

Notes on a certain class of analytic functions

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Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, the subclass $\mathcal{B}$ of $\mathcal{A}$ concerned with the class of uniformly convex functions or the class $\mathcal{S}_p$ is defined. By virtue of some properties of uniformly convex functions and the class $\mathcal{S}_p$, an extreme function of the class $\mathcal{B}$ and its power series are considered.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at ζ in $\mathbb{U}$ onto the convex arc (or the starlike arc with respect to $f(\zeta)$). These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [4] and [5], that is, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).$$

It is independently studied by Ma and Minda [3]. But the one variable characterization for the class $\mathcal{UST}$ is still open. Further, a function $f(z) \in \mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Rønning [4]. We easily know that the relation $f(z) \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_p$. In view of these classes, we introduce the subclass $\mathcal{B}$ of $\mathcal{A}$ consisting

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of all functions $f(z)$ which satisfy

$$\operatorname{Re} \left(\frac{z}{f(z)} \right) > \left| \frac{z}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).$$

We try to derive some properties of functions $f(z)$ belonging to the class $\mathcal{B}$.

Remark 1.1. For $f(z) \in \mathcal{B}$, we write $w(z) = \frac{f(z)}{z} = u + iv$, then w lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

Example 1.1. Let us consider the function $f(z) \in \mathcal{A}$ as given by

$$f(z) = z + \frac{1}{\sqrt{2}}z^2.$$

Then we easily see that the function $f(z)$ is not univalent. And $\frac{f(z)}{z}$ maps $\mathbb{U}$ onto the circular domain which is 1 as the center and $\frac{1}{\sqrt{2}}$ as the radius, that is, $f(z) \in \mathcal{B}$.

2 An extreme function for the class $\mathcal{B}$

In this section, we would like to exhibit an extreme function of the class $\mathcal{B}$ and its power series. For our results, we need to recall here some properties of the class $\mathcal{S}_p$.

Lemma 2.1. (Rønning [4]). The extremal function $f(z)$ for the class $\mathcal{S}_p$ is given by

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2.$$

By using the expansion of logarithmic part of $\frac{zf'(z)}{f(z)}$ in Lemma 2.1, we get

Lemma 2.2. (Ma and Minda [3]). The power series of $\frac{zf'(z)}{f(z)}$ is following

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \\ &= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n. \end{aligned}$$

The digamma function $\psi(z+1)$ is defined by

$$\psi(z+1) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \psi(z) + \frac{1}{z},$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

When z is natural number, we obtain

$$\psi(n+1) = \sum_{k=1}^n \frac{1}{k} - \gamma \quad (n \in \mathbb{N}),$$

where γ is Euler's constant and $-\gamma = \psi(1)$.

From Remark 1.1 and Lemma 2.1, we have the first result for the class $\mathcal{B}$.

Theorem 2.1. *The extreme function $f(z)$ for the class $\mathcal{B}$ is given by*

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.$$

Proof. Let us consider the function $\frac{f(z)}{z}$ as given by

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}.$$

It suffices to show that $\frac{f(z)}{z}$ maps $\mathbb{U}$ onto the interior of the domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0,$$

implying that $\frac{f(z)}{z}$ maps the unit circle onto the boundary of the domain. Taking $z = e^{i\theta}$, we obtain that

$$\begin{aligned} \frac{1}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2} &= \frac{1}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + e^{i\frac{\theta}{2}}}{1 - e^{i\frac{\theta}{2}}} \right) \right)^2} \\ &= \frac{1}{1 + \frac{2}{\pi^2} \left(\log i - \log \left(\tan \frac{\theta}{4} \right) \right)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\frac{1}{2} + \frac{2}{\pi^2} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^2 - i \frac{2}{\pi} \log \left(\tan \frac{\theta}{4} \right)} \\
&= \frac{\frac{1}{2} + \frac{2}{\pi^2} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^2}{\frac{1}{4} + \frac{6}{\pi^2} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^4} \\
&\quad + i \frac{\frac{2}{\pi} \log \left(\tan \frac{\theta}{4} \right)}{\frac{1}{4} + \frac{6}{\pi^2} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^4}.
\end{aligned}$$

Writing $\frac{f(z)}{z} = u + iv$, we see that

$$\log \left(\tan \frac{\theta}{4} \right) = \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}.$$

Thus we have

$$\begin{aligned}
v &= \frac{\frac{2}{\pi} \log \left(\tan \frac{\theta}{4} \right)}{\frac{1}{4} + \frac{6}{\pi^2} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left(\log \left(\tan \frac{\theta}{4} \right) \right)^4} \\
&= \frac{\frac{2}{\pi} \pi(u \pm \sqrt{u^2 - v^2})}{\frac{1}{4} + \frac{6}{\pi^2} \left(\frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v} \right)^2 + \frac{4}{\pi^4} \left(\frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v} \right)^4}.
\end{aligned}$$

Therefore, we arrive that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 = 0.$$

This completes the proof of the theorem. $\square$

Considering the power series of the function $f(z)$ in Theorem 2.1, we derive

Theorem 2.2. *The power series of the extreme function for the class $\mathcal{B}$ is given by*

$$\begin{aligned}
f(z) &= \frac{z}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2} \\
&= z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left(\frac{8}{\pi^2} \right)^p \sum_{\substack{p \\ \sum_{j=1}^p m_j = n-1}} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^n \quad (m_j \in \mathbb{N}).
\end{aligned}$$

Proof. Let us suppose that

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}$$

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

$$\begin{aligned} \frac{f(z)}{z} &= \frac{1}{1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n} \\ &= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n + \left(\frac{8}{\pi^2} \right)^2 \left\{ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n \right\}^2 \\ &\quad - \left(\frac{8}{\pi^2} \right)^3 \left\{ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n \right\}^3 + \dots \\ &\quad + (-1)^n \left(\frac{8}{\pi^2} \right)^n \left\{ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{2k-1} \right) z^n \right\}^n + \dots \\ &= 1 - \frac{8}{\pi^2} \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) z \\ &\quad + \left\{ -\frac{8}{\pi^2} \left(\frac{1}{2} \sum_{k=1}^2 \frac{1}{2k-1} \right) + \left(\frac{8}{\pi^2} \right)^2 \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) \right\} z^2 \\ &\quad + \left[-\frac{8}{\pi^2} \left(\frac{1}{3} \sum_{k=1}^3 \frac{1}{2k-1} \right) + \left(\frac{8}{\pi^2} \right)^2 \left\{ \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) \left(\frac{1}{2} \sum_{k=1}^2 \frac{1}{2k-1} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{1}{2} \sum_{k=1}^2 \frac{1}{2k-1} \right) \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) \right\} - \left(\frac{8}{\pi^2} \right)^3 \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right)^3 \right] z^3 \\ &\quad + \dots \\ &\quad + \left\{ -\frac{8}{\pi^2} \sum_{\sum_{j=1}^1 m_j = n} \left(\prod_{j=1}^1 \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) + \left(\frac{8}{\pi^2} \right)^2 \sum_{\sum_{j=1}^2 m_j = n} \left(\prod_{j=1}^2 \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) \right. \\ &\quad \left. + \left(\frac{8}{\pi^2} \right)^3 \sum_{\sum_{j=1}^3 m_j = n} \left(\prod_{j=1}^3 \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) + \dots + \left(\frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^p m_j = n} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) \right. \\ &\quad \left. + \dots + \left(\frac{8}{\pi^2} \right)^n \sum_{\sum_{j=1}^n m_j = n} \left(\prod_{j=1}^n \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) \right\} z^n + \dots \quad (m_j \in \mathbb{N}) \end{aligned}$$

$$\begin{aligned}
&= 1 - \frac{8}{\pi^2} \left(\frac{1}{1} \sum_{k=1}^1 \frac{1}{2k-1} \right) z + \sum_{p=1}^2 (-1)^p \left(\frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^p m_j=2} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^2 \\
&\quad + \sum_{p=1}^3 (-1)^p \left(\frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^p m_j=3} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^3 + \dots \\
&\quad + \sum_{p=1}^n (-1)^p \left(\frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^p m_j=n} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^n + \dots \\
&= 1 + \sum_{n=1}^{\infty} \sum_{p=1}^n (-1)^p \left(\frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^p m_j=n} \left(\prod_{j=1}^p \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^n.
\end{aligned}$$

This completes the proof of the theorem. $\square$

By using digamma function in Theorem 2.2, we have

Corollary 2.1. *The power series of the extreme function for the class $\mathcal{B}$ is rewritten as following*

$$\begin{aligned}
f(z) &= z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left(\frac{8}{\pi^2} \right)^p \times \\
&\quad \sum_{\sum_{j=1}^p m_j=n-1} \left\{ \prod_{j=1}^p \frac{1}{m_j} \left(\psi(m_j+1) - \frac{1}{2} \psi([m_j/2]+1) - \frac{1}{2} \psi(1) \right) \right\} z^n \quad (m_j \in \mathbb{N}),
\end{aligned}$$

where $[\]$ is the Gauss symbol

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