ is very large, the periods are given by

$$T(k) \sim 2 \int_{-(k^2/2)^{1/m}}^{(k^2/2)^{1/m}} \frac{dx}{\sqrt{(k^2/2) - |x|^m}} = C_m k^{-1+2/m}, \quad (1.12)$$

Note that, as $k \rightarrow \infty$, $T(k) \rightarrow \infty$ if $0 < m < 2$ and $T(k) \rightarrow 0$ if $m > 2$. Thus, if $m > 2$, for given $t > 0, x$ and y , the equation $x = q(t, y, k)$ for k has infinite number of solutions

with arbitrary large $|k|$ and, reflecting this, $E(t, x, y)$ is nowhere C^1 and is not in general bounded at infinity (see [Y4], [MY]). Thus, we cannot expect that (1.4) and (1.5) for the case $m \leq 2$ remain to hold for $m > 2$. Actually, the motivation for this work was to understand how this change of properties of $E(t, x, y)$ reflects on the local smoothing property and Strichartz inequality. We expect, nonetheless, $1/m$ times differentiability improving (1.2) because of the very relation (1.12) and the “physical” argument given for the free Schrödinger equation: If K is a compact set and the velocity of the particle in K is $\sim k$, it stays in K for $\leq C/k$ during one period and its period is $\sim Ck^{-1+2/m}$ for the energy is $\sim k^2$. Hence, it stays in K for $\leq CTk^{-2/m}$ during the time $[0, T]$ and we expect differentiability improving by $1/m$.

The rest of the paper is devoted to the proof of Theorem 1.2 and Theorem 1.3. We display the plan of the paper here outlining the proofs. We observe that we can find the fraction $k^{-2/m}$ mentioned above by looking at the motion of the particle only for one period which is $\sim k^{-1+2/m} \sim \lambda^{-(\frac{1}{2}-\frac{1}{m})}$ if the energy is $\lambda \sim k^2$. Hinted by this, we decompose the solution $u(t) = \sum_{j=0}^{\infty} e^{-itH} u_{0j}$ in such a way that u_{0j} is spectrally localized around $\lambda_j = 2^j$ with respect to H . It actually is easy to see that for proving (1.2) and (1.3), it is sufficient to show respectively

$$\int_0^{\varepsilon h_j} \|\Psi(x) e^{-itH} u_{0j}\|^2 dt \leq C \lambda_j^{-1/2} \|u_{0j}\|^2, \quad (1.13)$$

$$\left(\int_0^{\varepsilon h_j} \|e^{-itH} u_{0j}\|_p^\theta dt \right)^{\frac{1}{\theta}} \leq C \|u_{0j}\| \quad (1.14)$$

for some $\varepsilon > 0$ and $C > 0$ independent of j , where $h_j \equiv \lambda_j^{-(\frac{1}{2}-\frac{1}{m})}$ is virtually the period of the particle with energy λ_j .

In section 2 we prove some preparatory results such as approximation of $\phi(H)$ by a psuedo-differential operator (Φ DO for short). In section 3, we show that $e^{-itH} \phi_j(H)$, where $\phi_j(H)$ is the spectral localization around $H \sim \lambda_j$ is well approximated, at least for $|t| \leq \varepsilon h_j$, by $e^{-itH_j} \phi_j(H)$ generated by $H_j = -(1/2)\Delta + \chi(x/C_1 \lambda_j^{1/m})V(x)$. Here $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function such that $\chi(x) = 1$ when $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$, and C_1 is large enough so that $|x| \geq C_1 \lambda^{1/m}$ implies $V(x) > 5\lambda$ whenever $\lambda > 10^{10}$. The reason behind this is that classical particles of energy λ cannot enter the domain where $V(x) > \lambda$. For proving this and also for obtaining the expression of $e^{itH_j} \Psi^2(x) e^{-itH_j}$ as a Φ DO in section 5, we change the scale of time and convert the equations into the semi-classical form: If $s = t/h$ and $\tilde{H}_j = h_j^2 H_j$, then $e^{-itH_j} = e^{-is\tilde{H}_j/h}$. The point here is that $\tilde{V}_j(x) = h_j^2 \chi(x/C_1 \lambda_j^{1/m})V(x)$ satisfies the estimate $|\partial_x^\alpha \tilde{V}_j(x)| \leq C_\alpha$ for $|\alpha| \geq 2$ with C_α independent of j . It then follows that e^{-itH_j} has the integral kernel $E_j(t, x, y)$ of the form (1.11) for $|t| < \varepsilon h_j$, ε independent of j , and, its phase and amplitude functions

are estimated uniformly with respect to j . In particular, $|E_j(t, x, y)| \leq C|t|^{-n/2}$ with j -independent C and this implies (1.14). We give a more precise argument in section 4. In section 5, we use h - Φ DO calculus and express $e^{it\tilde{H}_j/h_j}\Psi^2(x)e^{-it\tilde{H}_j/h_j}$ as a h - Φ DO and prove (1.13) by following the argument for the free Schrödinger equation given above.

Incidentally the fact that the study of $e^{-itH}\phi_j(H)$ for one period of the bicharacteristics $|t| < \varepsilon h_j$ is sufficient for concluding the sharp local smoothing property is reminiscent of the similar fact for the sharp remainder estimate for the distribution of eigenvalues (see e.g. [Ta]) or the local decay property of the spectral projection operator at high energy ([Y5]) for H . See also [Bu] where similar argument is used for proving Strichartz inequalities for Schrödinger equations on compact manifolds.

2 Preliminaries

We write $S(m, g)$ for Hörmander's symbol class with slowly varying metrics g and g -continuous weight functions $m(x, \xi)$ (cf. [Ho], Chapter 18) and define the Φ DO $p(x, D) = Op(p)$ with symbol $p \in S(m, g)$ (we write $\sigma(P) = p(x, \xi)$ for the symbol of $P = p(x, D)$) by

$$p(x, D)u(x) = Op(p)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\xi} p(x, \xi) u(y) dy d\xi.$$

We use $S(m, g_0)$ and $S(m, g_1)$ where $g_0 = dx \otimes dx + d\xi \otimes d\xi$ and $g_1 = dx \otimes dx / \langle x \rangle^2 + d\xi \otimes d\xi / \langle \xi \rangle^2$. We recall a positive function m is g_1 -continuous if it satisfies $|\partial_x^\alpha \partial_\xi^\beta m(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} m(x, \xi)$ and $p \in S(m, g_1)$ if and only if

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} m(x, \xi).$$

We denote by $p \sharp q$ the symbols of $Op(p)Op(q)$. If $p \in S(m_1, g_1)$, $q \in S(m_2, g_1)$, we have

$$p \sharp q - \sum_{|\alpha| < N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi) \cdot \partial_x^\alpha q(x, \xi) \in S(\langle x \rangle^{-N} \langle \xi \rangle^{-N} m_1 m_2, g_1), \quad N = 1, 2, \dots, \quad (2.1)$$

$$\sigma(p(x, D)^*) - \sum_{|\alpha| < N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha p(x, \xi) \in S(\langle x \rangle^{-N} \langle \xi \rangle^{-N} m_1, g_1), \quad N = 1, 2, \dots \quad (2.2)$$

Similar relations hold for $S(m, g_0)$. The symbol class $S(m, g)$ is Fréchet space with natural seminorms and $p \mapsto p(x, D)$ is continuous from $S(1, g_0)$ or $S(1, g_1)$ to the Banach space of bounded operators in $L^2(\mathbb{R}^n)$.

We begin with the following lemma. We write $a(x, \xi) = (1/2)\xi^2 + V(x)$. We may and do assume in what follows that $V(x) > 1$ without losing the generality.

Lemma 2.1. Let $\delta > \gamma > 0$ and $\phi, \psi \in C_0^\infty([0, \infty))$ be such that

$$\text{supp } \psi(t) \subset [0, \gamma), \quad \phi(t) = 1 \text{ for } t \in [0, \delta].$$

Define $\Phi_\lambda(x, \xi) = \phi(a(x, \xi)/\lambda)$ for $\lambda > 1$. Then for any N , there exists C_N such that

$$\|H^N(1 - \Phi_\lambda(x, D))\psi(H/\lambda)H^N\|_{B(L^2)} \leq C_N \lambda^{-N}, \quad (2.3)$$

where the constant C_N is independent of $\lambda \geq 1$.

Proof. Write $\tilde{\Phi}_\lambda(x, \xi) = 1 - \Phi_\lambda(x, \xi)$. Take an almost analytic extension $\psi(z)$ of $\psi(t)$ such that $\psi(z)$ is supported by a compact subset of $|z| < \gamma$ and set $\psi_\lambda(z) = \psi(z/\lambda)$. We have

$$\tilde{\Phi}_\lambda(x, D)\psi(H/\lambda) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \psi_\lambda}{\partial \bar{z}}(z) \tilde{\Phi}_\lambda(x, D)(H - z)^{-1} dz \wedge d\bar{z}. \quad (2.4)$$

We construct a parametrix of $\tilde{\Phi}_\lambda(x, D)(H - z)^{-1}$ for $|z| < \gamma\lambda$. On the support of $\Phi_\lambda(x, \xi)$ we have

$$\lambda^{-1} |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha\beta} \min(\lambda^{-\min(|\alpha|/2 + |\beta|/m, 1)}, \langle \xi \rangle^{-|\alpha|} \langle x \rangle^{-|\beta|}) \quad (2.5)$$

with constants $C_{\alpha\beta}$ independent of $\lambda \geq 1$, and $\{\Phi_\lambda(x, \xi), \tilde{\Phi}_\lambda(x, \xi) : \lambda \geq 1\}$ is bounded in $S(1, g)$. We write $b(x, \xi, z) = a(x, \xi) - z$ and define $q_0, q_1, \dots$ inductively by

$$q_0 = \tilde{\Phi}_\lambda/b, \quad q_1 = i\partial_\xi q_0 \cdot \partial_x V/b, \quad q_j = \left(\sum_{|\alpha|+k=j, |\alpha| \geq 1} \frac{-i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha q_k \cdot \partial_x^\alpha V \right)/b, \quad j \geq 2. \quad (2.6)$$

It is obvious that q_j are of the forms

$$\sum_{k=1}^{N_j} \frac{a_{jk}(x, \xi)}{(a(x, \xi) - z)^k}$$

and $a_{jk}(x, \xi) = 0$ when $a(x, \xi) \leq \delta\lambda$. When $a(x, \xi) > \delta\lambda$ and $|z| < \gamma\lambda$, we have $|b(x, \xi, z)| \geq (\delta - \gamma)\lambda$ and $|\partial_x^\beta \partial_\xi^\alpha b^{-1}| \leq C_{\alpha\beta} (a + \lambda)^{-1} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|}$ with constants $C_{\alpha\beta}$ independent of $|z| \leq \gamma\lambda$ and $\lambda \geq 1$. Thus, for $j = 0, 1, \dots$,

$$\{(a + \lambda)q_j : |z| \leq \gamma\lambda, \lambda \geq 1\} \subset S(\langle x \rangle^{-j} \langle \xi \rangle^{-j}, g) \text{ is bounded.} \quad (2.7)$$

Denote $Q_j = Op(q_j)$, $j = 0, 1, \dots$. We have

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{\psi}_\lambda}{\partial \bar{z}} Q_j dz \wedge d\bar{z} = 0, \quad j = 0, 1, \dots, \quad (2.8)$$

because integration by parts shows

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \psi_\lambda}{\partial \bar{z}} \frac{a_{jk}(x, \xi)}{(a(x, \xi) - z)^j} dz \wedge d\bar{z} = \frac{1}{2\pi i (j-1)!} \int_{\mathbb{C}} \frac{\partial^j \psi_\lambda}{\partial \bar{z} \partial z^{j-1}} \frac{a_{jk}(x, \xi)}{a(x, \xi) - z} dz \wedge d\bar{z} \quad (2.9)$$

and, as $\psi^{(j-1)}(z)$ is a almost analytic extension of $\psi^{(j-1)}(x)$, (2.9) is equal to

$$\frac{\lambda^{-(j-1)}}{(j-1)!} \psi^{(j-1)} \left(\frac{a(x, \xi)}{\lambda} \right) a_{jk}(x, \xi) = 0.$$

By virtue of the product formula (2.1), we have

$$\begin{aligned} (q_0 + q_1 + \dots) \# b &= q_0 b - i \partial_\xi q_0 \cdot \partial_x V + \sum_{|\alpha|=2} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha q_0 \cdot \partial_x^\alpha V + \sum_{|\alpha|=3} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha q_0 \cdot \partial_x^\alpha V + \dots \\ &+ q_1 b - i \partial_\xi q_1 \cdot \partial_x V + \sum_{|\alpha|=2} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha q_1 \cdot \partial_x^\alpha \partial_x V + \dots \\ &+ q_2 b - i \partial_\xi q_2 \cdot V + \dots \end{aligned}$$

Hence (2.6) and (2.7) imply that, if we set $R_{\lambda,N}(z, x, D) = \tilde{\Phi}_\lambda(x, D) - (Q_0 + Q_1 + \dots + Q_N)(H - z)$, $N = 0, 1, \dots$, then $\{R_{\lambda,N}(z, x, \xi) : |z| \leq \gamma\lambda, \lambda \geq 1\}$ is bounded in $S(\langle x \rangle^{-(N+1)} \langle \xi \rangle^{-(N+1)}, g)$ and

$$\tilde{\Phi}_\lambda(x, D)(H - z)^{-1} = (Q_0 + Q_1 + \dots + Q_N) - R_{\lambda,N}(z, x, D)(H - z)^{-1} \quad (2.10)$$

It follows by the continuity property of Φ DOs that

$$\|H^{2N+1} R_{\lambda,(4N+1)m}(z, x, D) H^{2N+1}\| \leq C_N, \quad |z| \leq \gamma\lambda, \lambda \geq 1$$

and by inserting (2.10) into (2.4) and by using (2.8) that

$$\tilde{\Phi}_\lambda(x, D) \psi(H/\lambda) = \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{\psi}_\lambda}{\partial \bar{z}}(z) R_{\lambda,(4N+1)m}(z, x, D) (H - z)^{-1} dz \wedge d\bar{z} \quad (2.11)$$

for any $N = 1, 2, \dots$. It then follows that

$$\|H^{2N+1} \tilde{\Phi}_\lambda(x, D) \psi(H/\lambda) H^{2N+1}\| \leq C_N \lambda^{-1} \int_{\Omega_\lambda} |\Im z| \|(H - z)^{-1}\| |dz \wedge d\bar{z}| \leq C'_N \lambda,$$

which implies the lemma because

$$\|H^N \tilde{\Phi}_\lambda(x, D) \psi(H/\lambda) H^N\| \leq C_N \lambda^{-N-1} \|H^{2N+1} \tilde{\Phi}_\lambda(x, D) \psi(H/\lambda) H^{2N+1}\|.$$

by virtue of the support property of ψ . ■

Lemma 2.2. Let $\phi \in C_0^\infty([0, \infty))$ and $\Psi \in C_0^\infty(\mathbb{R}^n)$. Define, for $\lambda \geq 1$, $\Phi_\lambda(x, \xi) = \phi(a(x, \xi)/\lambda)$ and $K_\lambda(x, \xi) = \Psi(x)^2 \phi(a(x, \xi)/\lambda)^2$. Then, there exists a constant $C > 0$ such that for any $\lambda \geq 1$

$$\|\Phi_\lambda(x, D) - \Phi_\lambda(x, D)^*\|_{B(L^2)} \leq C \lambda^{-(\frac{1}{2} + \frac{1}{m})}, \quad (2.12)$$

$$\|\Phi_\lambda(x, D) \Psi^2(x) \Phi_\lambda(x, D)^* - K_\lambda(x, D)\|_{B(L^2)} \leq C \lambda^{-\frac{1}{2}}. \quad (2.13)$$

Proof. It follows from (2.2) and (2.5) that $\{\sigma(\Phi_\lambda^*) - \Phi_\lambda : \lambda\}$ is bounded in $S(\lambda^{-(1/2+1/m)}, g)$. This implies (2.12). The proof for (2.13) is similar. ■

We take $\psi_0, \psi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \psi_0(x), \psi(x) \leq 1$, $\text{supp } \psi \subset (2^{-1}, 2)$ and

$$\psi_0(x) + \sum_{j=1}^{\infty} \psi(x/2^j) = 1 \quad \text{for } x \in [0, \infty) \quad (2.14)$$

and set $\psi_j(x) = \psi(x/2^j)$, $j = 1, 2, \dots$. We let $\phi \in C_0^\infty((1/4, 4))$ be such that $\phi(x) = 1$ for $1/2 < x < 2$ and define, slightly abusing notation, $\Phi_j(x, \xi) = \phi(a(x, \xi)/2^j)$ for $j = 0, 1, \dots$. Note that $1/2 \leq \sum_{j=1}^{\infty} \psi_j(x)^2 \leq 1$.

Lemma 2.3. *Let $\Psi \in S(1, g)$. For any $N > 0$ there exists a constant $C_N > 0$ such that*

$$\|\Psi(x, D)u\|^2 \leq 72(\|\Psi(x, D)\phi_0(H)u\|^2 + \sum_{j=1}^{\infty} \|\Psi(x, D)\Phi_j(x, D)\psi_j(H)u\|^2) + C_N\|\langle H \rangle^{-N}u\|. \quad (2.15)$$

Proof. Take another $\tilde{\psi} \in C_0^\infty((1/2, 2))$ such that $\psi(x)\tilde{\psi}(x) = \psi(x)$ and set $\tilde{\psi}_j(t) = \tilde{\psi}(t/2^j)$. By virtue of Lemma 2.1, we have for any N ,

$$\|H^N(1 - \Phi_j(x, D))\tilde{\psi}_j(H)H^N\|_{B(L^2)} \leq C_N 2^{-jN}. \quad (2.16)$$

Write $u_j = \phi_j(H)u$. We have $u = \sum u_j = \sum \tilde{\psi}_j(H)u_j$ and by virtue of (2.16)

$$\begin{aligned} \|\Psi(x, D)u\|^2 &= \left\| \sum_{j=0}^{\infty} \Psi(x, D)\tilde{\psi}_j(H)u_j \right\|^2 \\ &\leq 2 \left\| \sum_{j=0}^{\infty} \Psi(x, D)\Phi_j(x, D)u_j \right\|^2 + C_N \sum_{j=0}^{\infty} 2^{-jN} \|u_j\|^2 \\ &\leq 2 \sum_{j,k=0}^{\infty} (\Phi_k(x, D)^* \Psi(x, D)^* \Psi(x, D) \Phi_j(x, D)u_j, u_k) + C_N \|H^{-N}u\|^2. \end{aligned} \quad (2.17)$$

Since $\{\Phi_j : j = 1, 2, \dots\}$ is bounded in $S(1, g)$ and the supports of Φ_j and Φ_k are disjoint from each other if $|j - k| \geq 5$. Hence, we see that $\{\Phi_k(x, D)^* \Psi(x, D)^* \Psi(x, D) \Phi_j(x, D) : |j - k| \geq 5\}$ is bounded in $S(\langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)$ for every $N = 1, 2, \dots$. It follows that, for any N ,

$$\|\langle H \rangle^N \Phi_k(x, D)^* \Psi(x, D) \Psi(x, D) \Phi_j(x, D) \langle H \rangle^N\|_{B(L^2)} \leq C_N$$

with constant independent of $|j - k| \geq 5$. Thus

$$\begin{aligned} & \left| \sum_{|j-k| \geq 5} (\Phi_k^* \Psi(x, D)^* \Psi(x, D) \Phi_j(x, D)u_j, u_k) \right| \\ & \leq C_N \sum_{j,k=0}^{\infty} 2^{-N(j+k)} \|u_j\| \|u_k\| \leq C_N \|\langle H \rangle^{-N}u\|^2. \end{aligned} \quad (2.18)$$

On the other hand Schwarz inequality implies

$$\begin{aligned}
& \left| \sum_{|j-k| \leq 4} (\Psi(x, D)\Phi_j(x, D)u_j, \Psi(x, D)\Phi_k(x, D)u_k) \right| \\
& \leq 2 \sum_{|j-k| \leq 4} (\|\Psi(x, D)\Phi_j(x, D)u_j\|^2 + \|\Psi(x, D)\Phi_k(x, D)u_k\|^2) \\
& \leq 36 \sum_{j=0}^{\infty} \|\Psi(x, D)\Phi_j(x, D)u\|^2
\end{aligned} \tag{2.19}$$

The lemma follows by combining (2.17), (2.18) and (2.19). ■

3 Approximation of propagator

We let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. We define

$$H_\lambda = -\frac{1}{2}\Delta + V_\lambda(x), \quad V_\lambda(x) = V(x)\chi(x/C_1\lambda^{\frac{1}{m}}),$$

Lemma 3.1. *Let $\psi \in C_0^\infty((0, \infty))$ be as in Lemma 2.1. Then, there exist constants $C_1 > 0$ and $\varepsilon > 0$ such that for any $N, \ell = 0, 1, \dots$*

$$\sup_{|t| \leq \varepsilon h} \|H^\ell(e^{-itH} - e^{-itH_\lambda})\psi(H/\lambda)\| \leq C_{N\ell}\lambda^{-N} \tag{3.1}$$

for a positive constant $C_{N\ell}$ independent of $\lambda \geq 1$.

For proving Lemma 3.1, we set $h = \lambda^{-(\frac{1}{2} - \frac{1}{m})}$ and convert the equation (1.1) into the semi-classical form considering h as a semi-classical parameter. Thus, we define,

$$H^h = h^2 H = \frac{-h^2}{2}\Delta + h^2 V(x), \quad \tilde{H}^h = h^2 H_\lambda = \frac{-h^2}{2}\Delta + h^2 V_\lambda(x) \tag{3.2}$$

and write $V^h(x) = h^2 V_\lambda(x)$. Then, (3.1) is equivalent to

$$\sup_{|t| \leq \varepsilon} \|H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\psi(H/\lambda)\| \leq C_{N\ell}\lambda^{-N}. \tag{3.3}$$

It is important to notice here that

$$|\partial_x^\alpha V^h(x)| \leq C_\alpha, \quad |\alpha| \geq 2, \tag{3.4}$$

where C_α is independent of $\lambda > 1$. The following theorem is due to Fujiwara ([F]). We write $(q^h(t, y, k), p^h(t, y, k))$ for the solutions of Newton's equations

$$\begin{aligned}
\dot{q}(t) &= p(t), & \dot{p}(t) &= -\nabla_q V^h(q), \\
q(0) &= y, & p(0) &= k,
\end{aligned} \tag{3.5}$$

corresponding to the Hamiltonian $\tilde{H}^h$.

Theorem 3.2. *There exists $\varepsilon > 0$ independent of $h > 1$ such that the following statements are satisfied.*

- (1) *For every $x, y \in \mathbb{R}^n$ and $0 < |t| < \varepsilon$, there exists a unique $k = k^h(t, x, y)$ such that $x = q^h(t, y, k)$; $s \mapsto q^h(s) = q^h(s, y, k^h(t, x, y))$ is a unique solution of (3.5) such that $q^h(t) = x$ and $q^h(0) = y$.*
- (2) *Define $S^h(t, x, y)$ for $0 < |t| < \varepsilon$ and $x, y \in \mathbb{R}^n$ by*

$$S^h(t, x, y) = \int_0^t \{(1/2)\dot{q}^h(s)^2 - V^h(q^h(s))\} ds. \quad (3.6)$$

Then $S^h(t, x, y)$ is real C^∞ and satisfies

$$\left| \partial_x^\alpha \partial_y^\beta \left(S^h(t, x, y) - \frac{(x - y)^2}{2t} \right) \right| \leq C_{\alpha\beta} |t|, \quad |\alpha + \beta| \geq 2. \quad (3.7)$$

- (3) *For $0 < |t| < \varepsilon$, the integral kernel $E^h(t, x, y)$ of $e^{-it\tilde{H}^h/h}$ can be written in the form*

$$E^h(t, x, y) = \frac{1}{(2\pi i t h)^{n/2}} e^{iS^h(t, x, y)/h} a^h(t, x, y) \quad (3.8)$$

and $a^h(t, x, y)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta (a^h(t, x, y) - 1)| \leq C_{\alpha\beta} |t h|, \quad |\alpha + \beta| \geq 0. \quad (3.9)$$

- (4) *For $\ell = 0, 1, \dots$, there exists a constant C_ℓ such that*

$$\sum_{|\alpha|+|\beta| \leq \ell} \|x^\alpha \partial_x^\beta e^{-it\tilde{H}^h/h} u\| \leq C_\ell \sum_{|\alpha|+|\beta| \leq \ell} \|x^\alpha \partial_x^\beta u\|. \quad (3.10)$$

- (5) *The constants $C_{\alpha\beta}$ and C_ℓ of (3.7), (3.9) and (3.10) do not depend on $h > 1$.*

Recall that $S^h(t, x, y)$ is a generating function of the flow determined by (3.5):

$$\frac{\partial S^h}{\partial x}(t, q^h(t, y, k), y) = p(t, y, k), \quad \frac{\partial S^h}{\partial y}(t, q^h(t, y, k), y) = -k. \quad (3.11)$$

We need the following lemma.

Lemma 3.3. *Let $\nu = th$ and $\tilde{S}^h(t, x, y) = tS^h(t, x, y)$, where S^h is defined by (3.6). Then, there exist $C_1 > 0$ and $\varepsilon > 0$ such that the following estimates are satisfied for (t, x, z, y, ξ) such that*

$$\Phi_\lambda(z, \xi/\nu) \neq 0, \quad |x| \geq C_1 \lambda^{\frac{1}{m}}, \quad y \in \mathbb{R}^n, \quad |t| \leq \varepsilon: \quad (3.12)$$

- (1) $\left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) + \xi \right| \geq \frac{1}{10}(|x| + C_1 \lambda^{\frac{1}{m}}).$

$$(2) \quad \left| \frac{\partial \tilde{S}^h}{\partial x}(t, x, z) \right| \leq 2 \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) + \xi \right|.$$

$$(3) \quad \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) + \xi \right| + |z - y| \geq 100^{-1}(|x| + |y| + |z| + C_1 \lambda^{\frac{1}{m}}).$$

Proof. Write $k = \xi/t$ for $t \neq 0$. When $\Phi_\lambda(z, \xi/\nu) \neq 0$, we have $|\xi| \leq 6|\nu|\sqrt{\lambda} = 6|t|\lambda^{\frac{1}{m}}$, $|k| = |\xi/t| \leq 6\lambda^{\frac{1}{m}}$ and $|z| \leq C_0\lambda^{\frac{1}{m}}$ for some constant C_0 . Since $|\partial_x V^h(x)| \leq C\lambda^{\frac{1}{m}}$, where $C = D_2(4C_1)^{m-1}$ depends only on C_1 , we have

$$|q^h(t, z, k)| = \left| z + tk - \int_0^t (t-s) \partial_x \tilde{V}_h(q^h(s, z, k)) ds \right| \leq C_0\lambda^{\frac{1}{m}} + 6\varepsilon\lambda^{\frac{1}{m}} + 3\varepsilon^2 C\lambda^{\frac{1}{m}}.$$

We choose $C_1 \geq (2D_2/D_1)^m$ such that $10^3 C_0 < C_1$ and then $0 < \varepsilon < 1$ such that $10^3(6 + 3C)\varepsilon < C_1$. We have

$$|q^h(t, z, k)| \leq 100^{-1} C_1 \lambda^{\frac{1}{m}}. \quad (3.13)$$

Let $\tilde{x} = q^h(t, z, k)$, $k = \xi/t$, so that $(\partial \tilde{S}^h / \partial z)(t, \tilde{x}, z) = -\xi$ (see (3.11)). Then, taking $\varepsilon > 0$ smaller if necessary, we have from (3.7) and (3.13) that

$$\begin{aligned} \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) + \xi \right| &= \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) - \frac{\partial \tilde{S}^h}{\partial z}(t, \tilde{x}, z) \right| \\ &= \left| \int_0^1 \frac{\partial^2 \tilde{S}^h}{\partial x \partial z}(t, \theta x + (1-\theta)\tilde{x}, z) d\theta \cdot (x - \tilde{x}) \right| \geq \frac{1}{2} |x - \tilde{x}| \geq 8^{-1}(|x| + C_1 \lambda^{\frac{1}{m}}) \end{aligned}$$

if $|x| \geq C_1 \lambda^{\frac{1}{m}}$ and (1) follows. By virtue of (3.11) and the conservation law of energy, we have

$$\frac{1}{2} \left(\frac{\partial S^h}{\partial x} \right) (t, x, z)^2 + \tilde{V}_h(x) = \frac{1}{2} \left(\frac{\partial S^h}{\partial z} \right) (t, x, z)^2 + \tilde{V}_h(z).$$

If $|x| \geq C_1 \lambda^{\frac{1}{m}}$ and $|z| \leq C_0 \lambda^{\frac{1}{m}}$, we have $\tilde{V}_h(z) \leq \tilde{V}_h(x)$. Hence,

$$\left| \frac{\partial \tilde{S}^h}{\partial x}(t, x, z) \right| \leq \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) \right| \leq \left| \frac{\partial \tilde{S}^h}{\partial z}(t, x, z) + \xi \right| + |\xi|$$

Since $|\xi| \leq 6|t|\lambda^{\frac{1}{m}} \leq 100^{-1}(|x| + C_1 \lambda^{\frac{1}{m}})$ if $\varepsilon < 10^{-3}$, statement (2) follows from (1). By the choice of C_1 , we have $|z| \leq C_0 \lambda^{\frac{1}{m}} \leq 10^{-3} C_1 \lambda^{\frac{1}{m}}$ and $10^{-1}|x| - |z| \geq 10^{-2}(|x| + |z|)$. It follows from (1) that the left hand side of (3) is bounded from below by

$$\begin{aligned} 10^{-1}(|x| + C_1 \lambda^{\frac{1}{m}}) + |z - y| &\geq 10^{-1}(|x| + C_1 \lambda^{\frac{1}{m}}) + |y| - |z| \\ &\geq 100^{-1}(|x| + |y| + |z| + C_1 \lambda^{\frac{1}{m}}). \end{aligned}$$

Proof of Lemma 3.1. By virtue of Lemma 2.1 and (3.10), it suffices to show

$$\sup_{|t| \leq \varepsilon} \|H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x, D)\| \leq C_{N\ell}\lambda^{-N} \quad (3.14)$$

Duhamel formula yields

$$H^\ell(e^{-itH^h/h} - e^{-it\tilde{H}^h/h})\Phi_\lambda(x, D)u = -ih \int_0^t H^\ell e^{-i(t-s)H} (V - V_\lambda) e^{-is\tilde{H}^h/h} \Phi_\lambda(x, D)u ds$$

and the operator $H^\ell(V - V_\lambda)$ can be written in the form $\sum_{|\alpha| \leq 2\ell} c_\alpha(x) \partial_x^\alpha$ where $c_\alpha(x)$ are supported by $\{x : |x| \geq C_1 \lambda^{1/m}\}$ and are bounded by $C \langle x \rangle^{m(\ell+1)}$. Hence, it suffices for proving the lemma to show that, for any M and $|\alpha| \leq \ell$,

$$\int_0^t \|\chi_{|x| \geq C_1 \lambda^{1/m}} \langle x \rangle^M \partial_x^\alpha e^{-it\tilde{H}^h/h} \Phi_\lambda(x, D)u\| dt \leq C_{M\ell} \lambda^{-N} \quad (3.15)$$

Introduce a new parameter $\nu = th$ and write $tS^h = \tilde{S}^h$. Then,

$$\begin{aligned} & e^{-it\tilde{H}^h/h} \Phi_\lambda(x, D)u(x) \\ &= \frac{1}{(2\pi i\nu)^{n/2} (2\pi\nu)^n} \int e^{i(\tilde{S}^h(t,x,z) + (z-y)\xi)/\nu} a^h(t, x, z) \Phi_\lambda(z, \xi/\nu) u(y) dy d\xi dz. \end{aligned} \quad (3.16)$$

We differentiate the right hand side of (3.16) by ∂_x^α and multiply by $\langle x \rangle^M$. This will produce several terms of the form

$$\frac{\langle x \rangle^M}{(2\pi i\nu)^{n/2} (2\pi\nu)^n} \int e^{iJ(t,x,z,y,\xi)/\nu} \prod_{j=1}^\ell \left(\frac{i}{\nu} \frac{\partial^{\alpha_j} \tilde{S}^h}{\partial x^{\alpha_j}} \right) \frac{\partial^\beta a^h}{\partial x^\beta}(t, x, z) \Phi_\lambda(z, \xi/\nu) u(y) dy d\xi dz, \quad (3.17)$$

where $\alpha_1 + \dots + \alpha_\ell + \beta = \alpha$ and $\alpha_j \neq 0$, and

$$J(t, x, z, y, \xi) = \tilde{S}^h(t, x, z) + (z - y)\xi.$$

When $|x| \geq C_1 \lambda^{\frac{1}{m}}$, $\Phi(z, \xi/\nu) \neq 0$ and $|t| < \varepsilon$, we have by virtue of Lemma 3.3

$$\left| \frac{\partial J}{\partial z} \right| \geq \frac{1}{10}(|x| + C_1 \lambda^{\frac{1}{m}}), \quad \left| \frac{\partial J}{\partial z} \right| + \left| \frac{\partial J}{\partial \xi} \right| \geq 10^{-3}(|x| + |y| + |z| + C_1 \lambda^{\frac{1}{m}}). \quad (3.18)$$

Define

$$L_0 = -i \left(\frac{\partial J}{\partial z} \right)^{-2} \frac{\partial J}{\partial z} \frac{\partial}{\partial z}, \quad L_1 = -i \left\{ \left(\frac{\partial J}{\partial z} \right)^2 + \left(\frac{\partial J}{\partial \xi} \right)^2 \right\}^{-1} \left\{ \frac{\partial J}{\partial z} \frac{\partial}{\partial z} + \frac{\partial J}{\partial \xi} \frac{\partial}{\partial \xi} \right\}.$$

First order differential operators L_0 and L_1 satisfy

$$\nu L_0 e^{iJ/\nu} = \nu L_1 e^{iJ/\nu} = e^{iJ/\nu}.$$

We apply to (3.17) ℓ times integration by parts by using L_0 and then N times integration parts by using L_1 . The factor $\nu^{-\ell}$ in the integrand of (3.17) is cancelled by ν^ℓ produced by L_0^ℓ and we obtain

$$\begin{aligned}
 (3.17) &= \frac{i^n \nu^N \langle x \rangle^M}{(2\pi i \nu)^{3n/2}} \int \{L_0^\ell L_1^N e^{iJ/\nu}\} b^h(t, x, z, \xi) u(y) dy d\xi dz \\
 &= \frac{i^n \nu^N \langle x \rangle^M}{(2\pi i \nu)^{3n/2}} \int e^{iJ/\nu} (L_1^*)^N (L_0^*)^\ell \{b^h(t, x, z, \xi)\} u(y) dy d\xi dz \\
 &= \frac{1}{(2\pi \nu i)^{n/2}} \int e^{i\tilde{S}^h(t, x, z)/\nu} F(x, z, \nu) dz.
 \end{aligned} \tag{3.19}$$

Here L_0^* and L_1^* are the transpose of L_0 and L_1 , respectively:

$$L_0 = i \frac{\partial}{\partial z} \cdot \left(\frac{\partial J}{\partial z} \right)^{-2} \frac{\partial J}{\partial z}, \quad L_1 = i \left\{ \frac{\partial}{\partial z} \cdot \frac{\partial J}{\partial z} + \frac{\partial}{\partial \xi} \cdot \frac{\partial J}{\partial \xi} \right\} \left\{ \left(\frac{\partial J}{\partial z} \right)^2 + \left(\frac{\partial J}{\partial \xi} \right)^2 \right\}^{-1}$$

and $b^h(t, x, z, \xi)$ and $F(t, x, z, \nu)$ are defined by

$$\begin{aligned}
 b^h(t, x, z, \xi) &= \prod_{j=1}^{\ell} \left(i \frac{\partial^{\alpha_j} \tilde{S}^h}{\partial x^{\alpha_j}} \right) \frac{\partial^{\beta} a^h}{\partial x^{\beta}} \Phi_{\lambda}(z, \xi/\nu), \\
 F(t, x, z, \nu) &= \frac{\langle x \rangle^M \nu^N}{(2\pi \nu)^n} \int e^{i(z-y)\xi/\nu} \{ (L_1^*)^N (L_0^*)^\ell b^h(t, x, z, \xi) \} u(y) dy d\xi
 \end{aligned} \tag{3.20}$$

Recall that Φ_{λ} is bounded in $S(1, g)$, hence

$$\nu^{|\beta|} |(\partial_z^{\alpha} \partial_{\xi}^{\beta}) \Phi_{\lambda}(z, \xi/\nu)| \leq C_{\alpha\beta} \langle z \rangle^{-|\alpha|} \langle \xi/\nu \rangle^{-|\beta|/2}, \tag{3.21}$$

(3.7) implies that the second or higher derivatives of J with respect to (x, z, y, ξ) are bounded uniformly with respect to $0 < |t| < \varepsilon$. It then follows by the help of (1) and (2) of Lemma 3.3 that

$$|\partial_x^{\alpha} \partial_z^{\beta} (L_0^*)^{\ell} b^h(t, x, z, \xi)| \leq C_{\alpha\beta}$$

and then, by virtue of (3.18),

$$c^h(t, x, z, \xi) = (L_1^*)^N \{ (L_0^*)^{\ell} b^h(t, x, z, \xi) \}$$

satisfies

$$\nu^N |\partial_x^{\alpha} \partial_z^{\beta} c^h(t, x, z, \xi)| \leq C_{\alpha\beta N} (|x| + |y| + |z| + C_1 \lambda^{\frac{1}{m}})^{-N} \tag{3.22}$$

with constants $C_{\alpha\beta}$ independent of (t, x, z, ξ) and $\lambda \geq 1$. Since $c^h(t, x, z, \xi)$ is supported by $|\xi| \leq C\lambda^{1/2}\nu$, we obtain, by replacing N by $4N$, $N \geq n$, that,

$$\begin{aligned}
 |\partial_x^{\alpha} \partial_z^{\beta} F(t, x, z, \nu)| &\leq \frac{C_{\alpha\beta N} \langle x \rangle^M}{(2\pi \nu)^n} \int_{|\xi| \leq C\lambda^{1/2}\nu} \langle \xi/\nu \rangle^{|\beta|} (|x| + |y| + |z| + C_1 \lambda^{\frac{1}{m}})^{-4N} |u(y)| dy d\xi \\
 &\leq C_N \langle x \rangle^{M-N} \langle z \rangle^{-N} \lambda^{-\frac{N}{m}} \frac{1}{(2\pi \nu)^n} \int_{|\xi| \leq C\lambda^{1/2}\nu} \langle \xi/\nu \rangle^{|\beta|} d\xi \int_{\mathbb{R}^n} \langle y \rangle^{-N} |u(y)| dy \\
 &\leq C_N \langle x \rangle^{M-N} \langle z \rangle^{-N} \lambda^{-\left(\frac{N}{m} - \frac{n+|\beta|}{2}\right)} \|u\|_2.
 \end{aligned}$$

Thus, if we set $G(t, x, z, \nu) = F(t, x, z, \nu)\langle z \rangle^n$, we have for any $N > \max(M, n)$ that

$$|\partial_x^\alpha \partial_z^\beta G(t, x, z, \nu)| \leq C_{\alpha\beta} \lambda^{(n-\frac{N}{m})} \|u\|_2, \quad |\alpha|, |\beta| \leq n \quad (3.24)$$

Hence, applying the L^2 continuity property of oscillatory integral operators to

$$\frac{1}{(2\pi\nu)^{n/2}} \int e^{i\tilde{S}^h(t,x,z)/\nu} F(t, x, z, \nu) dz = \frac{1}{(2\pi\nu)^{n/2}} \int e^{i\tilde{S}^h(t,x,z)/\nu} G(t, x, z, \nu) f(z) dz,$$

$f(z) = \langle z \rangle^{-n}$, we see from (3.24) that

$$\|(3.17)\| \leq C_N \lambda^{(n-\frac{N}{m})} \|u\|_2 \|f\|_2 \leq C'_N \lambda^{(n-\frac{N}{m})} \|u\|$$

This ends the proof of Lemma 3.1. ■

4 Proof of Strichartz inequality

We prove Theorem 1.3 in this section. We use the notation of the previous sections. Thus $\{\psi_j\}$ is the partition of unity of (2.14), $u_{0j} = \psi_j(H)u_0$ so that $u_0 = \sum_{j=0}^{\infty} u_{0j}$ and $\Phi_j(x, \xi) = \phi(a(x, \xi)/2^j)$. When $\lambda_j = 2^j$, we set the semi-classical parameter h_j by

$$h_j = \lambda_j^{-(\frac{1}{2}-\frac{1}{m})} = 2^{-j(\frac{1}{2}-\frac{1}{m})}$$

and denote $H_j = H^{h_j}$ and $\tilde{H}_j = \tilde{H}^{h_j}$, where H^h and $\tilde{H}^h$ are the operators defined by (3.2).

Lemma 4.1. *Let $p \in [2, \infty)$, $\theta \in (2, \infty]$ be such that $0 \leq \frac{2}{\theta} = n \left(\frac{1}{2} - \frac{1}{p} \right) < 1$. Then, there exists a constant $\varepsilon > 0$ and $C > 0$ independent of $j = 0, 1, \dots$ such that*

$$\left(\int_{|t| \leq \varepsilon h_j} \|e^{-itH} u_{0j}\|_p^\theta dt \right)^{1/\theta} \leq C \|u_{0j}\|_2. \quad (4.1)$$

Proof. By the elliptic estimate and the Sobolev embedding theorem, we have $\|u\|_p \leq C_p \|H^n u\|_2$ for any $1 \leq p \leq \infty$ and (4.1) holds for $j = 0$. We let $j \geq 1$. We have by Minskowski inequality

$$\begin{aligned} & \left(\int_{|t| \leq \varepsilon h_j} \|e^{-itH} u_{0j}\|_p^\theta dt \right)^{1/\theta} \\ & \leq \left(\int_{|t| \leq \varepsilon h_j} \|e^{-itH_j} u_{0j}\|_p^\theta dt \right)^{1/\theta} + \left(\int_{|t| \leq \varepsilon h_j} \|(e^{-itH} - e^{-itH_j}) u_{0j}\|_p^\theta dt \right)^{1/\theta} \end{aligned} \quad (4.2)$$

By virtue of Lemma 3.1, We have

$$\begin{aligned} & \sup_{|t| \leq \varepsilon h_j} \|(e^{-itH} - e^{-itH_j})u_{0j}\|_p \\ & \leq C \sup_{|t| \leq \varepsilon h_j} \|H^n(e^{-itH} - e^{-itH_j})\tilde{\psi}_j(H)u_{0j}\|_2 \leq C_N 2^{-jN} \|u_{0j}\|_2. \end{aligned} \quad (4.3)$$

Recall that $e^{-itH_j} = e^{-i(t/h_j)\tilde{H}_j/h_j}$ and $e^{-it\tilde{H}_j/h_j}$ has the integral kernel given by (3.8) with h_j in replace of h . Thus, e^{-itH_j} also has smooth integral kernel $\tilde{E}_j(t, x, y)$ which satisfies

$$|\tilde{E}_j(t, x, y)| \leq C|t|^{-n/2}, \quad |t| \leq \varepsilon h_j$$

with j -independent constant C . Thus, e^{-itH_j} satisfies (1.8) with constant independent of j and the theorem of Keel-Tao mentioned in the introduction implies

$$\left(\int_{|t| \leq \varepsilon h_j} \|e^{-itH_j}u_{0j}\|_p^\theta dt \right)^{1/\theta} \leq C \|u_{0j}\|_2. \quad (4.4)$$

Combining (4.2), (4.3) and (4.4), we obtain for (4.1). ■

Proof of Theorem 1.3. Given $T > 0$, find $L_j \equiv [T/\varepsilon h_j] + 1 \leq C_\varepsilon 2^{j(\frac{1}{2} - \frac{1}{m})}$ number of points

$$0 = t_0 < t_1 < \dots < t_{L_j} = T$$

such that $|t_k - t_{k-1}| < \varepsilon h_j$. Then, Lemma 4.1 implies

$$\begin{aligned} \int_0^T \|e^{-itH}u_{0j}\|_p^\theta dt &= \sum_{k=1}^{L_j} \int_{t_{k-1}}^{t_k} \|e^{-itH}u_{0j}\|_p^\theta dt \\ &= \sum_{k=1}^{L_j} \int_0^{t_k - t_{k-1}} \|e^{-itH}e^{i(t_k - t_{k-1})H}u_{0j}\|_p^\theta dt \\ &\leq \sum_{k=1}^{L_j} C \|u_{0j}\|_2^\theta \leq C_\varepsilon 2^{j(\frac{1}{2} - \frac{1}{m})} \|u_{0j}\|_2^\theta \leq C_\varepsilon \|\langle H \rangle^{\frac{1}{\theta}(\frac{1}{2} - \frac{1}{m})} u_{0j}\|^\theta. \end{aligned}$$

Minkowski's inequality and Schwatz' inequality then imply

$$\left(\int_0^T \|e^{-itH}u_0\|_p^\theta dt \right)^{1/\theta} \leq C \sum_{j=0}^{\infty} \|\langle H \rangle^{\frac{1}{\theta}(\frac{1}{2} - \frac{1}{m})} u_{0j}\| \leq C \|\langle H \rangle^\gamma u_0\|$$

for any $\gamma > \frac{1}{\theta}(\frac{1}{2} - \frac{1}{m})$. This concludes the proof of Theorem 1.3.

5 Proof of local smoothing property

In this section we prove Theorem 1.2. We use the notation of the previous section. In particular, $\lambda_j = 2^j$, $h_j = 2^{-j(\frac{1}{2}-\frac{1}{m})}$ is the corresponding semi-classical parameter and $U_j(t) = e^{-i(t/h_j)\tilde{H}_j/h}$. We fix a function $\Psi \in C_0^\infty(\mathbb{R}^n)$.

Lemma 5.1. *Suppose that there exists a constant C independent of $j = 0, 1, \dots$ and $u_0 \in L^2(\mathbb{R}^n)$ such that*

$$\int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x, D)^* e^{-itH_j} u_{0j}\|^2 dt \leq C\lambda_j^{-1/2} \|u_{0j}\|^2. \quad (5.1)$$

Then Theorem 1.2 follows.

Proof. We have from (5.1) and Lemma 3.1

$$\begin{aligned} & \int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x, D)^* e^{-itH} u_{0j}\|^2 dt \\ & \leq \int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x, D)^* e^{-itH_j} u_{0j}\|^2 dt + \int_0^{\varepsilon h_j} \|\Psi(x)\Phi_j(x, D)^* (e^{-itH} - e^{-itH_j}) \tilde{\Psi}_j(H) u_{0j}\|^2 dt \\ & \leq C\lambda_j^{-1/2} \|u_{0j}\|^2 + C_N \lambda_j^{-N}. \end{aligned}$$

As in the proof of Theorem 1.3, we take $L_j \leq C_\varepsilon \lambda_j^{(\frac{1}{2}-\frac{1}{m})}$ number of points $0 = t_0 < t_1 < \dots < t_{L_j} = T$ such that $|t_k - t_{k-1}| < \varepsilon h_j$. It then follows that

$$\begin{aligned} \int_0^T \|\Psi(x)\Phi_j(x, D)^* e^{-itH} u_{0j}\|^2 dt &= \sum_{k=1}^{L_j} \int_{t_{k-1}}^{t_k} \|\Psi(x)\Phi_j(x, D)^* e^{-itH} u_{0j}\|^2 dt \\ &= \sum_{k=1}^{L_j} \int_0^{t_k - t_{k-1}} \|\Psi(x)\Phi_j(x, D)^* e^{-itH} e^{i(t_k - t_{k-1})H} u_{0j}\|^2 dt \\ &\leq \sum_{k=1}^{L_j} C\lambda_j^{-1/2} \|u_{0j}\|_2 \leq C_\varepsilon \lambda_j^{-1/m} \|u_{0j}\|^2. \end{aligned} \quad (5.2)$$

Summing up (5.2) with respect to $j = 0, 1, \dots$ and applying (2.15), we conclude that

$$\begin{aligned} \int_0^T \|\Psi(x) e^{-itH} u_0\|^2 dt &\leq C \sum_{j=0}^{\infty} \int_0^T \|\Psi(x)\Phi_j(x, D)^* e^{-itH} u_{0j}\|^2 dt + C_{N,T} \|\langle H \rangle^{-N} u_0\|^2 \\ &\leq \sum_{j=0}^{\infty} C_\varepsilon \lambda_j^{-1/m} \|u_{0j}\|^2 + C_{N,T} \|\langle H \rangle^{-N} u_0\|^2 \leq C \|\langle H \rangle^{-1/2m} u_0\|^2, \end{aligned}$$

which implies Theorem 1.2. ■

We prove (5.1). Define $K_j(x, \xi) = \Psi(x)^2 \Phi_j(x, \xi)^2$. We have by virtue of (2.13) that

$$\|K_j(x, D) - \Phi_j(x, D)\Psi(x)^2\Phi_j(x, D)^*\|_{B(L^2)} \leq C\lambda_j^{-1/2}.$$

Introducing the semiclassical parameter h_j and the operator $\tilde{H}_j$ again, we rewrite (5.1)

$$\begin{aligned} & \int_0^{\epsilon h_j} \|\Psi(x)\Phi_j(x, D)^* e^{-itH_j} u_{0j}\|^2 dt \\ &= h_j \int_0^\epsilon \|\Psi(x)\Phi_j(x, D)^* e^{-it\tilde{H}_j/h_j} u_{0j}\|^2 dt \\ &\leq h_j \int_0^\epsilon (e^{it\tilde{H}_j/h_j} K_j(x, D) e^{-it\tilde{H}_j/h_j} u_{0j}, u_{0j}) dt + Ch_j \lambda_j^{-1/2}. \end{aligned} \quad (5.3)$$

We write $K_j(x, D)$ in the form of h - Φ DO by changing $\xi \rightarrow \xi/h_j$:

$$\begin{aligned} K_j(x, D)u(x) &= \frac{1}{(2\pi)^n} \int e^{i(x-y)\xi} \Psi^2(x) \phi^2\left(\frac{\xi^2/2 + V(x)}{\lambda_j}\right) u(y) dy d\xi \\ &= \frac{1}{(2\pi h_j)^n} \int e^{i(x-y)\xi/h_j} \Psi^2(x) \phi^2\left(\frac{\xi^2/2 + V^{h_j}(x)}{\lambda_j^{\frac{2}{m}}}\right) u(y) dy d\xi \\ &= \tilde{K}_j(x, h_j D)u(x), \end{aligned}$$

where $\tilde{K}_j(x, \xi) = \Psi^2(x) \phi^2((\xi^2/2 + V^{h_j}(x))/\lambda_j^{\frac{2}{m}})$. Notice that we have replaced $h_j^2 V(x)$ by $V^{h_j}(x)$ as they agree on the support of Ψ . It is obvious that $\{\tilde{K}_j(x, \xi) : j = 1, 2, \dots\}$ is a bounded set of $S(1, g_0)$, where $g_0 = dx^2 + d\xi^2$. We compute

$$K_j(t, x, h_j D) = e^{it\tilde{H}_j/h_j} K_j(x, D) e^{-it\tilde{H}_j/h_j}$$

following the standard procedure in h - Φ DO (see e.g. [Ro]). We have

$$\begin{aligned} 0 &= \frac{d}{dt} \{e^{-it\tilde{H}_j/h_j} K_j(t, x, h_j D) e^{it\tilde{H}_j/h_j}\} \\ &= e^{-it\tilde{H}_j/h_j} \left(\frac{\partial K_j}{\partial t}(t, x, h_j D) - \frac{i}{h_j} [\tilde{H}_j, K_j(t, x, h_j D)] \right) e^{it\tilde{H}_j/h_j} \end{aligned}$$

We ansatz that $K_j(t, x, h_j D)$ is an h - Φ DO and that it has an expansion

$$K_j(t, x, h_j D) \sim \sum_{n=0}^{\infty} h_j^n K_{jn}(t, x, h_j D).$$

Denote $\tilde{H}_j(x, \xi) = \xi^2/2 + V^{h_j}(x)$. Then, the symbol of the h - Φ DO in the brackets on the right is given by

$$\frac{\partial K_j}{\partial t}(t, x, \xi) - \frac{\partial \tilde{H}_j}{\partial \xi} \frac{\partial K_j}{\partial x} + \frac{\partial \tilde{H}_j}{\partial x} \frac{\partial K_j}{\partial \xi} + \sum_{|\alpha| \geq 2} h_j^{|\alpha|-1} \frac{(-i)^{|\alpha|+1}}{\alpha!} \left(\frac{\partial^\alpha \tilde{H}_j}{\partial \xi^\alpha} \frac{\partial^\alpha K_j}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{H}_j}{\partial x^\alpha} \frac{\partial^\alpha K_j}{\partial \xi^\alpha} \right)$$

We determine K_{jn} by inserting $K_j(t, x, \xi) = \sum_{n=0}^{\infty} h_j^n K_{jn}(t, x, \xi)$ into the right hand side, collecting the terms with the same order in h and set them $= 0$. The result is

$$\frac{\partial K_{j0}}{\partial t}(t, x, \xi) - \frac{\partial \tilde{H}_j}{\partial \xi} \frac{\partial K_{j0}}{\partial x} + \frac{\partial \tilde{H}_j}{\partial x} \frac{\partial K_{j0}}{\partial \xi} = 0 \quad (5.4)$$

and for $n = 1, 2, \dots$

$$\begin{aligned} & \frac{\partial K_{jn}}{\partial t}(t, x, \xi) - \frac{\partial \tilde{H}_j}{\partial \xi} \frac{\partial K_{jn}}{\partial x} + \frac{\partial \tilde{H}_j}{\partial x} \frac{\partial K_{jn}}{\partial \xi} \\ & + \sum_{k+|\alpha|=n+1, |\alpha| \geq 2} \frac{(-i)^{|\alpha|+1}}{\alpha!} \left(\frac{\partial^\alpha \tilde{H}_j}{\partial \xi^\alpha} \frac{\partial^\alpha K_{jk}}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{H}_j}{\partial x^\alpha} \frac{\partial^\alpha K_{jk}}{\partial \xi^\alpha} \right) = 0 \end{aligned} \quad (5.5)$$

Solve (5.4) and (5.5) inductively with the initial condition

$$K_{j0}(0, x, \xi) = \tilde{K}_j(x, \xi), \quad K_{jn}(0, x, \xi) = 0, \quad n = 1, 2, \dots$$

We denote the solutions of the initial value problem (3.5) with $h = h_j$ by $(q^j(t, y, k), p^j(t, y, k))$. Since the map $(x, \xi) \rightarrow (q^j(t, x, \xi), p^j(t, x, \xi))$ is a global diffeomorphism and the derivatives of $(q^j(t, x, \xi), p^j(t, x, \xi))$ with respect to (x, ξ) are bounded uniformly with respect to $|t| < \varepsilon$ and $j = 1, 2, \dots$, we find that

$$K_{j0}(t, x, \xi) = \tilde{K}_j(q^j(t, x, \xi), p^j(t, x, \xi)) \quad (5.6)$$

solves the equation (5.4) and $\{K_{j0} : j = 0, 1, \dots\}$ is bounded in $S(1, g_0)$. Evidently $K_{j0}(t, x, \xi) = 0$ unless $(q^j(t, x, \xi), p^j(t, x, \xi)) \in \text{supp } \tilde{K}_j$.

The equation (5.5) for $n = 1$ can be written in the form

$$\begin{aligned} & \frac{d}{dt} K_{j1}(t, q^j(-t, x, \xi), p^j(-t, x, \xi)) = R_{j1}(t, q^j(-t, x, \xi), p^j(-t, x, \xi)) \\ & \equiv \sum_{|\alpha|=2} \frac{i}{\alpha!} \left(\frac{\partial^\alpha \tilde{H}_j}{\partial \xi^\alpha} \frac{\partial^\alpha K_{j0}}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{H}_j}{\partial x^\alpha} \frac{\partial^\alpha K_{j0}}{\partial \xi^\alpha} \right) (t, q^j(-t, x, \xi), p^j(-t, x, \xi)) \end{aligned}$$

and may be solved in the form

$$K_{j1}(t, q^j(-t, x, \xi), p^j(-t, x, \xi)) = \int_0^t R_{j1}(s, q^j(-s, x, \xi), p^j(-s, x, \xi)) ds$$

or

$$K_{j1}(t, x, \xi) = \int_0^t R_{j1}(s, q^j(t-s, x, \xi), p^j(t-s, x, \xi)) ds.$$

Again $\{K_{j1}(t, x, \xi) : j = 1, 2, \dots, |t| < \varepsilon\}$ is bounded in $S(1, g_0)$ and $K_{j1}(t, x, \xi) = 0$ unless $(q^j(t, x, \xi), p^j(t, x, \xi)) \in \text{supp } \tilde{K}_j$. The latter can be seen from (5.6) and $K_{j0}(s, q^j(t-s, x, \xi), p^j(t-s, x, \xi)) = K_{j0}(t, x, \xi)$ which follows from the group property of the flow

$(y, k) \mapsto (q^j(t, y, k), p^j(t, y, k))$. We successively solve the equation (5.5) for $n = 2, 3, \dots$ in a similar fashion and find that solutions $K_{j0}, K_{j1}, \dots$ satisfy

$$\{K_{jn}(t, x, \xi) : j = 1, 2, \dots, |t| < \varepsilon\} \text{ is bounded in } S(1, g_0), \quad n = 0, 1, \dots, \quad (5.7)$$

$$K_{jn}(t, x, \xi) = 0 \text{ if } (q^j(t, x, \xi), p^j(t, x, \xi)) \notin \text{supp } \tilde{K}_j. \quad (5.8)$$

We define

$$K_j^N(t, x, \xi) = \sum_{n=0}^N h_j^n K_{jn}(t, x, \xi).$$

Lemma 5.2. *Let $K_j^N(t, x, \xi)$ be defined as above. Then, there exists $\varepsilon > 0$ such that the following estimates are satisfied:*

(1) *For any $N = 1, 2, \dots$, there exists a constant C_N such that for $j = 1, 2, \dots$,*

$$\sup_{|t| \leq \varepsilon} \|e^{it\tilde{H}_j/h_j} K_j(x, D) e^{-it\tilde{H}_j/h_j} - K_j^N(t, x, h_j D)\|_{B(L^2)} \leq C_N h_j^{N+1}. \quad (5.9)$$

(2) *For any $N = 1, 2, \dots$ and α, β , there exists a constant $C_{\alpha\beta N}$ such that for $j = 1, 2, \dots$*

$$\left\| \int_0^\varepsilon K_j^N(t, x, h_j D) dt \right\| \leq C_{\alpha\beta N} \lambda_j^{-\frac{1}{m}}. \quad (5.10)$$

Proof. By construction and the symbol calculus for h - Φ DO ([Ro]), it is standard to see that

$$\frac{\partial K_j^N}{\partial t}(t, x, h_j D) - \frac{i}{h_j} [\tilde{H}_j, K_j^N(t, x, h_j D)] \in OpS(h_j^{N+1}, g_0)$$

uniformly with respect to j and $|t| < \varepsilon$. Hence,

$$\|e^{-it\tilde{H}_j/h_j} K_j^N(t, x, h_j D) e^{it\tilde{H}_j/h_j} - K_j(x, h_j D)\| \leq C_N h_j^{N+1}$$

with j independent constant C_N . The statement (1) follows. For proving (5.10), it suffices to show

$$\left| \int_0^\varepsilon \partial_\xi^\alpha \partial_x^\beta K_j^N(t, x, \xi) dt \right| \leq C_{\alpha\beta N} \lambda_j^{-\frac{1}{m}}. \quad (5.11)$$

By virtue of (5.7) and (5.8), we know that $|\partial_\xi^\alpha \partial_x^\beta K_j^N(t, x, \xi)| \leq C_N$ with C_N independent of j , $|t| < \varepsilon$ and $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and that $K_j^N(t, x, \xi) = 0$ unless $\Psi(q^j(t, x, \xi)) \neq 0$. Thus, for proving (5.11), it clearly suffices to show by replacing $\varepsilon > 0$ by a smaller constant if necessary, that there exists a constant $C > 0$ independent of j such that

$$\tilde{K}_j(q^j(0, x, \xi), p^j(0, x, \xi)) \neq 0, \text{ then } \tilde{K}_j(q^j(t, x, \xi), p^j(t, x, \xi)) = 0 \text{ for } C\lambda_j^{-\frac{1}{m}} < |t| < \varepsilon.$$

This, however, is almost evident. First, we remark that $|\partial_x V^{h_j}(x)| \leq C\langle x \rangle$ with j independent constant $C > 0$. It follows that $1 + |\dot{q}^j(t)| + |\dot{p}^j(t)| \leq C(1 + |q^j(t)| + |p^j(t)|)$ and

$$\sup_{|t| \leq \varepsilon} (1 + |q^j(t)| + |p^j(t)|) \leq (1 + |q^j(0)| + |p^j(0)|) e^{C\varepsilon} \leq C\lambda_j^{\frac{1}{m}}.$$

The last inequality holds because $\tilde{K}_j((q^j(0), p^j(0)) \neq 0$ implies $p^j(0)^2/2 + V^{h_j}(q^j(0)) \sim \lambda_j^{\frac{2}{m}}$ and $q^j(0) \in \text{supp } \Psi$. Thus, $|p^j(0)| \geq C\lambda_j^{\frac{1}{m}}$ and

$$\sup_{|t| \leq \varepsilon} |p(t) - p(0)| \leq \int_0^\varepsilon |\partial_q \tilde{V}_{h_j}(q(s))| ds \leq C\varepsilon \lambda_j^{\frac{1}{m}} \leq 10^{-3} |p(0)|$$

if $\varepsilon > 0$ is sufficiently small. Thus, $p(t)$ changes its direction and the magnitude only by a small fraction and we clearly have $q^j(t) \notin \text{supp } \Psi$ if $|t| \geq 100 \text{diam}(\text{supp } \Psi)/|p(0)|$ when $|t| < \varepsilon$. ■

Completion of the proof of Theorem 1.2. By virtue of (5.9) and (5.10), we have

$$\left| \int_0^\varepsilon (e^{it\tilde{H}_j/h_j} K_j(x, D) e^{-it\tilde{H}_j/h_j} u_{0j}, u_{0j}) dt \right| \leq C_N h_j^N + \left| \left(\int_0^\varepsilon K_j^N(t, x, h_j D) dt \cdot u_{0j}, u_{0j} \right) \right| \leq C\lambda_j^{-1/m}.$$

We apply this to the right of (5.3) and obtain

$$\int_0^{\varepsilon h_j} \|\Psi(x) \Phi_j(x, D)^* e^{-itH_j} u_{0j}\|^2 dt \leq C h_j \lambda_j^{-1/2} = C\lambda_j^{-1/m} \quad (5.12)$$

which implies Lemma 5.1, hence, Theorem 1.2. ■

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