

Various Gauss fibers

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ABSTRACT. We find examples and constructions of nontrivial fiber structures of Gauss maps in positive characteristic. 3 types of constructions of projective varieties were announced in this talk: (A) Gauss fibers are the given projective variety, (B) Gauss fibers are hyperplane sections of the given projective variety, and (C) Gauss map is the given rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ with $dg \equiv 0$. Additional matters to the talk are included in this paper, for example, discussion of the differences of the constructions (A)-(C), and a generalization of Kaji and Rathmann's construction of Gauss map, related to (C), which is the given inseparable morphism $\mathbf{P}^1 \rightarrow \mathbf{P}^1$.

1. INTRODUCTION

In this paper, the base field K is an algebraically closed field and varieties are integral algebraic schemes over K .

Precisely, "Gauss fibers" mean (general) fibers of the Gauss maps. Definition of the Gauss map is as follows:

Definition 1.1. *Let $X \subset \mathbf{P}^N$ be a projective variety. The Gauss map γ on X is the rational map from X to the Grassmannian $\mathbf{G}(\dim X, N)$ such that $\gamma(p) = \mathbf{T}_p X$ for any smooth point $p \in X$, where $\mathbf{T}_p X$ is the projective embedded tangent space.*

Example 1.2. *If $X \subset \mathbf{P}^N$ is the hypersurface given by F , then*

$$\gamma = \left(\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_N} \right) : X \dashrightarrow \mathbf{G}(N-1, N) \cong \mathbf{P}^{N*}.$$

Remark 1.3. *The following facts are known.*

- (1) *If $\text{char}K = 0$ then general fibers of γ are linear spaces ([1],[5],[14]).
(They are one-points when $\dim = 0$.)*
- (2) *If $\text{char}K > 0$ then there is a variety whose general fibers of γ are two or more distinct points.*

The fact (1), when X is a curve, implies that multiple tangent lines (which have two or more distinct tangential points at X) are only finitely many. If X is a surface with $\dim \gamma(X) = 1$ then we can classify X to the two kinds of ruled surfaces, a cone or a tangent surface. (A cone surface is the join of a curve and one point, and a tangent surface is covered by tangent lines of some curve.) These surfaces are called developable surfaces.

A. H. Wallace gave examples of the kind of (2) ([13]). Kleiman-Laksov also found interesting examples ([10]). It seems to be difficult to construct smooth varieties of this kind, but H. Kaji ([7],[8]), J. Rathmann ([12]) and A. Noma ([11]) constructed such varieties.

The author found the following example.

Example 1.4. $XZ^6 - (Y^6 + W^6)W = 0 \subset \mathbf{P}^3$. *If $\text{char}K = 2$ (resp. $\text{char}K = 3$) then general Gauss fibers are plane elliptic curves (resp. plane smooth conics) ([2]).*

In the author's best knowledge, this is the first example whose general Gauss fibers are not finite unions of linear spaces. Furthermore, the author found constructions of varieties with non-linear smooth Gauss fibers in positive characteristic.

- (A) Construction of a projective variety whose general fibers of the Gauss map are the given projective variety ([3]).
- (B) Construction of a projective variety whose general fibers of the Gauss map are hyperplane sections of the given (general) projective variety ([4]).
- (C) Construction of a projective variety whose Gauss map is the given rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ with $dg \equiv 0$.

The main purpose of this paper is an introduction of these constructions and explanation of differences of each constructions.

2. CONSTRUCTION (C)

Let $g = (g_0, \dots, g_n) : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ be the given rational map such that $\frac{\partial g_j}{\partial x_i} \equiv 0$ for all i, j , and let X be the closure of image of the rational map $i : \mathbf{A}^n \dashrightarrow \mathbf{P}^{n+1}$; $(x_1, \dots, x_n) \dashrightarrow (1 : x_1 : \dots : x_n : -g_0 - x_1g_1 - \dots - x_ng_n)$. Then, $T_{i(x)}X$ is spanned by the row vectors of the following matrices;

$$\begin{pmatrix} 1 & x_1 & \dots & x_n & -g_0 - x_1g_1 - \dots - x_ng_n \\ 0 & 1 & \dots & 0 & -g_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -g_n \end{pmatrix} \sim \begin{pmatrix} & -g_0 \\ & -g_1 \\ I_{n+1} & \vdots \\ & -g_n \end{pmatrix}$$

where I_{n+1} is the $(n+1) \times (n+1)$ unit matrix. Hence $\gamma : X \dashrightarrow \mathbf{P}^{n+1*}$ is given by $(g_0 : \dots : g_n : 1)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A}^n & \xrightarrow{g} & \mathbf{A}^{n+1} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & \mathbf{P}^{n+1*} \end{array}$$

Remark 2.1. *Example 1.4 is given by this construction: $n = 2$, $g_0 = g_2 = g_3 = 0$ and $g_1 = x_1^6 + x_2^6$ (with suitable coordinates).*

If we consider the rational map $g : \mathbf{A}^n \dashrightarrow \mathbf{A}^{n+1}$ as the rational map g' from $\mathbf{P}^n$ to $\mathbf{P}^{n+1}$, then the above varieties can be got as the image of a suitable linear projection of the graph $\Gamma_{g'} \subset \mathbf{P}^{n^2+3n+1}$ of g' which is embedded by Segre embedding.

Now we study the graph Γ_g of a rational map $g : \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ with $dg \equiv 0$. Let $X \subset \mathbf{P}^{nm+n+m}$ be the image of the graph Γ_g by Segre embedding $\mathbf{P}^n \times \mathbf{P}^m \subset \mathbf{P}^{nm+n+m}$. Then we have the commutative

diagram

$$\begin{array}{ccc}
 & \Gamma_g & \\
 p_1 \swarrow & \downarrow p_2 & \\
 \mathbf{P}^n & \xrightarrow[g]{} & \mathbf{P}^m
 \end{array}
 \quad \cong \quad
 \begin{array}{ccc}
 & X & \\
 & \downarrow \gamma & \\
 & Y & \\
 & \downarrow & \\
 & \mathbf{G}(n, nm + n + m) &
 \end{array}
 \quad \xrightarrow{h}$$

where h is an embedding given by $h(t) = \mathbf{P}^n \times t$. This implies that the Gauss map γ can be identified with the projection p_2 , hence generically identified with g . Precisely, X is the closure of the image of the rational map $\mathbf{P}^n \dashrightarrow \mathbf{P}^{nm+n+m}$:

$$(1 : x_1 : \cdots : x_n) \mapsto (1 : x_1 : \cdots : x_n : g_1 : \cdots : g_m : \cdots : x_i g_j : \cdots).$$

We can check easily that varieties given by (C) can be got as the image of a suitable linear projection of the graph of $\mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$.

The latter construction is a generalization of Kaji and Rathmann's for inseparable morphisms $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ ([7],[12]).

In the latter construction, it is very interesting that the Gauss map can be defined at any point of X , hence it is a morphism, and the tangent variety of X is $\mathbf{P}^n \times \mathbf{P}^m$ if g is dominant. The second fact implies that any Segre variety of two projective spaces is the tangent variety of some variety, and that the classical fact $X \subset \text{SingTan} X$ in characteristic 0 ([1]) does not hold in positive characteristic.

3. CONSTRUCTION (A)

3.1. Concept of (A) or (B). Let $Y \subset \mathbf{P}^k$ be a given projective variety of codimension r . We move $\mathbf{P}^k$ "inseparably" in $\mathbf{P}^N$ ($N \gg k$). Then, Y moves in conformity to the projective space $\mathbf{P}^k$, and constructs X . We will have the diagram

$$\begin{array}{ccc}
 \mathbf{A}^r \times \mathbf{P}^k & \xrightarrow{\eta} & \mathbf{P}^N \\
 \cup & & \cup \\
 \mathbf{A}^r \times Y & \xrightarrow{\eta|_{\mathbf{A}^r \times Y}} & X
 \end{array}$$

such that η is inseparable (onto its image) and $\eta|_{\mathbf{A}^r \times Y}$ is birational.

The idea for our construction (A) or (B) is based on “circular surfaces” ([6]) studied in differential geometry or real singularity theory. Conceptually, our variety with (A) could be called a “developable” circular surface.

3.2. Construction (A) (plane curve’s case). Let $p > 0$ be the characteristic, and let $\rho_0, \rho_1, \rho_2 : \mathbf{A}^1 \rightarrow \mathbf{P}^3$ be morphisms (which form a frame) as follows,

$$\begin{aligned}\rho_0 &= (1 \ 0 \ u \ u^p) \\ \rho_1 &= (0 \ 1 \ 0 \ 0) \\ \rho_2 &= (0 \ 0 \ 1 \ 0)\end{aligned}$$

Let $\eta : \mathbf{A}^1 \times \mathbf{P}^2 \rightarrow \mathbf{P}^3$ be

$$(u) \times (1 : y_1 : y_2) \mapsto [\rho_0 + y_1\rho_1 + y_2\rho_2] = (1 : y_1 : u + y_2 : u^p).$$

We may assume that $y_1 - a$ is a local parameter at a smooth point $(1 : a : b) \in Y$. (We can always take this coordinates by linear transforms of $\mathbf{P}^2$.) Let X be the closure of $\eta(\mathbf{A}^1 \times Y)$, and let $\tau := \eta|_{\mathbf{A}^1 \times Y} : \mathbf{A}^1 \times Y \rightarrow X$. Then the following proposition holds.

Proposition 3.1. *The morphism τ is birational, and $T_{\tau(u,y)}X = \eta(u \times \mathbf{P}^2)$ for a general point (u, y) .*

Proof. The differentials of τ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & dy_2/dy_1 & 0 \end{pmatrix}$$

(upper row is a list of the differentials by u , lower is the differentials by y_1). We find the separability of τ by this matrix and, because τ is generically one-to-one, birationality of τ .

$T_{\tau(u,y)}X$ is spanned by the row vectors of the following matrices:

$$\begin{pmatrix} 1 & y_1 & u + y_2 & u^p \\ 0 & 0 & 1 & 0 \\ 0 & 1 & dy_2/dy_1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & u^p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This coincides with $\eta(u \times \mathbf{P}^2)$. □

3.3. Generalized form. Let $k \geq 2$ and $r < k$ be positive integers, and let $Y \subset \mathbf{P}^k$ be a closed subvariety of codimension r . We take the morphisms $\rho_0, \dots, \rho_k : \mathbf{A}^r \rightarrow \mathbf{P}^{k+r}$ as follows,

$$\begin{aligned}\rho_0 &= (1 \ 0 \ \dots \ 0 \ u_1 \ \dots \ u_r \ u_1^p \ \dots \ u_r^p) \\ \rho_1 &= (0 \ 1 \ \dots \ 0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 0) \\ &\vdots \\ \rho_k &= (0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 1 \ 0 \ \dots \ 0)\end{aligned}$$

Remark 3.2. *The form can be more generalized (see [3]). The base space $\mathbf{A}^r$ which moves projective planes and the how of the moving $\{\rho_i\}$ are more free and formulated to some extent. Furthermore, we can also construct varieties whose general fibers are two or more Y s for the suitable moving $\{\rho_i\}$.*

4. CONSTRUCTION (B)

4.1. Construction (B) (surface case). Let $p > 0$ be the characteristic, and let $\rho_0, \rho_1, \rho_2, \rho_3 : \mathbf{A}^2 \rightarrow \mathbf{P}^6$ be morphisms (which form a frame) as follows:

$$\begin{aligned}\rho_0 &= (1 \ 0 \ 0 \ u \ u^p \ v \ 0) \\ \rho_1 &= (0 \ 1 \ 0 \ 0 \ 0 \ 0 \ v) \\ \rho_2 &= (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0) \\ \rho_3 &= (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)\end{aligned}$$

Let $\eta : \mathbf{A}^2 \times \mathbf{P}^3 \rightarrow \mathbf{P}^6$ be

$$\begin{aligned}(u, v) \times (1 : y_1 : y_2 : y_3) &\mapsto [\rho_0 + y_1\rho_1 + y_2\rho_2 + y_3\rho_3] \\ &= (1 : y_1 : y_2 : u + y_3 : u^p : v : vy_1).\end{aligned}$$

We may assume that $y_1 - a$, $y_2 - b$ are a local parameter at a smooth point $(1 : a : b : c) \in Y$. Let X be the closure of $\eta(\mathbf{A}^2 \times Y)$, and let $\tau := \eta|_{\mathbf{A}^2 \times Y} : \mathbf{A}^2 \times Y \rightarrow X$. Then the following proposition holds.

Proposition 4.1. *The morphism τ is birational, and γ is generically identified with the morphism $\mathbf{A}^2 \times Y \rightarrow \mathbf{A}^3 : (u, v) \times (y_1, y_2, y_3) \mapsto (u^p, v, y_1)$.*

Proof. The differentials of τ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & y_1 \\ 1 & 0 & dy_3/dy_1 & 0 & 0 & v \\ 0 & 1 & dy_3/dy_2 & 0 & 0 & 0 \end{pmatrix}$$

(the first row is a list of the differentials by u , the second is by v , the third, fourth are the differentials by y_1, y_2 respectively). We find the separability of τ by this matrix and, because τ is generically one-to-one, birationality of τ .

$T_{\tau(u,y)}X$ is spanned by the row vectors of the following matrices:

$$\begin{pmatrix} 1 & y_1 & y_2 & u + y_3 & u^p & v & vy_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_1 \\ 0 & 1 & 0 & dy_3/dy_1 & 0 & 0 & v \\ 0 & 0 & 1 & dy_3/dy_2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & u^p & 0 & -vy_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_1 \\ 0 & 1 & 0 & 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This implies our 2nd assertion. $\square$

Corollary 4.2. *The set $\{\gamma^{-1}(p)\}_{p \in X_{sm}}$ almost coincides with $\{Y_\lambda\}_{\lambda \in K}$, where Y_λ is the hyperplane section $Y \cap \{Y_1 - \lambda Y_0 = 0\}$ (except the line $Y_0 = Y_1 = 0$).*

Example 4.3. *Let $\text{char} K > 2$. Let $Y \subset \mathbf{P}^3$ be the surface given by $Y_2^2 Y_0 - Y_3(Y_3 - Y_0)(Y_3 - Y_1)$, and let Y_λ be the hyperplane section $Y \cap \{Y_1 - \lambda Y_0 = 0\}$. Then, the set of all Gauss fibers of X almost coincides with $\{Y_\lambda\}$.*

4.2. Generalized form. Let $k \geq 2$ and $r < k$ be positive integers, and let $Y \subset \mathbf{P}^k$ be a closed subvariety of codimension r . We take the morphisms $\rho_0, \dots, \rho_k : \mathbf{A}^{r+1} \rightarrow \mathbf{P}^{k+r+2}$ as follows,

$$\begin{aligned} \rho_0 &= (1 \ 0 \ \dots \ 0 \ u_1 \ \dots \ u_r \ u_1^p \ \dots \ u_r^p \ u_{r+1} \ 0) \\ \rho_1 &= (0 \ 1 \ \dots \ 0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 0 \ 0 \ u_{r+1}) \\ &\vdots \\ \rho_k &= (0 \ 0 \ \dots \ 0 \ 0 \ \dots \ 1 \ 0 \ \dots \ 0 \ 0 \ 0) \end{aligned}$$

5. ELLIPTIC CURVES AS GAUSS FIBERS

In this section, we discuss the differences of our constructions. We recall some properties of our constructions.

Remark 5.1. *A variety X given by each constructions has the following properties:*

- (A) *X is birational to the product of two some varieties and one of which varieties Y is the general fiber of the Gauss map.*
- (B) *X is birational to the product of two some varieties and the differential $d\gamma$ of the Gauss map is not identically zero.*
- (C) *X is rational and the differential $d\gamma$ of the Gauss map is identically zero.*

Let $\text{char}K = p > 3$. Then we get projective varieties whose Gauss fibers are elliptic curves if we take Y or g as follows.

- (A) Let $Y \subset \mathbf{P}^2$ be given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$.
- (B) Let $Y \subset \mathbf{P}^3$ be given by $Y_2^2 Y_0 - Y_3(Y_3 - Y_0)(Y_3 - Y_1) = 0$ (Example 4.3).
- (C) Let $g : \mathbf{A}^2 \rightarrow \mathbf{A}^3$ be $(x_1, x_2) \mapsto (x_1^{3p} + x_2^{3p}, 0, 0)$.

We call X_a (resp. X_b, X_c) constructed by (A) (resp. (B), (C)) with the above Y (resp. Y, g).

X_a can not be constructed by (B) nor (C), because $d\gamma \equiv 0$ and this is not rational.

X_b can not be constructed by (A) nor (C), because isomorphic classes of Gauss fibers vary and $d\gamma$ is not identically 0.

X_c can not be constructed by (A) nor (B), because Gauss fibers are elliptic curves and this is rational, and $d\gamma \equiv 0$.

REFERENCES

- [1] G. Fischer and J. Piontowski, Ruled varieties. Friedr. Vieweg & Sohn, Braunschweig, 2001.
- [2] S. Fukasawa, Developable varieties in positive characteristic, Hiroshima Math. J., **35**(2005), 167–182.

- [3] S. Fukasawa, Varieties with non-linear Gauss fibers, *Math. Ann.*, **334**(2006), 235–239.
- [4] S. Fukasawa, Varieties with nonconstant Gauss fibers, to appear in *Hiroshima Math. J.*
- [5] P. Griffiths and J. Harris, Algebraic geometry and local differential geometry, *Ann. Sci. École Norm. Sup. (4)* **12**(1979), 355–452.
- [6] S. Izumiya, K. Saji and N. Takeuchi, Circular surfaces, preprint, EPrint Series of Department of Mathematics, Hokkaido University.
- [7] H. Kaji, On the tangentially degenerate curves, *J. London Math. Soc. (2)*, **33**(1986), 430–440.
- [8] H. Kaji, On the Gauss maps of space curves in characteristic p , *Compositio Math.*, **70**(1989), 177–197.
- [9] S. L. Kleiman, Tangency and duality. CMS Conf. Proc. 6 (Proc. 1984 Vancouver Conf. in Alg. Geom.), 163–226, Amer. Math. Soc., 1986.
- [10] S. L. Kleiman, Multiple tangents of smooth plane curves (after Kaji), *Algebraic geometry: Sundance 1988*, 71–84, *Contemp. Math.*, 116, Amer. Math. Soc., Providence, RI, 1991.
- [11] A. Noma, Gauss maps with nontrivial separable degree in positive characteristic, *J. Pure Appl. Algebra*, **156**(2001), 81–93.
- [12] J. Rathmann, The uniform position principle for curves in characteristic p , *Math. Ann.*, **276**(1987), 565–579.
- [13] A. H. Wallace, Tangency and duality over arbitrary fields, *Proc. London Math. Soc. (3)*, **6**(1956), 321–342.
- [14] F. L. Zak, Tangents and secants of algebraic varieties. *Transl. Math. Monographs*, 127. American Mathematical Society, Providence, RI, 1993.

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