

A stratified Whitney jet and a tempered-stratified ultradistribution

By

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§ 1. Introduction

A tempered distribution and a Whitney function appear in various aspects of analysis such as the Riemann-Hilbert correspondence [2], the theory of asymptotic analysis and etc. As they are defined by non-local properties, for example, a tempered distribution on an open set $U \subset X$ is, by definition, a distribution on U which extends to the whole space X , tempered distributions and Whitney functions do not form sheaves in a usual topological space X . This is one of reasons why they were difficult to be managed. However they are still sheaves on the subanalytic site X_{sa} . The recent development of the theory for ind-sheaves and sheaves on subanalytic sites by M.Kashiwara-P.Schapira [5] and L.Prelli [11], etc. enables us to apply sheaf theoretic methods (Grothendieck's six operations) to such an object. Therefore a tempered distribution and a Whitney function now can be managed well in a functorial way.

It is reasonable to expect tempered ultradistributions and Whitney functions of Gevrey class to form sheaves on X_{sa} also. Unfortunately this expectation is not true as it is shown in Examples 2.3 and 2.4 of this note.

Our aim is to construct sheaves on X_{sa} corresponding to tempered ultradistributions and Whitney functions of Gevrey class. We introduce, in the note, *a tempered-stratified ultradistribution* and *a stratified Whitney jet of Gevrey class*. They can be regarded as alternatives of a tempered ultradistribution and a Whitney jet of Gevrey class due to the fact that their sections on an open subanalytic subset with the smooth boundary coincide with those of a tempered ultradistribution and a Whitney jet of Gevrey class respectively.

Received August 6, 2009.

2000 Mathematics Subject Classification(s): 46M20, 46F05, 32B20, 32C38

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The stratified Whitney jets of Gevrey class certainly form a sheaf on the subanalytic site X_{sa} . While, for a tempered ultradistribution, we can construct the corresponding sheaf on X_{sa} only if $\dim X \leq 2$. We discuss, in the last section of the note, the possibility of its construction on a higher dimensional manifold.

This note is a short summary of our paper [10] in which the complete proofs for propositions and theorems here are given.

§ 2. Preparation

Let X be a real analytic manifold. We always assume, in this note, a manifold to be countable at infinity. We denote by $\text{Mod}(\mathbb{C}_X)$ the category of sheaves on X with values in $\mathbb{C}$ -vector spaces, and by $\mathcal{C}^\infty$ the sheaf of infinitely differentiable functions on X .

Let A be a locally closed subset in X , and $\mathcal{J}_A(X)$ designates the set of continuous sections over A of the jet vector bundle J_X of X . We have the canonical morphism $j_A : \mathcal{C}^\infty(X) \rightarrow \mathcal{J}_A(X)$ and the natural restriction map $J_{B,A} : \mathcal{J}_A(X) \rightarrow \mathcal{J}_B(X)$ for a locally closed subset $B \subset A$. If X is the n -dimensional Euclidean space with coordinates $(x_1, \dots, x_n)$, then an element in $\mathcal{J}_A(X)$ is identified with a family of continuous functions $\{\varphi_\alpha(x)\}_{\alpha \in (\{0\} \cup \mathbb{N})^n}$ on A , and j_A is given by

$$j_A(\varphi) = \left\{ \frac{\partial^\alpha \varphi}{\partial x^\alpha} \Big|_A \right\}_{\alpha \in (\{0\} \cup \mathbb{N})^n} \quad \text{for } \varphi \in \mathcal{C}^\infty(X).$$

In what follows, the symbol $*$ denotes (s) or $\{s\}$ for some $s > 1$. Let us recall the definition of the sheaf $\mathcal{C}^*$ of ultradifferentiable functions of class $*$ in X . For an open subset V in the n -dimensional Euclidean space with coordinates $(x_1, \dots, x_n)$, an element $f(x) \in \mathcal{C}^{(s)}(V)$ (resp. $f(x) \in \mathcal{C}^{\{s\}}(V)$) is an infinitely differentiable function with the following growth condition. For any compact set $K \subset V$ and for any $h > 0$ (resp. some $h > 0$), there exists a constant C such that

$$\sup_{x \in K} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq C h^{|\alpha|} (|\alpha|!)^s \quad \text{for any } \alpha \in (\{0\} \cup \mathbb{N})^n,$$

respectively. The definition naturally extends to that of an ultradifferentiable function on a real analytic manifold.

Definition 2.1. For a locally closed subset A in X , the set $j_A(\mathcal{C}^*(X)) \subset \mathcal{J}_A(X)$ is called Whitney jets of Gevrey class $*$ over A , and we denote it by $\mathcal{W}_A^*(X)$.

For an open subset V in X , the vector space $\mathcal{C}^*(V)$ can be endowed with a locally convex topology in a natural way. The set $\mathcal{D}b^*(V)$ of ultradistributions of class $*$ on V is, by definition, the strong dual of $\mathcal{C}^*(V)$. It is well-known that a family $\{\mathcal{D}b^*(V)\}_V$ forms the sheaf $\mathcal{D}b^*$ of ultradistributions of class $*$ on X .

Given a closed set $Z \subset X$ and $F \in \text{Mod}(\mathbb{C}_X)$ we denote by $\Gamma_Z(F)$ the subsheaf of F of sections supported by Z .

Definition 2.2. Let U be an open set in X . The set $\mathcal{D}b_X^{*t}(U)$ of tempered ultradistributions of class $*$ over U is defined by

$$\mathcal{D}b_X^{*t}(U) := \frac{\Gamma(X; \mathcal{D}b^*)}{\Gamma_{X \setminus U}(X; \mathcal{D}b^*)}.$$

Let $\mathcal{F}$ be a presheaf on X , and let U and V be open subanalytic subsets in X with U , V , $U \cap V$ and $U \cup V$ being locally cohomologically trivial. We now consider the following sequence of Abelian groups

$$(2.1) \quad 0 \rightarrow \mathcal{F}(U \cup V) \rightarrow \mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(U \cap V) .$$

If $\mathcal{F}$ is the presheaf of either tempered distributions or Whitney functions, it follows from the Łojasiewicz inequality [9] for subanalytic subsets that the sequence (2.1) is exact. The exactness of (2.1) implies that tempered distributions and Whitney functions form sheaves on the subanalytic site X_{sa} . We refer the reader to [5] and [11] for the precise definitions of a subanalytic site and a sheaf defined on it.

While, for the presheaf either $\mathcal{F}(U) := \mathcal{D}b_X^{*t}(U)$ or $\mathcal{F}(U) := \mathcal{W}_U^*(X)$, the sequence (2.1) is not exact as the following examples show (see [10] for the details).

Example 2.3. Let $l \geq 1$. Set

$$U_1 := \{(x, y) \in \mathbb{R}^2; y > x^{2l+1}\} ,$$

$$U_2 := \mathbb{R} \times \mathbb{R}_{<0} .$$

Let $L_{loc}(U)$ denote the set of locally integrable functions in an open subset U . Define a function $u_i(x, y) \in L_{loc}(U_i)$ ($i = 1, 2$) by

$$u_1(x, y) := \begin{cases} \exp\left(\frac{1}{y - x^{2l+1}}\right) & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{>0}) \\ 0 & \text{for } (x, y) \in U_1 \cap (\mathbb{R} \times \mathbb{R}_{\leq 0}) \end{cases}$$

$$u_2(x, y) := 0 \quad \text{for any } (x, y) \in U_2 .$$

Then we have $u_1 \in \mathcal{D}b_X^{(2)t}(U_1)$. Clearly $u_2 \in \mathcal{D}b_X^{(2)t}(U_2)$. As $u_1|_{U_1 \cap U_2} = u_2|_{U_1 \cap U_2}$, there exists $u \in L_{loc}(U_1 \cup U_2)$ such that $u|_{U_1} = u_1$ and $u|_{U_2} = u_2$, but we can show $u \notin \mathcal{D}b_X^{(2)t}(U_1 \cup U_2)$.

Example 2.4. Let $m \geq 2$ and set

$$U_1 := \{(x, y) \in \mathbb{R}^2; y < 0\},$$

$$U_2 := \{(x, y) \in \mathbb{R}^2; x < 0 \text{ or } y > x^m\}.$$

Define the jets $G_1 = \{g_{1,\alpha}\} \in \mathcal{J}_{U_1}(X)$ and $G_2 = \{g_{2,\alpha}\} \in \mathcal{J}_{U_2}(X)$ by

$$g_{1,\alpha}(x, y) := \begin{cases} 0 & (x, y) \in U_1 \cap \{x \leq 0\} \\ \frac{\partial^\alpha}{\partial x^\alpha} \exp\left(-\frac{1}{x}\right) & (x, y) \in U_1 \cap \{x > 0\} \end{cases}$$

$$g_{2,\alpha}(x, y) := 0.$$

Then we have $G_k \in \mathcal{W}_{U_k}^{\{2\}}(X)$ ($k = 1, 2$). As $G_1 = G_2$ in $U_1 \cap U_2$, we find $G \in \mathcal{J}_{U_1 \cup U_2}(X)$ with $G = G_k$ in U_k for $k = 1, 2$. However $G \notin \mathcal{W}_{U_1 \cup U_2}^{\{2\}}(X)$.

These examples indicate that the presheaves $\{\mathcal{D}b_X^{*t}(U)\}_U$ and $\{\mathcal{W}_U^*(X)\}_U$ do not form sheaves on the subanalytic site X_{sa} .

At the end of the subsection, we give some definitions and results which are needed later on. Let X be a real analytic manifold of dimension n and A a locally closed subanalytic subset of X . Throughout this paper, a subanalytic stratification $\{A_\alpha\}_\alpha$ of A is that of X which is finer than the partition $\{A, X \setminus A\}$.

Definition 2.5. We say that A is 1-regular at $p \in X$ if there exist a neighborhood $U \subset X$ of p , a neighborhood $V \subset \mathbb{R}^n$ of the origin and an isomorphism $\psi : (U, p) \rightarrow (V, 0)$ satisfying the following condition. There exist a positive constant $\kappa > 0$ and a compact neighborhood $K \subset V$ of the origin such that for any $x_1, x_2 \in \psi(A \cap U) \cap K$ there exists a subanalytic curve l in $\psi(A \cap U)$ joining x_1 and x_2 and satisfying the estimate

$$|l| \leq \kappa |x_1 - x_2|,$$

where $|l|$ stands for the length of l .

The set A is said to be 1-regular if it is 1-regular at any point $p \in X$.

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a stratification of A . Then $\{A_\alpha\}_{\alpha \in \Lambda}$ is called a *1-regular stratification* if each stratum is 1-regular, connected and relatively compact. A 1-regular stratification always exists thanks to the following proposition due to K. Kurdyka.

Proposition 2.6 ([8]).

1. Let $Z \subset X$ be a locally closed subanalytic subset and $\{X_\alpha\}$ a stratification of Z . There exists a 1-regular stratification of Z finer than $\{X_\alpha\}$.
2. Let $U \subset X$ be a subanalytic open set. There exists an open covering $\{U_j\}_{j \in J}$ of U such that, for any $j \in J$, U_j is a subanalytic 1-regular set and that, for any compact set K in X , a finite number of U_j only intersects K .

§ 3. A stratified Whitney jet and a stratified ultradistribution

In this section, we first give the definitions of a stratified Whitney jet and a stratified ultradistribution. Then we investigate their properties and establish a relation between them.

§ 3.1. A stratified Whitney jet of Gevrey class

Let A be a locally closed subanalytic subset.

Definition 3.1. We say that $F \in \mathcal{J}_A(X)$ is a stratified Whitney jet of class $*$ over A if for any compact subanalytic set K there exists a subanalytic stratification $\{A_\alpha\}_{\alpha \in \Lambda}$ of A such that $j_{A_\alpha \cap K, A}(F) \in \mathcal{W}_{A_\alpha \cap K}^*(X)$ holds for any $\alpha \in \Lambda$ with $A_\alpha \subset A$.

We denote by $\mathcal{SW}_A^*(X)$ the set of stratified Whitney jets of class $*$ over A . If $\{A_\alpha\}_{\alpha \in \Lambda}$ is a stratification of A , then we denote by $\mathcal{SW}_{\{A_\alpha\}}^*(X)$ the subset of $\mathcal{SW}_A^*(X)$ defined by:

$$\mathcal{SW}_{\{A_\alpha\}}^*(X) = \{F \in \mathcal{J}_A(X); j_{A_\alpha, A}(F) \in \mathcal{W}_{A_\alpha}^*(X) \text{ for any } \alpha \in \Lambda \text{ with } A_\alpha \subset A\}.$$

The following proposition is fundamental to understand a stratified Whitney jet.

Proposition 3.2. *Let A be a locally closed subanalytic set in X and $\{A_\alpha\}$ a 1-regular stratification of A . Then we have $\mathcal{SW}_A^*(X) = \mathcal{SW}_{\{A_\alpha\}}^*(X)$. In particular, if A is 1-regular, then $\mathcal{SW}_A^*(X) = \mathcal{W}_A^*(X)$.*

The reason why we introduce a stratified Whitney jet is explained by the following proposition.

Proposition 3.3. *Let A_1 and A_2 be locally closed subanalytic subsets in X and set $A = A_1 \cup A_2$. Assume that both A_1 and A_2 are either closed or open in A .*

1. *The sequence of Abelian groups*

$$(3.1.1) \quad 0 \rightarrow \mathcal{SW}_A^*(X) \rightarrow \mathcal{SW}_{A_1}^*(X) \oplus \mathcal{SW}_{A_2}^*(X) \rightarrow \mathcal{SW}_{A_1 \cap A_2}^*(X)$$

is exact.

2. *If $A_1 \cap A_2$ is 1-regular or if $\dim X \leq 2$ and A is closed, then the third arrow of (3.1.1) is surjective.*

§ 3.2. A stratified ultradistribution

The vector space $\mathcal{SW}_A^*(X)$ of stratified Whitney jets has also a natural locally convex topology. We introduce, in the subsection, the set of stratified ultradistributions which is the strong dual of $\mathcal{SW}_A^*(X)$ for a compact subanalytic set A .

Let X be a real analytic manifold and A a closed subanalytic subset in X . For a stratification $\{A_\alpha\}$ of A , let us define stratified ultradistributions along $\{A_\alpha\}$.

Definition 3.4. An ultradistribution $u \in \mathcal{D}b^*(X)$ is said to be stratified along $\{A_\alpha\}$ if u can be written in the form:

$$u = \sum_{\alpha \text{ with } A_\alpha \subset A} u_\alpha, \quad u_\alpha \in \Gamma_{\overline{A_\alpha}}(X; \mathcal{D}b^*).$$

We set

$$\mathcal{S}Db_{\{A_\alpha\}}^*(X) := \{u \in \mathcal{D}b^*(X); u \text{ is stratified along } \{A_\alpha\}\} \subset \Gamma_A(X; \mathcal{D}b^*).$$

We define the set $\mathcal{S}Db_{[A]}^*(X)$ of *stratified ultradistributions of class $*$ along A* by

$$\mathcal{S}Db_{[A]}^*(X) := \varprojlim_{\substack{\text{stratification} \\ \{A_\alpha\} \text{ of } A}} \mathcal{S}Db_{\{A_\alpha\}}^*(X).$$

Note that, since for any stratification $\{A_\alpha\}$ there exists a 1-regular stratification finer than $\{A_\alpha\}$, we have

$$\mathcal{S}Db_{[A]}^*(X) = \varprojlim_{\substack{\text{1-regular stratification} \\ \{A_\alpha\} \text{ of } A}} \mathcal{S}Db_{\{A_\alpha\}}^*(X).$$

We have the similar propositions as those for a stratified Whitney jet.

Proposition 3.5. *Let A be a closed subanalytic subset in X and $\{A_\alpha\}$ a 1-regular stratification of A . Then we have $\mathcal{S}Db_{[A]}^*(X) = \mathcal{S}Db_{\{A_\alpha\}}^*(X)$. In particular, if A is 1-regular, we have $\mathcal{S}Db_{[A]}^*(X) = \Gamma_A(X; \mathcal{D}b^*)$.*

Proposition 3.6. *Let A_1 and A_2 be closed subanalytic sets. If $A_1 \cap A_2$ is 1-regular or if $\dim X \leq 2$ holds, then the sequence of Abelian groups*

$$0 \rightarrow \mathcal{S}Db_{[A_1 \cap A_2]}^*(X) \rightarrow \mathcal{S}Db_{[A_1]}^*(X) \oplus \mathcal{S}Db_{[A_2]}^*(X) \rightarrow \mathcal{S}Db_{[A_1 \cup A_2]}^*(X) \rightarrow 0$$

is exact.

These results come from the corresponding ones in the previous subsection by the following duality.

Theorem 3.7. *Let X be an orientable real analytic manifold, and $A \subset X$ a compact subanalytic set. Then, algebraically, we have*

$$\mathcal{S}Db_{[A]}^*(X) \simeq (\mathcal{S}W_A^*(X))'$$

where $(\mathcal{S}W_A^(X))'$ denotes the topological dual space of $\mathcal{S}W_A^*(X)$.*

§ 4. Sheaves of Gevrey classes on subanalytic sites

§ 4.1. The sheaf of the stratified Whitney jets on X_{sa}

Let X be a real analytic manifold. The presheaf of stratified Whitney jets of class $*$ is defined by

$$\mathcal{SW}_{X_{sa}}^*(U) := \mathcal{SW}_U^*(X),$$

where U is a subanalytic open subset of X . Note that, since $\Gamma(\overline{U}; \mathcal{D}_X)$ acts on $\mathcal{SW}_U^*(X)$, $\mathcal{SW}_{X_{sa}}^*$ is a $\varrho! \mathcal{D}_X$ -module where $\mathcal{D}_X$ is the sheaf of linear differential operators in X and $\varrho : X \rightarrow X_{sa}$ is the canonical morphism (see [5] and [11]).

Propositions 3.2 and 3.3 can be rephrased in terms of sheaves on subanalytic sites as follows.

Proposition 4.1. *The presheaf $\mathcal{SW}_{X_{sa}}^*$ of stratified Whitney jets is a sheaf on the subanalytic site X_{sa} . If $U \subset X$ is a 1-regular open subanalytic set, then we have*

$$\mathcal{SW}_{X_{sa}}^*(U) = \mathcal{W}_U^*(X) \simeq \mathcal{W}_{\overline{U}}^*(X) \simeq \frac{\mathcal{C}^*(X)}{\mathcal{I}_{X, \overline{U}}^*(X)},$$

where $\mathcal{I}_{X, \overline{U}}^*$ denotes the subsheaf of $\mathcal{C}^*$ consisting of functions vanishing on $\overline{U}$ up to infinite order.

We define the presheaf $\mathcal{W}_{X_{sa}}^*$ by $\mathcal{W}_{X_{sa}}^*(U) := \mathcal{W}_U^*(X)$ for a subanalytic open subset U .

Corollary 4.2. *We have $\mathcal{W}_{X_{sa}}^{*a} \simeq \mathcal{SW}_{X_{sa}}^*$ where $\mathcal{W}_{X_{sa}}^{*a}$ denotes the sheafification of the presheaf $\mathcal{W}_{X_{sa}}^*$ on the subanalytic site X_{sa} .*

§ 4.2. A tempered-stratified ultradistribution

For $U \subset X$ a subanalytic open set, we define the set of *tempered-stratified ultradistributions* on U as

$$\mathcal{Db}_{X_{sa}}^{*ts}(U) := \frac{\mathcal{SDb}_{[X]}^*(X)}{\mathcal{SDb}_{[X \setminus U]}^*(X)} = \frac{\mathcal{Db}^*(X)}{\mathcal{SDb}_{[X \setminus U]}^*(X)}.$$

Theorem 4.3. *Let U be an open subanalytic subset of X .*

1. *The ring $\Gamma(\overline{U}; \mathcal{D}_X)$ acts on $\mathcal{Db}_{X_{sa}}^{*ts}(U)$.*
2. *Let V be an open subanalytic subset of X . Then we have the following exact sequence.*

$$\mathcal{Db}_{X_{sa}}^{*ts}(U \cup V) \rightarrow \mathcal{Db}_{X_{sa}}^{*ts}(U) \oplus \mathcal{Db}_{X_{sa}}^{*ts}(V) \rightarrow \mathcal{Db}_{X_{sa}}^{*ts}(U \cap V) \rightarrow 0.$$

Further, if $\dim X \leq 2$, then the first morphism of the above sequence is injective.

3. If $X \setminus U$ is 1-regular, then $\mathcal{D}b_{X_{sa}}^{*ts}(U)$ coincides with the sections of tempered ultra-distributions of class $*$ on U , that is,

$$\mathcal{D}b_{X_{sa}}^{*ts}(U) = \mathcal{D}b_X^{*t}(U) .$$

As an immediate corollary of the above theorem, we have

Corollary 4.4. *If $\dim X \leq 2$, then $\mathcal{D}b_{X_{sa}}^{*ts}$ is a sheaf on the subanalytic site X_{sa} .*

§ 5. Higher dimensional case

Let $\mathcal{G}$ be a sheaf on the subanalytic site X_{sa} , and let $r_{V,U}$ denote the restriction morphism of $\mathcal{G}$ for $V \subset U$ open subanalytic subsets. Assume that $\mathcal{G}$ satisfies the following conditions.

1. If an open subanalytic subset U has smooth boundary, then $\mathcal{G}(U) \simeq \mathcal{D}b^{*t}(U)$.
2. For any open subanalytic subset U , the restriction morphism $r_{U,X} : \mathcal{G}(X) \rightarrow \mathcal{G}(U)$ is surjective, i.e. $\mathcal{G}$ is quasi-injective.

If $\dim X \leq 2$, our $\mathcal{D}b_{X_{sa}}^{*ts}$ satisfies Conditions 1. and 2. above. Moreover the sheaf of tempered distributions on X_{sa} also satisfies Condition 2. Note that, as tempered distributions themselves form a sheaf on the subanalytic site X_{sa} , it does not make sense to ask for Condition 1. For the possibility of a general construction of a sheaf of tempered ultradistributions on X_{sa} , we have the following result.

Proposition 5.1. *If $\dim X > 2$, then there exists no sheaf $\mathcal{G}$ on the subanalytic site X_{sa} that satisfies the above conditions 1. and 2.*

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