

Tropical Jacobian and the generic fiber of the ultra-discrete periodic Toda lattice are isomorphic

By

Rei INOUE* and Tomoyuki TAKENAWA**

Abstract

We prove that the general isolevel set of the ultra-discrete periodic Toda lattice is isomorphic to the tropical Jacobian associated with the tropical spectral curve. This result implies that the theta function solution obtained in the authors' previous paper is the complete solution. We also propose a method to solve the initial value problem.

§ 1. Introduction

§ 1.1. Background and aim

The periodic box and ball system (pBBS) [13] and the ultra-discrete periodic Toda lattice (UD-pToda) [9] are important examples of integrable cellular automata, and they are closely related [4]. The initial value problem of the pBBS was solved in many ways: by the elementary combinatorics [7], by the combinatorial Bethe ansatz associated with the quantum group [6], and by the ultradiscretization of the solution to the discrete periodic Toda lattice [3]. On the way to solve this problem in [6, 5, 3] they introduced what now called the tropical Jacobian and the tropical Riemann theta function, and they found that there is a bijection between the general isolevel set of the pBBS and the lattice points of the tropical Jacobian.

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*Faculty of Pharmaceutical Sciences, Suzuka University of Medical Science,
3500-3 Minami-tamagaki, Suzuka, Mie, 513-8670, Japan.

e-mail: reiiy@suzuka-u.ac.jp

**Faculty of Marine Technology, Tokyo University of Marine Science and Technology,
2-1-6 Etchu-jima, Koto-ku, Tokyo, 135-8533, Japan.

e-mail: takenawa@kaiyodai.ac.jp

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In [1] we explored the UD-pToda and the pBBS from the viewpoint of the tropical geometry. (See [8] and references therein for the detail of the tropical geometry.) We conjectured that there is an isomorphism between the general isolevel set of the UD-pToda and the tropical Jacobian of the tropical spectral curve. The aim in this article is to prove this conjecture (Theorem 1.3), as a sequel of [1, 2]. As a consequence, we solve the initial value problem of the UD-pToda.

§ 1.2. Ultra-discrete periodic Toda lattice

Fix a positive integer g . The ultra-discrete periodic Toda lattice (UD-pToda) is given by the phase space $\mathcal{T}$:

$$\mathcal{T} = \{\tau = (Q_0, \dots, Q_g, W_0, \dots, W_g) \in \mathbb{R}^{2(g+1)} \mid \sum_{n=0}^g Q_n < \sum_{n=0}^g W_n\},$$

and the piecewise-linear map on $\mathcal{T}$:

$$(1.1) \quad T : \mathcal{T} \rightarrow \mathcal{T}; \quad (Q_n^t, W_n^t)_{n=0, \dots, g} \mapsto (Q_n^{t+1}, W_n^{t+1})_{n=0, \dots, g},$$

where

$$(1.2) \quad \begin{aligned} Q_n^{t+1} &= \min[W_n^t, Q_n^t - X_n^t], & W_n^{t+1} &= Q_{n+1}^t + W_n^t - Q_n^{t+1}, \\ X_n^t &= \min_{k=0, \dots, g} \left[\sum_{l=1}^k (W_{n-l}^t - Q_{n-l}^t) \right]. \end{aligned}$$

Here we assume the periodicity $Q_{n+g+1}^t = Q_n^t$ and $W_{n+g+1}^t = W_n^t$. The map T has conserved quantities given by the tropical polynomials $C_i(\tau)$ ($i = -1, 0, \dots, g$) on $\mathcal{T}$, which come from the coefficients of the characteristic polynomial of the Lax matrix through the ultra-discrete limit. For example, we have

$$(1.3) \quad \begin{aligned} C_{-1}(\tau) &= \sum_{n=0}^g (Q_n + W_n), & C_0(\tau) &= \min \left[\sum_{n=0}^g Q_n, \sum_{n=0}^g W_n \right], \\ C_{g-1}(\tau) &= \min \left[\min_{i \neq j} (Q_i + Q_j), \min_{i \neq j} (W_i + W_j), \min_{j \neq i-1, i} (Q_i + W_j) \right], \\ C_g(\tau) &= \min_{n=0, \dots, g} [Q_n, W_n]. \end{aligned}$$

See [1] for a detail of $C_i(\tau)$. For a fixed $C = (C_{-1}, C_0, \dots, C_g) \in \mathbb{R}^{g+2}$, we define the isolevel set $\mathcal{T}_C$ as

$$(1.4) \quad \mathcal{T}_C = \{\tau \in \mathcal{T} \mid C_i(\tau) = C_i \ (i = -1, 0, \dots, g)\}.$$

§ 1.3. Bilinear equation

Let $\mathcal{S}_t$ ($t \in \mathbb{Z}$) be a subset of infinite dimensional space:

$$\mathcal{S}_t = \{T_n^t \in \mathbb{R} \mid n \in \mathbb{Z}\},$$

where T_n^t has a quasi-periodicity; i.e. T_n^t satisfies $T_{n+g+1}^t = T_n^t + c_n^t$, where c_n^t satisfies

$$(1.5) \quad c_n^t = an + bt + c$$

for some $a, b, c \in \mathbb{R}$. Fix $L \in \mathbb{R}$ such that $2b - a < (g + 1)L$ and define a map ϕ from $\mathcal{S}_t \times \mathcal{S}_{t+1}$ to $\mathcal{S}_{t+1} \times \mathcal{S}_{t+2}$ as $\phi : (T_n^t, T_n^{t+1})_{n \in \mathbb{Z}} \mapsto (T_n^{t+1}, T_n^{t+2})_{n \in \mathbb{Z}}$ with

$$(1.6) \quad T_n^{t+2} = 2T_n^{t+1} - T_n^t + X_{n+1}^t,$$

where we define a function on $\mathcal{S}_t \times \mathcal{S}_{t+1}$:

$$(1.7) \quad X_n^t = \min_{l=0, \dots, g} [lL + 2T_{n-l-1}^{t+1} + T_n^t + T_{n-1}^t - (2T_{n-1}^{t+1} + T_{n-l}^t + T_{n-l-1}^t)].$$

Proposition 1.1. *[2, Lemma 3.2, Proposition 3.4]*

(i) Let σ_t be a map $\sigma_t : \mathcal{S}_t \times \mathcal{S}_{t+1} \rightarrow \mathcal{T}$ given by

$$(1.8) \quad W_n^t = L + T_{n-1}^{t+1} + T_{n+1}^t - T_n^t - T_n^{t+1} + d, \quad Q_n^t = T_{n-1}^t + T_n^{t+1} - T_{n-1}^{t+1} - T_n^t + d,$$

where $d \in \mathbb{R}$. Then the following diagram is commutative:

$$(1.9) \quad \begin{array}{ccc} \mathcal{S}_t \times \mathcal{S}_{t+1} & \xrightarrow{\sigma_t} & \mathcal{T} \\ \downarrow \phi & & \downarrow T \\ \mathcal{S}_{t+1} \times \mathcal{S}_{t+2} & \xrightarrow{\sigma_{t+1}} & \mathcal{T} \end{array}.$$

(ii) The UD-pToda (1.2) can be transformed into a Hirota type bilinear equation for the quasi-periodic function T_n^t ,

$$(1.10) \quad T_n^{t-1} + T_n^{t+1} = \min[2T_n^t, T_{n-1}^{t+1} + T_{n+1}^{t-1} + L].$$

§ 1.4. Main result

We assume a generic condition for $C = (C_{-1}, C_0, \dots, C_g) \in \mathbb{R}^{g+2}$:

$$(1.11) \quad C_{-1} > 2C_0, \quad C_i + C_{i+2} > 2C_{i+1} \quad (i = 0, \dots, g-2), \quad C_{g-1} > 2C_g.$$

Define $L, \lambda_0, \lambda_1, \dots, \lambda_g$ and $p_1, \dots, p_g$ by

$$(1.12) \quad L = C_{-1} - 2(g+1)C_g, \quad \lambda_0 = C_g, \quad \lambda_i = C_{g-i} - C_{g-i+1} \quad (1 \leq i \leq g),$$

$$(1.13) \quad p_0 = L, \quad p_i = L - 2 \sum_{j=1}^g \min[\lambda_i - \lambda_0, \lambda_j - \lambda_0] \quad (1 \leq i \leq g).$$

Fix $C \in \mathbb{R}^{g+2}$ which satisfies the generic condition (1.11). Then we have

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_g, \quad p_0 > p_1 > \cdots > p_g > 0.$$

Define a positive definite symmetric matrix $K \in M_g(\mathbb{R})$ as

$$(1.14) \quad K_{ij} = \begin{cases} p_{i-1} + p_i + 2(\lambda_i - \lambda_{i-1}) & (i = j) \\ -p_i & (j = i + 1) \\ -p_j & (i = j + 1) \\ 0 & (\text{otherwise}). \end{cases}$$

In fact, K is the period matrix for the tropical curve Γ_C determined by C . See [1] for the definition of Γ_C . In parallel with the complex algebraic geometry, we have the tropical Jacobian $J(K) = \mathbb{R}^g / (\mathbb{Z}^g K)$ and the tropical Riemann theta function $\Theta(\mathbf{Z}) = \Theta(\mathbf{Z}; K)$ on $\mathbb{R}^g$:

$$\Theta(\mathbf{Z}) = \min_{\mathbf{m} \in \mathbb{Z}^g} \left[\frac{1}{2} \mathbf{m} K \mathbf{m}^\perp + \mathbf{m} \mathbf{Z}^\perp \right], \quad (\mathbf{Z} \in \mathbb{R}^g).$$

Proposition 1.2. [2, Corollary 2.13, Theorem 3.5]

(i) Set $\vec{\lambda} = (\lambda_1 - \lambda_0, \lambda_2 - \lambda_1, \dots, \lambda_g - \lambda_{g-1})$ and $\mathbf{e}_1 = (1, 0, \dots, 0)$. For $\mathbf{Z}_0 \in \mathbb{R}^g$, the function T_n^t given by

$$(1.15) \quad T_n^t = \Theta(\mathbf{Z}_0 - nL\mathbf{e}_1 + t\vec{\lambda}),$$

satisfies (1.10).

(ii) Let ι_t be the map $\mathbb{R}^g \rightarrow \mathcal{S}_t \times \mathcal{S}_{t+1}$ defined by

$$\mathbf{Z}_0 \mapsto (T_n^t = \Theta(\mathbf{Z}_0 - nL\mathbf{e}_1 + t\vec{\lambda}), T_n^{t+1} = \Theta(\mathbf{Z}_0 - nL\mathbf{e}_1 + (t+1)\vec{\lambda}))_{n \in \mathbb{Z}},$$

then the following diagram is commutative:

$$(1.16) \quad \begin{array}{ccccc} \mathbb{R}^g & \xrightarrow{\iota_t} & \mathcal{S}_t \times \mathcal{S}_{t+1} & \xrightarrow{\sigma_t} & \mathcal{T}_C \\ \downarrow \text{id.} & & \downarrow \phi & & \downarrow T, \\ \mathbb{R}^g & \xrightarrow{\iota_{t+1}} & \mathcal{S}_{t+1} \times \mathcal{S}_{t+2} & \xrightarrow{\sigma_{t+1}} & \mathcal{T}_C \end{array}$$

where we have $d = C_g$ for σ_t (1.8). (See Remark 2.) Namely, (1.15) gives a solution to (1.2) through (1.8).

Our goal of this article is to show the following theorem

Theorem 1.3. Let $\iota_\sigma : J(K) \rightarrow \mathcal{T}_C$ be the map induced by $\sigma_t \circ \iota_t : \mathbb{R}^g \rightarrow \mathcal{T}_C$. Then the map ι_σ is isomorphic.

We will prove Theorem 1.3 in the following order: we show $\sigma_t|_{\text{Im} \iota_t}$ is injective (Proposition 2.1) in §2.2, and ι_t is injective (Proposition 2.5) in §2.3. Finally we prove Theorem 1.3 in §2.4. To help to understand the proof of the theorem (especially for

§2.3), we show some idea and an example in §2.1. In §2.5 we summarize the initial value problem for UD-pToda.

§ 1.5. Notations and Remarks

We use the following notations of vectors in $\mathbb{R}^g$:

$$\begin{aligned}\vec{\lambda} &= (\lambda_1 - \lambda_0, \lambda_2 - \lambda_1, \dots, \lambda_g - \lambda_{g-1}), \\ \vec{g} &= (g, g-1, \dots, 1), \\ \mathbf{e}_i &: \text{the } i\text{-th vector of standard basis of } \mathbb{R}^g, \\ \mathbb{I} &= (1, 1, \dots, 1) = \mathbf{e}_1 + \dots + \mathbf{e}_g, \\ \mathbb{I}_k &= (1, \dots, 1, 0, \dots, 0) = \mathbf{e}_1 + \dots + \mathbf{e}_k.\end{aligned}$$

Remark 1. For $C \in \mathbb{R}^{g+2}$ with the condition (1.11), we have the following relations:

$$(1.17) \quad L > 2 \sum_{i=1}^g (\lambda_i - \lambda_0), \quad \vec{g} \cdot \vec{\lambda}^\perp = \sum_{i=1}^g (\lambda_i - \lambda_0) = C_0 - (g+1)C_g$$

and

$$\sum_{j=1}^g K_{ij} > 0, \quad \mathbb{I}K = p_g \mathbf{e}_g + L \mathbf{e}_1 + 2\vec{\lambda}, \quad \vec{g}K = (g+1)L \mathbf{e}_1.$$

Remark 2. The tropical Riemann theta function $\Theta(\mathbf{Z})$ is quasi-periodic:

$$\Theta(\mathbf{Z} + \mathbb{I}K) = -\frac{1}{2} \mathbb{I}K \mathbb{I}^\perp - \mathbb{I} \mathbf{Z}^\perp + \Theta(\mathbf{Z}) \quad (\mathbb{I} \in \mathbb{Z}^g)$$

and even: $\Theta(-\mathbf{Z}) = \Theta(\mathbf{Z})$. Thus, if we set T_n^t as (1.15), it satisfies the following quasi-periodicity;

$$(1.18) \quad T_{n+g+1}^t = T_n^t + c_n^t, \quad c_n^t = an + bt + c = \vec{g} \cdot (\mathbf{Z}_0 - nL \mathbf{e}_1 + \vec{\lambda}t - \frac{1}{2} \vec{g}K)^\perp,$$

which yields

$$a = -gL, \quad b = \sum_{i=1}^g (\lambda_i - \lambda_0).$$

Then we obtain $d = C_g$ at (1.8), via (1.12) and

$$C_{-1} = \sum_{n=0}^g (Q_n^t + W_n^t) = (g+1)L + 2(g+1)d + a.$$

§ 2. Proof of Theorem 1.3

In this section, we assume $C_g = 0$ without loss of generality. For $\mathbf{m} = (m_1, m_2, \dots, m_g) \in \mathbb{Z}^g$, set the fundamental region $D_{\mathbf{m}}$ of $\Theta(\mathbf{Z})$ as $D_{\mathbf{m}} = \{\mathbf{Z} \in \mathbb{R}^g ; \Theta(\mathbf{Z}) = \frac{1}{2}\mathbf{m}K\mathbf{m}^\perp + \mathbf{m}\mathbf{Z}^\perp\}$ which is explicitly written as

$$(2.1) \quad D_{\mathbf{m}} = \{\mathbf{Z} \in \mathbb{R}^g \mid -\mathbf{l}\mathbf{Z}^\perp \leq \mathbf{l}K(\mathbf{m} + \frac{1}{2}\mathbf{l})^\perp$$

for any $\mathbf{l} = \pm(\mathbf{e}_j + \mathbf{e}_{j+1} + \dots + \mathbf{e}_k)$ and $1 \leq j \leq k \leq g\}$

from Lemma 2.9 of [2]. For a fixed $\mathbf{Z}_0 \in \mathbb{R}^g$, we write $\mathbf{Z}_n^t = \mathbf{Z}_0 - nL\mathbf{e}_1 + t\vec{\lambda}$. Due to the structure of the fundamental regions, it is easy to see that if $\mathbf{Z}_n^t \in D_{\mathbf{m}}$, then

$$(2.2) \quad \mathbf{Z}_{n+g+1}^t \in D_{\mathbf{m}+\vec{g}}, \quad \mathbf{Z}_n^t = \mathbf{Z}_{n+g+1}^t \pmod{\mathbb{Z}^g K}.$$

§ 2.1. Idea and example

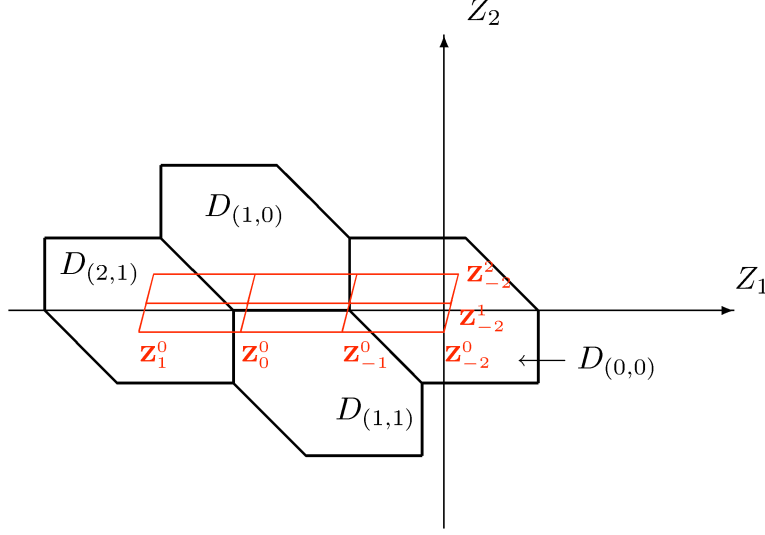
We first recall the relation between the UD-pToda and the pBBS. The pBBS is a cellular automaton that the finite number of balls move in a periodic array of L boxes each of which has one ball at most [13]. This system has conserved quantities parameterized by a non-decreasing array $\lambda = (\lambda_1, \dots, \lambda_g) \in (\mathbb{Z}_{>0})^g$. We write 0 and 1 for “an empty box” and “an occupied box” respectively. Let $B_L \simeq \{0, 1\}^{\times L}$ be the phase space of L -periodic BBS and $B_{L,\lambda} \subset B_L$ be a set of the states whose conserved quantity is λ . Then the injection $\beta : B_{L,\lambda} \hookrightarrow \mathcal{T}_C \cap \mathbb{Z}^{2g+2}$; $b \mapsto (Q_0, W_0, Q_1, W_1, \dots, Q_g, W_g)$ is defined as follows:

1. if the leftmost entry of b is 1, then set $Q_0 = \sharp$ (the first consecutive 1's from the left), otherwise set $Q_0 = 0$.
2. Set $W_i = \sharp$ (the $(i-1)$ -th consecutive 0's from the left) for $i = 0, \dots, g$. If $Q_0 > 0$, set $Q_i = \sharp$ (the $(i-1)$ -th consecutive 1's from the left), otherwise set $Q_i = \sharp$ (the i -th consecutive 1's from the left) for $i = 1, \dots, g$.

The following shows how β works;

$$\begin{array}{ccccccc} \underbrace{1 \dots 1}_{Q_0} \underbrace{0 \dots 0}_{W_0} & \dots & \underbrace{0 \dots 0}_{W_{g-1}} \underbrace{1 \dots 1}_{Q_g} & (W_g = 0) \\ \underbrace{0 \dots 0}_{W_0} \underbrace{1 \dots 1}_{Q_1} & \dots & \underbrace{1 \dots 1}_{Q_g} \underbrace{0 \dots 0}_{W_g} & (Q_0 = 0). \end{array}$$

The map β is not surjective but induces an isomorphism from $B_{L,\lambda}$ to $\mathcal{T}_C \cap \mathbb{Z}^{2g+2}$ modulo the shift $(Q_0, W_0, Q_1, W_1, \dots, Q_g, W_g) \mapsto (Q_1, W_1, \dots, Q_g, W_g, Q_0, W_0)$. See Section 4 of [1] for the details.

Figure 1. $D_{\mathbf{m}}$ and $\mathbf{Z}_n^t$

Let us consider the UD-pToda of $g = 2$ with the initial values

$$(2.3) \quad (Q_0, W_0, Q_1, W_1, Q_2, W_2) = (0, 1, 2, 7, 4, 0)$$

at $t = 0$. Thus we have

$$(X_0^0, X_1^0, X_2^0) = (-4, -3, 0), \quad L = 14, \quad \vec{\lambda} = (1, 4), \quad K = \begin{pmatrix} 26 & -10 \\ -10 & 20 \end{pmatrix}.$$

We assume that T_n^t is written as $T_n^t = \Theta(\mathbf{Z}_n^t = \mathbf{Z}_0 - nL\mathbf{e}_1 + \vec{\lambda}t)$ for all $n, t \in \mathbb{Z}$. From $X_2^0 = 0$ and (1.6), the points $(\mathbf{Z}_1^0, \Theta(\mathbf{Z}_1^0))$, $(\mathbf{Z}_1^1, \Theta(\mathbf{Z}_1^1))$, $(\mathbf{Z}_1^2, \Theta(\mathbf{Z}_1^2))$ are colinear and thus the points $\mathbf{Z}_1^0, \mathbf{Z}_1^1, \mathbf{Z}_1^2$ belong to the same region $D_{\mathbf{m}}$. Thus the points $\mathbf{Z}_{-2}^0, \mathbf{Z}_{-2}^1, \mathbf{Z}_{-2}^2$ belong to the same region $D_{\mathbf{m}-(2,1)}$ from (2.2). By assuming $\mathbf{m} = (2, 1)$, in this case we can fortunately find $\mathbf{Z}_0 = (-28, -3)$ by a heuristic way. See Fig. 1. The following is the time evolution of (2.3):

t	b^t	$(Q_0^t, W_0^t, Q_1^t, W_1^t, Q_2^t, W_2^t)$
0	${}_s01100000001111$	$(0, 1, 2, 7, 4, 0)$
1	${}_s10011111000000$	$(1, 2, 5, 6, 0, 0)$
2	${}_s01000000111110$	$(1, 6, 5, 1, 0, 1)$

where ${}_s$ denotes the place of Q_0^t in the corresponding state of the pBBS.

Observations. From Fig. 1 we see the following facts:

- $\mathbf{Z}_n^t \in D_{\mathbf{m}}$ implies $\mathbf{Z}_{n+1}^t \in D_{\mathbf{m}}, D_{\mathbf{m}+(1,0)}$ or $D_{\mathbf{m}+(1,1)}$ (Lemma 2.2),
- $\mathbf{Z}_{-1}^0, \mathbf{Z}_0^0 \in D_{(1,1)}$ while $Q_0^0 = W_2^0 = 0$ (Lemma 2.4).

From these observations we see the following: even if we do not know $\mathbf{Z}_0$, once the quantities of T_n^0 and the domain $D_{\mathbf{m}_n}$ to which $\mathbf{Z}_n^0$ belongs are given for all $n \in \mathbb{Z}/3\mathbb{Z}$, we have two independent linear equations for $\mathbf{Z}_0$. Actually, from $\mathbf{Z}_{-1}^0 \in D_{(1,1)}$ and $\mathbf{Z}_1^0 \in D_{(2,1)}$ we have

$$\begin{aligned} \frac{1}{2}(1,1)K(1,1)^\perp + (1,1)(\mathbf{Z}_0 + (L,0))^\perp &= T_{-1}^0, \\ \frac{1}{2}(2,1)K(2,1)^\perp + (2,1)(\mathbf{Z}_0 - (L,0))^\perp &= T_1^0, \end{aligned}$$

and $\mathbf{Z}_0$ is uniquely determined.

Therefore our problem is how to determine the region $D_{\mathbf{m}_n}$ to which $\mathbf{Z}_n^t$ belongs when T_n^t are given. For this purpose we investigate what happens when we replace $W_{n_0}^0$ with $\tilde{W}_{n_0}^0 = W_{n_0}^0 + \delta$ by using $\delta \gg 1$, for n_0 as $X_{n_0+1}^0 = 0$. We set $\tilde{W}_n^0 = W_n^0$ for $n \neq n_0$ and $\tilde{Q}_n^0 = Q_n^0$ for all n . We write $\tilde{\mathbf{Z}}_n^t$ for the corresponding points in $\mathbb{R}^g$.

In the previous example we have $n_0 = -2$ and set $\delta = 10$:

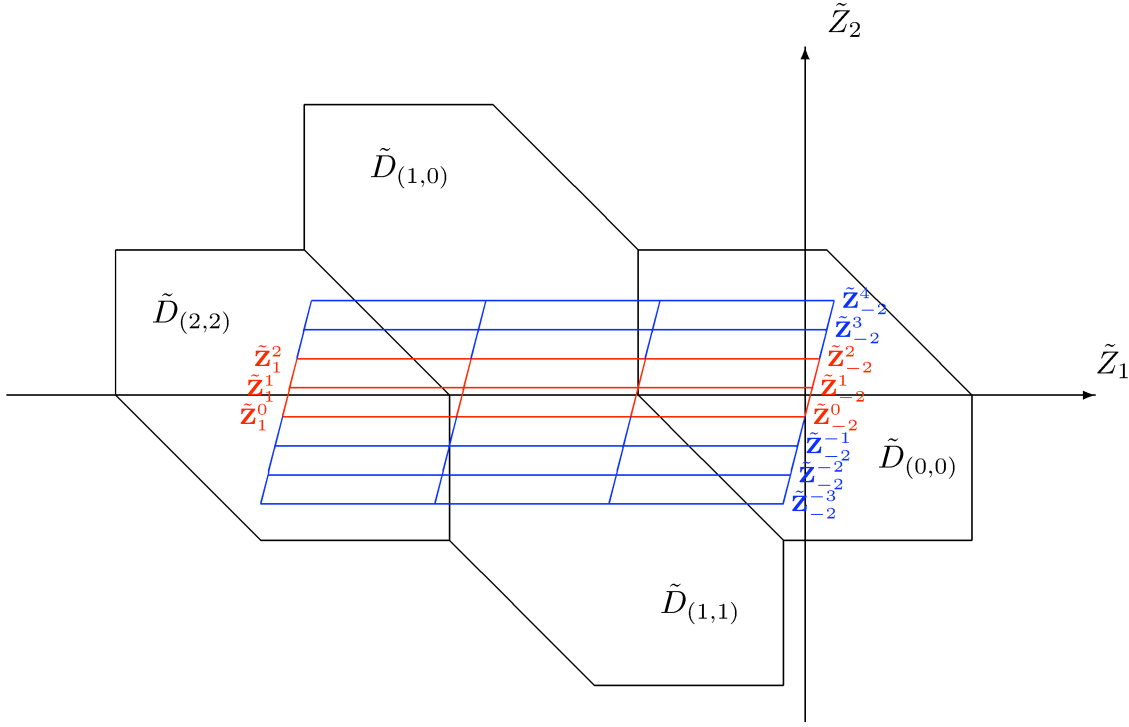
t	b^t	$(\tilde{Q}_0^t, \tilde{W}_0^t, \tilde{Q}_1^t, \tilde{W}_1^t, \tilde{Q}_2^t, \tilde{W}_2^t)$
-3	000111110000000000000000 _s 10	(1, 1, 0, 3, 5, 14)
-2	000000000011111000000000 _s 1	(1, 0, 0, 9, 5, 9)
-1	1000000000000000111110000 _s	(0, 0, 1, 14, 5, 4)
0	_s 01100000000000000000001111	(0, 1, 2, 17, 4, 0)
1	_s 100111110000000000000000	(1, 2, 5, 16, 0, 0)
2	_s 010000001111100000000000	(1, 6, 5, 11, 0, 1)
3	00 _s 10000000000011111000000	(1, 11, 5, 6, 0, 2)

Observations.

- By assuming $\tilde{\mathbf{Z}}_{-2}^0 \in \tilde{D}_{(0,0)}$, we can find $\tilde{\mathbf{Z}}_0 = (-48, -3)$, thus we have $\mathbf{Z}_{-2}^0 = \tilde{\mathbf{Z}}_{-2}^0$.
- For $t = 1, 2, 3$ we have $\mathbf{m}_{-2} = (0, 0)$ (by the assumption), $\mathbf{m}_{-1} = (0, 0)$, $\mathbf{m}_0 = (1, 0)$ and $\mathbf{m}_1 = (2, 1)$.

The second observation implies $\tilde{\mathbf{Z}}_0^t \in D_{(1,0)}$ and $\tilde{\mathbf{Z}}_1^t \in D_{(2,1)}$. Then $\tilde{\mathbf{Z}}_0$ is uniquely determined in the same way as the above, once $\{T_n^t\}_n$ is given. By using the first observation $\mathbf{Z}_0$ is uniquely fixed by $\tilde{\mathbf{Z}}_0$.

Remark 3. This idea of replacing $W_{n_0}^0$ with a huge number $W_{n_0}^0 + \delta$ (where $X_{n_0+1} = 0$) would be understandable by considering the corresponding pBBS. In the pBBS the equation $X_{n_0+1}^t = 0$ means that the “load of carrier” [10] is zero at the beginning “1” of $Q_{n_0+1}^t$, and hence $\tilde{Q}_n^{t+1}$ and $\tilde{W}_n^{t+1}$ does not change from the original ones except $\tilde{W}_{n_0}^{t+1}$. Moreover, for a large t the clusters of 1’s in pBBS will align coherently in the order of their sizes $\lambda_1, \lambda_2, \dots, \lambda_g$ thanks to δ . This indicates $\tilde{Q}_n^t = \lambda_{n-n_0-1}$ for $n = n_0 + 1, \dots, n_0 + g + 1$ and $\tilde{X}_n^t = 0$ for all n in the UD-pToda. On the other hand from $\tilde{X}_n^t = 0$ and (1.8) we obtain $\tilde{Q}_n^t = (\mathbf{m}_n - \mathbf{m}_{n-1})\lambda^\perp$. By making use of these properties the region $D_{\mathbf{m}_n}$ to which $\tilde{\mathbf{Z}}_n^t$ belongs is obtained for a large t as Lemma 2.8.

Figure 2. $\tilde{D}_{\mathbf{m}}$ and $\tilde{\mathbf{Z}}_n^t$

§ 2.2. Injectivity of $\sigma_t|_{\text{Im}\iota_t}$

Our goal in this subsection is to show the following:

Proposition 2.1. *The map $\sigma_t|_{\text{Im}\iota_t}$ is injective.*

Consider $(Q_n^t, W_n^t)_n \in \mathcal{T}_C$ in the image of $\sigma_t \circ \iota_t$, i.e. Q_n^t and W_n^t are written in terms of the tropical Riemann theta function via (1.8) and $T_n^t = \Theta(\mathbf{Z}_n^t = \mathbf{Z}_0 - n\mathbf{L}e_1 + t\vec{\lambda})$ with some $\mathbf{Z}_0 \in \mathbb{R}^g$.

Lemma 2.2. *If $\mathbf{Z} \in D_{\mathbf{m}}$, then $\mathbf{Z} - n\mathbf{L}e_1 + t\vec{\lambda}$ ($n = 0, -1$, $t = 0, 1, 2$) belongs to $D_{\mathbf{m}'}$, where $m'_1 - m_1 \in \{0, -1\}$.*

Proof. Without loss of generality we can assume $\mathbf{Z} \in D_0$, namely $-\frac{1}{2}\mathbf{l}K\mathbf{l}^\perp \leq -\mathbf{l}\mathbf{Z}^\perp \leq \frac{1}{2}\mathbf{l}K\mathbf{l}^\perp$ for all $\mathbf{l} \in \mathbb{Z}^g$. Define

$$f(\mathbf{l}, \mathbf{m}') = -\mathbf{l}(\mathbf{Z} - n\mathbf{L}e_1 + t\vec{\lambda})^\perp - \mathbf{l}K(\frac{1}{2}\mathbf{l} + \mathbf{m}')^\perp.$$

We consider the following four cases; (a-1): $m'_1 = -2$, (a-2): $m'_1 < -2$, (b-1): $m'_1 = 1$ or (b-2): $m'_1 > 1$. We show that in case (a-1) or (b-1) there exists $\mathbf{l} \in \mathbb{Z}^g$ such that $f(\mathbf{l}, \mathbf{m}') \geq 0$ (i.e. $\mathbf{Z} - n\mathbf{L}e_1 + t\vec{\lambda}$ is not in the interior of $D_{\mathbf{m}'}$), and that in case (a-2) or

(b-2) there exists $\mathbf{l} \in \mathbb{Z}^g$ such that $f(\mathbf{l}, \mathbf{m}') > 0$ (i.e. $\mathbf{Z} - nL\mathbf{e}_1 + t\vec{\lambda}$ is not in $D_{\mathbf{m}'}$), for $n = 0, -1$ and $t = 0, 1, 2$.

(a-1) or (a-2): Take $\mathbf{l}$ as $l_i = 1$ if $m'_i \leq -2$ and $l_i = 0$ if $m'_i \geq -1$. Then we get

$$\begin{aligned} f(\mathbf{l}, \mathbf{m}') &\geq -\mathbf{l}K\mathbf{l}^\perp + nL - t\mathbf{l}\vec{\lambda}^\perp - \mathbf{l}K\mathbf{m}'^\perp \quad (\text{since } -\mathbf{l}\mathbf{Z}^\perp \geq -\frac{1}{2}\mathbf{l}K\mathbf{l}^\perp) \\ &= nL - t\mathbf{l}\vec{\lambda}^\perp + \mathbf{l}K(\mathbb{I} - (\mathbf{m}' + \mathbf{l} + \mathbb{I}))^\perp, \end{aligned}$$

where we have $-\mathbf{l}K(\mathbf{m}' + \mathbf{l} + \mathbb{I})^\perp \geq 0$ since $-(m'_i + l_i + 1)\mathbf{l}K\mathbf{e}_i^\perp \geq 0$ for all i . Actually, if $m'_i \leq -2$, then $-(m'_i + l_i + 1) \geq 0$ and $\mathbf{l}K\mathbf{e}_i^\perp = K_{ii} + \sum_{j \neq i} l_j K_{ij} > 0$, and if $m'_i \geq -1$, then $-(m'_i + l_i + 1) \leq 0$ and $\mathbf{l}K\mathbf{e}_i^\perp = \sum_{j \neq i} l_j K_{ij} \leq 0$. Using $\mathbf{l}K\mathbb{I}^\perp = \mathbf{l}((L, 0, \dots, 0, p_g) + 2\vec{\lambda})$, we obtain

$$\begin{aligned} nL - t\mathbf{l}\vec{\lambda}^\perp + \mathbf{l}K(\mathbb{I} - (\mathbf{m}' + \mathbf{l} + \mathbb{I}))^\perp &\geq nL + \mathbf{l}((L, 0, \dots, 0, p_g) + (2-t)\vec{\lambda})^\perp \\ &= (1+n)L + l_g p_g + (2-t)\mathbf{l}\vec{\lambda}^\perp \quad (\text{since } l_1 = 1) \end{aligned}$$

which is non-negative for $n = 0, -1$ and $t = 0, 1, 2$. Thus the claim follows. In the case of (a-2), $f(\mathbf{l}, \mathbf{m}') > 0$ can be shown similarly.

(b-1) or (b-2): Take $\mathbf{l}$ as $l_i = -1$ if $m'_i \geq 1$ and $l_i = 0$ if $m'_i \leq 0$. In the same way as (a-1) we can show $f(\mathbf{l}, \mathbf{m}') \geq 0$ and $f(\mathbf{l}, \mathbf{m}') > 0$ respectively. $\square$

Lemma 2.3. (i) For any $t \in \mathbb{Z}$, there exists $n_0 \in \mathbb{Z}/(g+1)\mathbb{Z}$ such that $X_{n_0+1}^t = 0$.

(ii) Fix $\mathbf{Z}_0 \in \mathbb{R}^g$ and set $\mathbf{Z}_n^t = \mathbf{Z}_0 - nL\mathbf{e}_1 + \vec{\lambda}t$. If $\mathbf{Z}_0 \in D_{\mathbf{m}}$, there exists an unique $n_1 \in \{1, \dots, g+1\}$ such that $\mathbf{Z}_{n_1}^0, \mathbf{Z}_{n_1-1}^0 \in D_{\mathbf{m}'}$ with $m'_1 - m_1 = n_1 - 1$.

Proof. (i) Set the sequence $\{a_k\}_{k=1,2,3,\dots}$ as $a_k = \sum_{l=1}^k (W_{-l}^t - Q_{-l}^t)$ and find the term which is smaller than all rear terms. From the assumption $\sum_n Q_n^t < \sum_n W_n^t$, such term exists within the first $g+1$ terms. Let a_{n_0} be a such term and set the sequence $\{b_k\}_{k=1,2,3,\dots}$ as $b_k = a_{n_0+k} - a_{n_0} = \sum_{l=1}^k (W_{-n_0-l}^t - Q_{-n_0-l}^t)$. Then $b_k \geq 0$ holds for all k , thus we have $X_{-n_0}^t = \min[0, b_1, b_2, \dots, b_g] = 0$.

(ii) When $\mathbf{Z}_0^0 = \mathbf{Z}_0 \in D_{\mathbf{m}}$, we have $\mathbf{Z}_{g+1}^0 \in D_{\mathbf{m}+\vec{g}}$. From Lemma 2.2, each point $\mathbf{Z}_n^0$ ($n = 1, \dots, g+1$) is in $D_{\mathbf{m}_n}$ with $(\mathbf{m}_n)_1 - (\mathbf{m}_{n-1})_1 \in \{0, 1\}$. Thus the claim follows from the pigeonhole principle. $\square$

Lemma 2.4. For $(Q_n^0, W_n^0)_n \in \mathcal{T}_C$, there uniquely exists $n_1 \in \mathbb{Z}/(g+1)\mathbb{Z}$ such that either i) or ii) in the following is satisfied:

$$i) Q_{n_1}^0 = 0, \quad ii) W_{n_1-1}^0 = 0, \quad Q_{n_1-1}^0 > 0 \text{ and } Q_{n_1}^0 > 0.$$

Further, let $\mathbf{Z}_0$ be a point in $\mathbb{R}^g$ such that $\sigma_0 \circ \iota_0(\mathbf{Z}_0) = (Q_n^0, W_n^0)_n$. Then we have the following properties: n_1 coincides with that of Lemma 2.3 (ii); if i) is satisfied, then

$\mathbf{Z}_{n_1}^0, \mathbf{Z}_{n_1-1}^0 \in D_{\mathbf{m}}, \mathbf{Z}_{n_1}^1, \mathbf{Z}_{n_1-1}^1 \in D_{\mathbf{m}'},$ where $\mathbf{m}' - \mathbf{m} \in \{0, -1\}^g$ and $m_1 = m'_1$; if ii) is satisfied, then $\mathbf{Z}_{n_1}^0, \mathbf{Z}_{n_1-1}^0 \in D_{\mathbf{m}}, \mathbf{Z}_{n_1-1}^1, \mathbf{Z}_{n_1-2}^1 \in D_{\mathbf{m}'},$ where $\mathbf{m}' - \mathbf{m} \in \{0, -1\}^g$ and $m'_1 - m_1 = -1$.

Proof. Step 1: We show the first claim on $(Q_n^0, W_n^0)_n \in \mathcal{T}_C$. Recall that we have assumed $C_{g-1} > 2C_g = 0$, where C_{g-1} and C_g are given as (1.3). From $C_g = 0$, there exists n_1 such that $Q_{n_1}^0 = 0$ or $W_{n_1-1}^0 = 0$. (a) Suppose $Q_{n_1}^0 = 0$, then

$$\min[\min_{i \neq n_1} Q_i, \min_{j \neq n_1-1, n_1} W_j] \geq C_{g-1} > 0,$$

and therefore $Q_i > 0$ for $i \neq n_1$ and $W_j > 0$ for $j \neq n_1 - 1, n_1$. (b) Suppose $W_{n_1-1}^0 = 0$, then

$$\min[\min_{j \neq n'_1} W_j, \min_{i \neq n'_1, n'_1+1} Q_i] \geq C_{g-1} > 0,$$

and therefore $W_j > 0$ for $j \neq n'_1$ and $Q_i > 0$ for $i \neq n'_1, n'_1 + 1$.

The case where $Q_{n_1} = W_{n'_1} = 0$ may occur only if $n_1 = n'_1$ or $n_1 = n'_1 + 1$. Both cases belong to i) not to ii).

Step 2: We assume $\mathbf{Z}_0 \in D_{\mathbf{0}}$ without loss of generality. In the following we define $D_{(n_1,*)}$ as

$$D_{(n_1,*)} = \bigcup_{n' \in \mathbb{Z}^g \text{ s.t. } n'_1 = n_1} D_{\mathbf{n}'}$$

From Lemma 2.3 (ii), there exists unique $n_1 \in \{1, \dots, g+1\}$ such that $\mathbf{Z}_{n_1}^0, \mathbf{Z}_{n_1-1}^0 \in D_{\mathbf{m}}$ with $m_1 = n_1 - 1$. From Lemma 2.2 we have $\mathbf{Z}_{n_1-2}^0, \mathbf{Z}_{n_1-2}^1 \in D_{(m_1-1,*)}$. Further, we have $\mathbf{Z}_{n_1}^1 \in D_{(m_1,*)}$, indeed if $\mathbf{Z}_{n_1}^1 \in D_{(m_1-1,*)}$, then $\mathbf{Z}_{n_1}^1, \mathbf{Z}_{n_1-1}^1, \mathbf{Z}_{n_1-2}^1 \in D_{(m_1-1,*)}$, which is a contradiction. Thus $\mathbf{Z}_{n_1-1}^1$ belongs to $D_{(m_1,*)}$ or $D_{(m_1-1,*)}$. If $\mathbf{Z}_{n_1-1}^1 \in D_{(m_1,*)}$, then

$$\begin{aligned} Q_{n_1}^0 &= (\Theta(\mathbf{Z}_{n_1-1}^0) - \Theta(\mathbf{Z}_{n_1}^0)) - (\Theta(\mathbf{Z}_{n_1-1}^1) - \Theta(\mathbf{Z}_{n_1}^1)) \\ &= m_1 L - m_1 L = 0, \end{aligned}$$

and if $\mathbf{Z}_{n_1-1}^1 \in D_{(m_1-1,*)}$ then

$$\begin{aligned} W_{n_1-1}^0 &= L + (\Theta(\mathbf{Z}_{n_1}^0) - \Theta(\mathbf{Z}_{n_1-1}^0)) - (\Theta(\mathbf{Z}_{n_1-1}^1) - \Theta(\mathbf{Z}_{n_1-2}^1)) \\ &= L - m_1 L + (m_1 - 1)L = 0. \end{aligned}$$

Step 3. Since both n_1 in Step 1 and 2 are unique, they coincide. $\square$

Proof of Prop. 2.1. It is enough to show the claim for $t = 0$. From the equation (1.8) with the quasi-periodicity (1.18), we have

$$(2.4) \quad T_{n'+1}^0 - T_{n'}^0 = T_{n+1}^0 - T_n^0 - (n' - n)L + \sum_{j=n+1}^{n'} (Q_j^0 + W_j^0) \quad (n, n' \in \mathbb{Z}, n \leq n'),$$

$$(2.5) \quad T_n^1 - T_{n-1}^1 = T_n^0 - T_{n-1}^0 + Q_n^0 \quad (n \in \mathbb{Z}).$$

Take n_0 as $X_{n_0+1}^0 = 0$ (Lemma 2.3) and n_1 as Lemma 2.4 with $1 < n_1 - n_0 \leq g + 1$. Without loss of generality, we can assume $\mathbf{Z}_{n_0}^0 \in D_{\mathbf{0}}$. From $X_{n_0+1}^0 = 0$ and (1.6), we have $T_{n_0}^2 - T_{n_0}^1 = T_{n_0}^1 - T_{n_0}^0$ and therefore we can also assume $\mathbf{Z}_{n_0}^1, \mathbf{Z}_{n_0}^2 \in D_{\mathbf{0}}$. From Lemma 2.4, we have $\mathbf{Z}_{n_1}^0, \mathbf{Z}_{n_1-1}^0 \in D_{\mathbf{m}}$ with $m_1 = n_1 - n_0 - 1$.

From (2.4), $T_{n_0}^0 = 0$ and $T_{n_1}^0 - T_{n_1-1}^0 = -L(n_1 - 1 - n_0)$, we have

$$T_n^0 = -\frac{1}{2}(n - n_0)(n - n_0 - 1)L - S_n,$$

where

$$S_n = \begin{cases} \sum_{k=1}^{n-n_0} \sum_{j=n_0+k}^{n_1-1} (Q_j^0 + W_j^0) & (n_0 \leq n \leq n_1) \\ \sum_{j=n_0+1}^{n_1-1} (j - n_0)(Q_j^0 + W_j^0) - \sum_{j=n_1}^{n-1} (n - j)(Q_j^0 + W_j^0) & (n \geq n_1) \end{cases}.$$

We also have T_n^t for $n < n_0$ by the quasi-periodicity. Further, from (2.5) and $T_{n_0}^0 = T_{n_0}^1 = 0$, we have

$$T_n^1 = T_n^0 + \sum_{j=n_0+1}^n Q_j^0.$$

□

§ 2.3. Injectivity of ι_t

In this subsection we show

Proposition 2.5. *The map ι_t is injective.*

We take n_0 as $X_{n_0+1}^0 = 0$ (Lemma 2.3), and set $\mathbf{Z}_{n_0}^0 \in D_{\mathbf{0}}$ without loss of generality. Then $\mathbf{Z}_{n_0}^1$ also belongs to $D_{\mathbf{0}}$. Take also n_1 as Lemma 2.4 with $1 < n_1 - n_0 \leq g + 1$. From (2.1), Lemmas 2.2 and 2.3, we have

$$\begin{aligned} \mathbf{Z}_{n_0}^0 &\in D_{\mathbf{0}} \\ \mathbf{Z}_n^0 &\in D_{\mathbf{m}_n} \quad (n = 1, \dots, g) \\ \mathbf{Z}_{n_0+g+1}^0 &\in D_{\vec{g}}, \end{aligned}$$

where

$$\mathbf{m}_n = \begin{cases} \sum_{k=1}^n \mathbb{I}_{s(k)} & (n_0 + 1 \leq n \leq n_1 - 1) \\ \sum_{k=1}^{n_1-1} \mathbb{I}_{s(k)} & (n_1 \leq n \leq n_0 + g + 1) \end{cases}$$

for some permutation $s \in \mathfrak{S}_g$. Note that $\sum_{k=1}^g \mathbb{I}_k = \sum_{k=1}^g \mathbb{I}_{\sigma(k)} = \vec{g}$.

Let δ be a large positive number. Set $\tilde{\mathbf{Z}}_n^t$ as

$$(2.6) \quad \tilde{\mathbf{Z}}_n^t = \mathbf{Z}_n^t - \delta(n - n_0)\mathbf{e}_1 = \mathbf{Z}_{n_0}^0 - (L + \delta)(n - n_0)\mathbf{e}_1 + t\vec{\lambda}$$

and $\tilde{K}$ as

$$\tilde{K} = K + \delta J, \quad J = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix},$$

where the blank parts in J means zero. For $\mathbf{m} \in \mathbb{R}^g$, let $\tilde{D}_{\mathbf{m}}$ be the fundamental regions determined by $\tilde{K}$.

Lemma 2.6. *If $\mathbf{Z}_n^t \in D_{\mathbf{m}}$, then $\tilde{\mathbf{Z}}_n^t$ belongs to $\tilde{D}_{\mathbf{m}}$ for all $n \in \mathbb{Z}$ and $t = 0, 1$.*

Proof. From (2.1) it is enough to show

$$(2.7) \quad -\mathbf{l}(\mathbf{Z}_n^t - \delta(n - n_0)\mathbf{e}_1)^\perp \leq \mathbf{l}(K + \delta J)(\mathbf{m} + \frac{1}{2}\mathbf{1})^\perp,$$

under the constraints

$$-\mathbf{l}(\mathbf{Z}_n^t)^\perp \leq \mathbf{l}K(\mathbf{m} + \frac{1}{2}\mathbf{1})^\perp,$$

for any $\mathbf{l} = \mathbb{I}_j - \mathbb{I}_i$ ($0 \leq i, j \leq g$), $n_0 + 1 \leq n \leq n_0 + g + 1$ and $t = 0, 1$, where $\mathbf{m}$ is $\sum_{k=1}^{n-n_0} \mathbb{I}_{s(k)}$ or $\sum_{k=1}^{n-n_0-1} \mathbb{I}_{s(k)}$. "R.h.s - l.h.s" of (2.7) is greater than or equal to

$$\delta \mathbf{l}J(\mathbf{m} + \frac{1}{2}\mathbf{1})^\perp - \delta(n - n_0)l_1,$$

which can be shown to be nonnegative by using the formulae

$$(2.8) \quad J\mathbb{I}_k = \begin{cases} \mathbf{e}_1 + \mathbf{e}_k - \mathbf{e}_{k+1} & (1 \leq k \leq g-1) \\ \mathbf{e}_1 + \mathbf{e}_g & (k = g), \end{cases} \quad \frac{1}{2}\mathbf{l}K\mathbf{1}^\perp = 1,$$

and

$$\mathbf{l}J\mathbf{m}^\perp \geq \begin{cases} n - n_0 - 1 & (l_1 = 1) \\ -1 & (l_1 = 0) \\ -n + n_0 - 1 & (l_1 = -1) \end{cases}.$$

□

Lemma 2.7. *Define $\tilde{Q}_n^t$ and $\tilde{W}_n^t$ as*

$$\begin{aligned} \tilde{Q}_n^t &= \tilde{\Theta}(\tilde{\mathbf{Z}}_{n-1}^t) + \tilde{\Theta}(\tilde{\mathbf{Z}}_n^{t+1}) - \tilde{\Theta}(\tilde{\mathbf{Z}}_{n-1}^{t+1}) - \tilde{\Theta}(\tilde{\mathbf{Z}}_n^t), \\ \tilde{W}_n^t &= L + \delta + \tilde{\Theta}(\tilde{\mathbf{Z}}_{n-1}^{t+1}) + \tilde{\Theta}(\tilde{\mathbf{Z}}_{n+1}^t) - \tilde{\Theta}(\tilde{\mathbf{Z}}_n^t) - \tilde{\Theta}(\tilde{\mathbf{Z}}_n^{t+1}), \end{aligned}$$

where $\tilde{\Theta}(\tilde{Z}) = \Theta(\tilde{Z}; \tilde{K})$. Then $\tilde{Q}_n^0 = Q_n^0$ and $\tilde{W}_n^0 = W_n^0$ hold except $\tilde{W}_{n_0}^0 = W_{n_0}^0 + \delta$.

Proof. From Lemma 2.6 we obtain the following formula;

$$\tilde{\Theta}(\tilde{\mathbf{Z}}_n^t) - \Theta(\mathbf{Z}_n^t) = \begin{cases} -\frac{\delta}{2}(n - n_0)(n - n_0 - 1) & (n_0 \leq n \leq n_0 + g + 1) \\ 0 & (n = n_0 - 1) \end{cases}$$

for $t = 0, 1$. Indeed we have

$$\begin{aligned}
& \tilde{\Theta}(\tilde{\mathbf{Z}}_n^t) - \Theta(\mathbf{Z}_n^t) \\
&= \frac{1}{2} \mathbf{m}(K + \delta J) \mathbf{m}^\perp + \mathbf{m}(\mathbf{Z}_n^t - \delta(n - n_0) \mathbf{e}_1)^\perp - \frac{1}{2} \mathbf{m} K \mathbf{m}^\perp - \mathbf{m} \mathbf{Z}_n^{t\perp} \\
&= \frac{1}{2} \delta \mathbf{m} J \mathbf{m}^\perp - \delta(n - n_0) m_1 \\
&= \begin{cases} \frac{\delta}{2}((n - n_0)^2 + (n - n_0)) - \delta(n - n_0)^2 & (m_1 = n - n_0) \\ \frac{\delta}{2}((n - n_0 - 1)^2 + (n - n_0 - 1)) - \delta(n - n_0)(n - n_0 - 1) & (m_1 = n - n_0 - 1) \end{cases} \\
&= -\frac{\delta}{2}(n - n_0)(n - n_0 - 1),
\end{aligned}$$

for $n_0 \leq n \leq n_0 + g + 1$, and the case of $n = n_0 - 1$ is similarly shown. Thus we have

$$\begin{aligned}
\tilde{Q}_n^0 &= \Theta(\mathbf{Z}_{n-1}^0) + \Theta(\mathbf{Z}_n^1) - \Theta(\mathbf{Z}_{n-1}^1) - \Theta(\mathbf{Z}_n^0) \\
&\quad - \frac{\delta}{2}((n - n_0 - 1)(n - n_0 - 2) + (n - n_0)(n - n_0 - 1)) \\
&\quad - (n - n_0 - 1)(n - n_0 - 2) - (n - n_0)(n - n_0 - 1) \\
&= Q_n^0
\end{aligned}$$

for $n_0 + 1 \leq n \leq n_0 + g + 1$,

$$\begin{aligned}
\tilde{W}_n^0 &= L + \delta + \Theta(\mathbf{Z}_{n-1}^1) + \Theta(\mathbf{Z}_{n+1}^0) - \Theta(\mathbf{Z}_n^0) - \Theta(\mathbf{Z}_n^1) \\
&\quad - \frac{\delta}{2}((n - n_0 - 1)(n - n_0 - 2) + (n - n_0 + 1)(n - n_0)) \\
&\quad - (n - n_0)(n - n_0 - 1) - (n - n_0)(n - n_0 - 1) \\
&= W_n^0
\end{aligned}$$

for $n_0 + 1 \leq n \leq n_0 + g$, and

$$\begin{aligned}
\tilde{W}_{n_0}^0 &= L + \delta + \Theta(\mathbf{Z}_{n_0-1}^1) + \Theta(\mathbf{Z}_{n_0+1}^0) - \Theta(\mathbf{Z}_{n_0}^0) - \Theta(\mathbf{Z}_{n_0}^1) \\
&= W_{n_0}^0 + \delta.
\end{aligned}$$

□

Lemma 2.8. Set $\mathbf{m}_n = \sum_{k=1}^{n-n_0-1} \mathbb{I}_k$. For $1 \ll t \ll \delta$, we have

$$\tilde{\mathbf{Z}}_n^t \in \begin{cases} \tilde{D}_0 & \text{for } n = n_0, \\ \tilde{D}_{\mathbf{m}_{n-n_0-1}} & \text{for } n_0 + 1 \leq n \leq n_0 + g + 1. \end{cases}$$

Proof. From (2.1) it is enough to show that

$$-\mathbf{l}(\mathbf{Z}_{n_0}^0 - (n - n_0)(L + \delta) \mathbf{e}_1 + t \vec{\lambda})^\perp \leq \mathbf{l}(K + \delta J) \left(\sum_{k=1}^{n-n_0-1} \mathbb{I}_k + \frac{1}{2} \mathbf{1} \right)^\perp$$

for any $\mathbf{l} = \mathbb{I}_j - \mathbb{I}_i$ ($0 \leq i, j \leq g$). Putting together all terms independent of δ and t into c , this is equivalent to

$$(2.9) \quad c + t\mathbf{l}\vec{\lambda}^\perp + \delta \left(\mathbf{l}J \left(\sum_{k=1}^{n-n_0-1} \mathbb{I}_k + \frac{1}{2}\mathbf{l} \right)^\perp - (n-n_0)l_1 \right) \geq 0.$$

From formula (2.8) we have $J\mathbf{m}_n = (n-n_0)\mathbf{e}_1 - \mathbf{e}_{n-n_0}$, and thus (2.9) becomes

$$(2.10) \quad c + t\mathbf{l}\vec{\lambda}^\perp + \delta(-l_{n-n_0} + 1) \geq 0.$$

If $l_{n-n_0} = 0$ or -1 , (2.10) holds obviously for $1 \ll t \ll \delta$, while if $l_{n-n_0} = 1$, (2.10) holds for large t since $\mathbf{l}\vec{\lambda}^\perp > 0$. $\square$

Proof of Prop. 2.5. For $(Q_n^0, W_n^0)_n \in \text{Im } \sigma_0 \circ \iota_0$ and a large positive number δ , $(\tilde{Q}_n^0, \tilde{W}_n^0)_n$ is determined by using Lemma 2.7, and thus $\tilde{T}_n^t$ are uniquely determined by Proposition 2.1. Further, from Lemma 2.8 we know the region $\tilde{D}_{\mathbf{m}_n}$ the point $\tilde{\mathbf{Z}}_n^t$ belongs to for $1 \ll t \ll \delta$, thus we obtain g independent linear equations for $\tilde{\mathbf{Z}}_0$, which uniquely determine $\tilde{\mathbf{Z}}_0$ (as explained in the example). Finally we get a unique $\mathbf{Z}_0$ through $\mathbf{Z}_{n_0}^0 = \tilde{\mathbf{Z}}_{n_0}^0$ and (2.6). $\square$

§ 2.4. Proof of Theorem 1.3

From Prop. 2.1 and Prop. 2.5, ι_σ is injective. We show that ι_σ is surjective. Since ι_σ is continuous both for $\mathbf{Z}_0$ and $C = (C_{-1}, C_0, \dots, C_g)$, it is enough to show that ι_σ is a surjective map from a dense subset of $J(K)$ to a dense subset of $\mathcal{T}_C$ for a dense subset of

$$\{C \in \mathbb{R}^{g+2} \mid C \text{ satisfies the generic condition (1.11)}\}.$$

Suppose that C belongs to $\mathbb{Q}^g$. Let N be a common multiple of denominators of C_i 's. Let $J(K)_N$ denote $J(K) \cap (\frac{1}{N}\mathbb{Z})^g$ and $(\mathcal{T}_C)_N = (\mathcal{T}_C) \cap (\frac{1}{N}\mathbb{Z})^{2g+2}$. Here $J(K)_N$ and $(\mathcal{T}_C)_N$ are isomorphic to $J(NK)_1$ and $(\mathcal{T}_{NC})_1$ (the set of integer points of the phase space with the conserved quantities $NC = N(C_{-1}, \dots, C_g)$) respectively. From Proposition 4.4 of [1], we have $|(\mathcal{T}_{NC})_1| = (g+1)|B_{NL, N\lambda}|$ where $B_{NL, N\lambda}$ denotes the isolevel set of the corresponding pBBS. From Proposition 2.2 of [12] it is easily proved by using “10’ elimination” (or Theorem 3.11 of [6]) that the cardinal number $|B_{NL, N\lambda}|$ of $B_{NL, N\lambda}$ is $N^g p_0 p_1 \cdots p_{g-1}$, which is a $(g+1)$ -th part of $|J(NK)_1|$ from Lemma 2.5 of [1]. Therefore, since ι_σ is injective, ι_σ induces a surjection from $J(K)_N$ to $(\mathcal{T}_C)_N$. Since N can be taken as large as one likes and $\mathbb{Q}^g$ is dense in $\mathbb{R}^g$, ι_σ is a surjection from $J(K)$ to $\mathcal{T}_C$ for any C satisfying the generic condition (1.11).

§ 2.5. Initial value problem

Using the arguments of this section, we can solve the initial value problem for the UD-pToda. Indeed, T_n^0 and T_n^1 can be determined from $(Q_n^0, W_n^0)_n$ as the proof of Proposition 2.1. Set δ be a large positive number as $\delta \simeq gL$ — from the properties of the BBS (soliton like behavior [11]), the clusters of 1's align coherently in the order of their sizes for $t \simeq L$. For $t \simeq L$, Lemma 2.8 gives the region $\tilde{D}_{\mathbf{m}_n}$ to which the point $\tilde{\mathbf{Z}}_n^t$ belongs. As explained in the example we obtain g independent linear equations for $\tilde{\mathbf{Z}}_0$. Solving these, we obtain $\tilde{\mathbf{Z}}_0$, and thus $\mathbf{Z}_0$ by $\mathbf{Z}_{n_0}^0 = \tilde{\mathbf{Z}}_{n_0}^0$. Then the solution to the initial value problem is given by (1.15) with (1.8).

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