

A personal note on real Whittaker model

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In this note, I explain an “intuitive” understanding of the Jacquet integral for class-1 representation, in terms of the Helgason correspondence.

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The basic reference is $[K^2MO^2T]$.

§1. Helgason correspondence

Let us recall basic ingredient on the Helgason correspondence. Let G be a connected real semisimple Lie group with finite center, K a maximal compact subgroup of G . Then G/K is a Riemannian symmetric space. Let P be a minimal parabolic subgroup, and let $P = MAN$ be a Langlands decomposition of P . We denote by $\mathfrak{a}$ the Lie algebra of A . Similar German letters represents the Lie algebras of the Lie group denoted by the corresponding Latin letter. We denote by θ the Cartan involution with respect to the Cartan decomposition.

1.1. Space of hyperfunctions on the boundary

Let $\mathfrak{a}_{\mathbb{C}}^*$ be the complexification of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$. For any linear form λ on $\mathfrak{a}_{\mathbb{C}}^*$, we denote by L_{λ} the line bundle on G/P associated to the character of $P = MAN : man \mapsto e^{(\rho - \lambda)(\log a)}$. Here $\log : A \rightarrow \mathfrak{a}$ is the inverse of the exponential map $\exp : \mathfrak{a} \rightarrow A$.

We denote by $\mathcal{B}(G/P, L_{\lambda})$ the space of all L_{λ} -valued hyperfunctions on G/P . Then $\mathcal{B}(G/P, L_{\lambda})$ is canonically identified with the space of all hyperfunctions ψ on G such that $\psi(x \cdot man) = e^{(\lambda - \rho)(\log a)} \psi(x)$ ($m \in M, a \in A, u \in N$). For any $g \in G$ and $\psi \in \mathcal{B}(G/P, L_{\lambda})$, we put $\pi_{\lambda}(g)\psi(x) = \psi(g^{-1}x)$, then $\pi_{\lambda}(g)\psi \in \mathcal{B}(G/P, L_{\lambda})$, and π_{λ} defines a representation of G on $\mathcal{B}(G/P, L_{\lambda})$.

1.2. Space of analytic eigenfunctions on G/K

Let $D(G/K)$ be the ring of invariant differential operators, and $\mathcal{A}(G/K)$ be the space of all analytic functions on G/K .

For any algebra homomorphism X of $D(G/K)$ into $\mathbb{C}$, we denote by $\mathcal{A}(G/K; \mathfrak{M}(X))$ the space of all analytic functions on G/K satisfying the system $\mathfrak{M}(X)$ of the differential equations:

$$\mathfrak{M}(X) : \Delta u = X(\Delta)u \quad (\Delta \in D(G/K)).$$

For any $g \in G$, and $f \in \mathcal{A}(G/K)$, we put $(\pi(g)f)(x) = f(g^{-1}x)$. Then it is clear that $\mathcal{A}(G/K; \mathfrak{M}(X))$ is stable by $\pi(g)$ ($g \in G$). Thus we obtain a representation of G on $\mathcal{A}(G/K; \mathfrak{M}(X))$.

The space of K -invariant elements in $\mathcal{A}(G/K; \mathfrak{M}(X))$ is one-dimensional (a fact of spherical function), and there is a generator $\Phi_X \in \mathcal{A}(G/K; \mathfrak{M}(X))^K$ normalized by the condition $\Phi_X(eK) = 1$ at the origin $eK \in G/K$. Then for any $f \in \mathcal{A}(G/K; \mathfrak{M}(X))$

$$\int_K f(k^{-1}x) dk = f(eK)\Phi_X(x).$$

Here the Haar measure of K is normalized such that $\int_K dk = 1$.

1.3. Matching of parameters

Let $W(A)$ be the Weyl group for $(\mathfrak{g}, \mathfrak{a})$. Then there is a natural identification $\mathfrak{a}_{\mathbb{C}}^*/W(A) \xrightarrow{\sim} \text{Hom}(D(G/K), \mathbb{C})$ defined as follows:

Consider a direct sum decomposition:

$$(*) \quad U(\mathfrak{g}_{\mathbb{C}}) = \bar{n}_{\mathbb{C}} \cdot U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{a}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}}) \cdot \mathfrak{k}_{\mathbb{C}}.$$

Here $\bar{n}_{\mathbb{C}} = \theta(n_{\mathbb{C}})$, θ being the Cartan involution. Let $\mathfrak{h} : U(\mathfrak{a}_{\mathbb{C}}) \rightarrow U(\mathfrak{g}_{\mathbb{C}})$ be the algebra homomorphism induced by $\mathfrak{h}(H) = H - \rho(H)$ ($\forall H \in \mathfrak{a}_{\mathbb{C}}$). Here ρ is half of the sum of all positive roots. Let δ be the projection $U(\mathfrak{g}_{\mathbb{C}}) \rightarrow U(\mathfrak{a}_{\mathbb{C}})$ with respect to the above decomposition $(*)$. Then the composition $\mathfrak{h} \circ \delta$ induces an algebra isomorphism

$$\gamma : D(G/K) \xrightarrow{\sim} I_{W(A)}(\mathfrak{a}_{\mathbb{C}}),$$

via the canonical inclusion

$$D(G/K) \hookrightarrow U(\mathfrak{g}_{\mathbf{C}})^K = \{Y \in U(\mathfrak{g}_{\mathbf{C}}) | \text{Ad}(k) \cdot Y = Y, \forall k \in K\}.$$

Here $I_{W(A)}(\mathfrak{a}_{\mathbf{C}})$ denotes the set of $W(A)$ -invariants in $U(\mathfrak{a}_{\mathbf{C}}) = S(\mathfrak{a}_{\mathbf{C}})$. Therefore we have

$$\begin{aligned} \text{Hom}(D(G/K), \mathbf{C}) &= \text{Hom}(I_{W(A)}(\mathfrak{a}_{\mathbf{C}}), \mathbf{C}) \\ &= \mathfrak{a}_{\mathbf{C}}^*/W(A). \end{aligned}$$

For any $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, we denote by $\chi_{\lambda} : D(G/K) \rightarrow \mathbf{C}$ the corresponding algebra homomorphism. The preimage of χ_{λ} is given by $I_{\lambda}\{w\lambda | w \in W(A)\}$.

When $\chi = \chi_{\lambda}$, the function $\Phi_{\chi_{\lambda}}$ is given by

$$\Phi_{\lambda}(gK) = \Phi_{\chi_{\lambda}}(gK) = \int_K e^{(\lambda-\rho)H(gk)} dk.$$

Here for $g \in G$, $H(g) \in \mathfrak{a}$ is the unique element such that $g = k \cdot (\exp H(g)) \cdot n$ for some $k \in K$, $n \in N$.

1.4. Poisson transformation

The poisson transformation P_{λ} is defined as follows. For any $\phi \in \mathcal{B}(G/P, L_{\lambda})$, we put

$$(P_{\lambda}\phi)(gK) = \int_K \phi(gK) dk.$$

Then we can show that

$$(P_{\lambda}\phi)(gK) = \int_K e^{-(\lambda+\rho)(H(g^{-1}k))} \phi(gK) dk.$$

Put $\Psi_{\lambda}(gK) = e^{-(\lambda+\rho)(H(g^{-1}))}$. Then $\psi_{\lambda} \in \mathcal{A}(G/K; \mathfrak{M}(\chi))$. For any $\psi \in \mathcal{B}(G/P, L_{\lambda})$, we can consider the restriction ψ_K . Then $\psi_K(mk) = \psi_K(k)$ for any $m \in M$, $k \in K$. This show that ψ_K is canonically identified with a hyperfunction on K/M . Conversely for any hyperfunction $\phi \in \mathcal{B}(K/M) = \mathcal{B}(K)^M$, we put

$$\psi(kan) = e^{(\lambda-\rho)(\log a)} \phi(k) \quad (k \in K, a \in A, \text{ and } n \in N).$$

Clearly $\psi \in \mathcal{B}(G/P, L_{\lambda})$ and $\psi_K = \phi$. Thus we have an identification $\mathcal{B}(G/P, L_{\lambda}) \cong \mathcal{B}(K/M)$.

Via this identification, P_{λ} is written as

$$(P_{\lambda}\phi)(gK) = \int_K \pi(k) \Psi_{\lambda}(gK) \phi(k) dk \quad (\text{for } \phi \in \mathcal{B}(K/M)).$$

Hence $\Delta P_{\lambda}\phi = \chi_{\lambda}(\delta) \cdot P_{\lambda}\phi$ for all $\Delta \in D(G/K)$. Since $D(G/K)$ contains the Laplacian, which is elliptic, $P_{\lambda}\phi$ is real analytic.

Proposition. *The Poisson transform P_λ is a G -homomorphism of*

$$\mathcal{B}(G/P, L_\lambda) \rightarrow \mathcal{A}(G/K; \mathfrak{M}(\chi)).$$

1.5. Boundary value map β_λ

We do not give the precise definition of β_λ . But we note the following: Let $\mathfrak{M}(\chi)$ be the system of the differential equations defined by

$$\mathfrak{M}(\chi) : \Delta n = \chi(\Delta)n \quad (\Delta \in D(G/K)).$$

Then $\mathfrak{M}(\chi)$ has regular singularity along the edge $K/M \times \{0\}$. Moreover the roots of the indicial equation of $\mathfrak{M}(\chi)$ coincide with $\{S(\lambda); \lambda \in I_\chi\}$.

Anyway what is important for us is β_λ gives a G -homomorphism $\beta_\lambda : \mathcal{A}(G/K; \mathfrak{M}(\chi_\lambda)) \rightarrow \mathcal{B}(G/P; L_\lambda)$.

Under the assumption

$$(A) : -\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbf{N}, \quad \text{for } \alpha \in \Sigma^+,$$

P_λ and β_λ are mutually inverse up to scalar multiple.

§2. Class-1 Whittaker function

Let (π_λ, H_λ) be the class-1 principal series representation on a Hilbert space H_λ associated to $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Choose a character $\psi : N \rightarrow \mathbb{C}^*$, and let $\mu : (\pi_\lambda, H_\lambda^\infty) \rightarrow C_\psi^\infty(N \backslash G)$ be the Whittaker functional.

Let $\Phi \in H_\lambda^K$ be the generator of H_λ^K . Put $w(g) = \mu(\pi_\lambda(g)\Phi)$. Then

$$\begin{aligned} \forall g \in G, \forall n \in N, \quad W(ng) &= \psi(n)W(g); \\ \forall g \in G, \forall k \in K, \quad W(nk) &= W(g). \end{aligned}$$

Thus we can regard W as a function on G/K . Then W belongs to $\mathcal{A}(G/K; \mathfrak{M}(X_\lambda))$. Let us consider a subspace $\mathcal{A}_\psi(N \backslash G/K; \mathfrak{M}(X_\lambda))$ of $\mathcal{A}(G/K; \mathfrak{M}(X_\lambda))$, defined by

$$\{f(x) \in \mathcal{A}(G/K; \mathfrak{M}(X_\lambda)) \mid f(n \cdot x) = \psi(n)f(x)\}.$$

Then β_λ and P_λ induces an isomorphism

$$\mathcal{A}_\psi(N \backslash G/K; \mathfrak{M}(X_\lambda)) \cong \mathcal{B}_\psi(N \backslash G/K; L_\lambda).$$

Here $\mathcal{B}_\psi(N \backslash G/P : L_\lambda)$ is a subspace of function $\phi \in \mathcal{B}(G/P : L_\lambda)$ satisfying

$$\phi(ng) = \psi(n)\phi(g).$$

Now let us consider the space $\mathcal{B}_\psi(N \backslash G/P : L_\lambda)$. Using Bruhat decomposition

$$G = \bigcup_{w \in W(A)} NwP.$$

Hence $\phi \in \mathcal{B}_\psi(N \backslash G/P : L_\lambda)$ is determined by its “values” at $w \in W(A)$. Hence

$$\dim_{\mathbb{C}} \mathcal{B}_\psi(N \backslash G/P : L_\lambda) = \dim_{\mathbb{C}} \mathcal{A}_\psi(N \backslash G/K; \mathfrak{M}(X_\lambda)) = |W(A)|.$$

Let w_0 be the longest element of the Weyl group $W(A)$. Then the Poisson Ivan formation of the constant function 1 on $w_0 : P_\lambda(1)$ given an element of $\mathcal{A}_\psi(G/K; \mathfrak{M}(X_\lambda))$. This should be the Jacquet’s integral representation of w .

2.1. Example

Set $G = SL_2(\mathbf{R})$. $P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}$, $K = SO(2)$. Then,

$$G = KP; G/P = KP/P.$$

$$B(G/P; L_\lambda) \ni f : f(gP) = e^{\lambda - \rho_P}(p) f(g).$$

Here ρ_P = half sum of positive roots. Put $\psi(n) = \exp(2\pi i c x)$ for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$, with a fixed $c \in \mathbf{R}$. Then assume that f satisfies the conical condition:

$$f/ng) = \psi(n)f(g).$$

Write

$$\rho = \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1}x \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \in P.$$

Then

$$f(np) = \psi(n)e^{\lambda - \rho_P}(p) = \exp(2\pi i c x)(\sqrt{y})^{\lambda-1}.$$

Identify $G/P \cong \mathbf{P}^1(\mathbf{R})$ with K , via

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix},$$

$$\& (\cos \theta : -\sin \theta) \sim (-\cot \theta : 1) \text{ in } \mathbf{P}^1(\mathbf{R}).$$

Then the Poisson transformation is written as follows for $u \in \mathcal{B}(G/P, L_\lambda)$.

$$P_\lambda(u)(a) = \int_K u(ak) dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(ar_\theta) d\theta$$

Here write

$$ar_\theta = n_1 w a_2 n_2, \text{ i.e.}$$

$$ar_\theta = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}.$$

Then $s_1 = -y \cot \theta$. Set $u = f$, then

$$\begin{aligned} P_\lambda(f) \left(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \right) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-2\pi i c y \cot \theta) \times (\sqrt{y}^{-1} \sin \theta)^{(\lambda - \rho_P)} f(w) d\theta \\ &= f(w) \cdot y^{-\left(\frac{\lambda - \rho_P}{2}\right)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-2\pi i c y \cdot \cot \theta) \cdot (\sin \theta)^{(\lambda - \rho_P + 2)} \sin^{-2} \theta \cdot d\theta. \end{aligned}$$

Write $t = -\cot \theta$. Then $dt = \frac{d\theta}{\sin^2 \theta}$, $\sin \theta = \frac{1}{\sqrt{1+\cot^2 \theta}}$. Hence

$$P_\lambda(f) \left(\begin{pmatrix} \sqrt{y} & \\ & \sqrt{y} \end{pmatrix} \right) = f(w) \cdot y^{-\left(\frac{\lambda-\rho_P}{2}\right)} \int_{-\infty}^{\infty} \exp(2\pi i c y \cdot t) (1+t^2)^{-\frac{1}{2}(\lambda-\rho_P)+2} dt.$$

This is the Jacquet integral. General case is discussed similarly.

Remark I This correspondence of "algebraic Whittaker function" and Bruhat geometry seems to be valid in a very wide context. It seems applicable also four generalized Whittaker model, or Gelfand-Graev representation. Wallach [W] already obtains some results in this direction.

Remark II Using the contour integral representation of an element in $\mathcal{B}_\psi(N \setminus G/P : L_\lambda)$ which has a support of lower dimension i.e. $\text{supp}(\psi) \subset N \setminus G/P - \{w_0 P\}$, then it seems possible to have an integral representation of the all elements in $A_\psi(N \setminus G/K; \mathfrak{M}(X_\lambda))$ when λ satisfies the assumption (A).

Reference

- [K²MO²] Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, *Ann. of Math.*, **107** (1978), 1-39.
- [W] N. Wallach, Lie Algebra Cohomology and Holomorphic continuation of Generalized Jacquet Integrals, *Advanced Studies in Pure Math.*, **14** (1988), 123-151.

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Remark III. Goodman - Wallach [G-W] は代数的 Whittaker vectors $\text{Hom}_{(G,K)}(H_\lambda^K, C_c^\infty(M \backslash G))$ と H_λ^ω (analytic vectors) との等価性を示した (実際はより狭く Gelfand class に対する). [W] は $\{w_0 P\}$ に対する δ の π -distribution i.e. $\text{Hom}_G(H_\lambda^\omega, C_c^\infty(M \backslash G))$ の連続性を示した (重複度 1).

[G-W] R. Goodman and N.R. Wallach, Whittaker vectors and conical vectors, *J. Funct. Anal.* **39** (1980), 199-279.

[W2] N.R. Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. in *Lecture Notes in Math.* No 1024 (1983) pp. 287-369. Springer