

LOCATION OF BLOW UP POINTS OF LEAST ENERGY SOLUTIONS TO THE BREZIS-NIRENBERG EQUATION

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1. Introduction. Let Ω be a smooth bounded domain in $\mathbf{R}^N$, $N \geq 4$ and $p = \frac{N+2}{N-2}$. In this article, we return to the well-studied problem (P_ε) :

$$\begin{cases} -\Delta u = u^p + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\varepsilon > 0$ is a parameter.

The exponent p is called the critical Sobolev exponent in the sense that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is continuous but not compact. So from the variational view point, this problem belongs to the limit case of the Palais-Smale compactness condition, and the classical arguments do not apply to the questions related to the existence or nonexistence and multiplicity of solutions of this problem.

In pioneering work [3], Brezis and Nirenberg proved that, in spite of possible failure of the Palais-Smale compactness condition, (P_ε) has at least one non-trivial solution on a general bounded domain Ω when $\varepsilon \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

On the other hand when $\varepsilon = 0$, it is known that problem (P_0) reflects the topology and the geometry of the domain Ω . Pohozaev showed that if Ω is star-shaped, then (P_0) has no non-trivial solutions [7]. In other cases Bahri and Coron [1] proved that (P_0) has a solution when Ω has non-trivial topology in the sense that $H_d(\Omega, \mathbf{Z}_2) \neq \{0\}$ for some positive integer d , where $H_d(\Omega, \mathbf{Z}_2)$ denotes the d -th homology group of Ω with $\mathbf{Z}_2$ coefficients. Furthermore Ding [5] and Passaseo [8] proved that even if Ω is contractible, (P_0) can still have a solution if the geometry of Ω is non-trivial in some sense.

Because of the different nature of the problem when $\varepsilon > 0$ and $\varepsilon = 0$, it is interesting to study the asymptotic behavior of solutions u_ε of (P_ε) as $\varepsilon \rightarrow 0$. In this direction, Han [9] and Rey [12][13] proved independently the following result, which had been conjectured previously by Brezis and Peletier [4].

Theorem 0. (Han [9], Rey [12]) Let u_ε be a solution of problem (P_ε) and assume

$$\frac{\int_{\Omega} |\nabla u_\varepsilon|^2 dx}{(\int_{\Omega} |u_\varepsilon|^{p+1} dx)^{\frac{2}{p+1}}} = S + o(1) \quad \text{as } \varepsilon \rightarrow 0,$$

where S is the best Sobolev constant in $\mathbf{R}^N$:

$$S = \pi N(N-2) \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{2}{N}}.$$

Then we have (after passing to a subsequence):

(1) There exists $a_\infty \in \Omega$ (interior point) such that

$$|\nabla u_\varepsilon|^2 \rightharpoonup^* S^{\frac{N}{2}} \delta_{a_\infty} \quad \text{as } \varepsilon \rightarrow 0$$

in the sense of Radon measures of the compact space $\overline{\Omega}$, where δ_a is the Dirac measure supported by $a \in \mathbf{R}^N$.

(2) The a_∞ above is a critical point of the (positive) Robin function $H(a, a)$ on Ω :

$$\nabla_a H(a_\infty, a_\infty) = 0,$$

where $H(x, a)$ is the regular part of the Green's function $G(x, a)$:

$$H(x, a) := \frac{1}{(N-2)\omega_N} |x-a|^{2-N} - G(x, a),$$

in which $\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the $(N-1)$ dimensional volume of S^{N-1} and

$$\begin{cases} -\Delta_x G(x, a) = \delta_a(x), & x \in \Omega, \\ G(x, a)|_{x \in \partial\Omega} = 0. \end{cases}$$

(3) We have an exact blow up rate of the L^∞ -norm of u_ε as $\varepsilon \rightarrow 0$:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2(N-4)}{N-2}} = (N(N-2))^{\frac{N-4}{2}} \frac{(N-2)^3 \omega_N}{2C_N} H(a_\infty, a_\infty), \quad \text{if } N \geq 5,$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \|u_\varepsilon\|_{L^\infty(\Omega)} = 4\omega_4 H(a_\infty, a_\infty), \quad \text{if } N = 4,$$

where

$$C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)}.$$

In this article, we restrict our attention to a particular family of solutions to (P_ε) , namely the solutions $(\bar{u}_\varepsilon)_{\varepsilon \in (0, \lambda_1)}$ obtained by the method of Brezis and Nirenberg. We call $(\bar{u}_\varepsilon)$ the *least energy solutions* to the problem (P_ε) .

Before stating our main result, we recall the construction of least energy solutions by Brezis and Nirenberg.

For $\varepsilon \in (0, \lambda_1)$, define

$$S_\varepsilon := \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_{\Omega} |\nabla u|^2 dx - \varepsilon \int_{\Omega} u^2 dx \right\}. \quad (1.1)$$

Since the constraint on $\|u\|_{L^{p+1}(\Omega)}$ is not preserved under weak convergence in $H_0^1(\Omega)$, it is not obvious that S_ε is achieved or not. By using the fact that $S_\varepsilon < S$ if $\varepsilon > 0$, Brezis-Nirenberg proved that any minimizing sequence for (1.1) is compact in $H_0^1(\Omega)$ and (1.1) is achieved by some positive function $v_\varepsilon^0 \in H_0^1(\Omega)$. Furthermore if $\varepsilon < \lambda_1$, then it follows $S_\varepsilon > 0$ and

$$\bar{u}_\varepsilon := S_\varepsilon^{\frac{N-2}{4}} v_\varepsilon^0 \quad (1.2)$$

is a solution to (P_ε) .

By Global Compactness Theorem of Struwe [14], we know that the least energy solutions $\bar{u}_\varepsilon$ blow up at exactly one point in $\bar{\Omega}$ as $\varepsilon \rightarrow 0$. That is, there exist $\lambda_\varepsilon > 0$ with $\lambda_\varepsilon \rightarrow 0$ ($\varepsilon \rightarrow 0$) and $a_\varepsilon \in \Omega$ with $\lambda_\varepsilon / \text{dist}(a_\varepsilon, \partial\Omega) \rightarrow 0$ ($\varepsilon \rightarrow 0$) such that

$$\|\nabla(\bar{u}_\varepsilon - \alpha_N P U_{\lambda_\varepsilon, a_\varepsilon})\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (1.3)$$

where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$.

Here for $\lambda > 0$ and $a \in \Omega$, we define

$$U_{\lambda, a}(x) := \left(\frac{\lambda}{\lambda^2 + |x - a|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbf{R}^N \quad (1.4)$$

and $P U_{\lambda, a} := U_{\lambda, a} - \varphi_{\lambda, a} \in H_0^1(\Omega)$, where $\varphi_{\lambda, a}$ is the harmonic extension of $U_{\lambda, a}|_{\partial\Omega}$ to Ω :

$$\begin{cases} -\Delta \varphi_{\lambda, a} = 0 & \text{in } \Omega, \\ \varphi_{\lambda, a}|_{\partial\Omega} = U_{\lambda, a}|_{\partial\Omega}. \end{cases} \quad (1.5)$$

We call any accumulation point of $(a_\varepsilon)_{\varepsilon > 0}$ a *blow up point* of $(\bar{u}_\varepsilon)$. Note that if $a_\infty \in \bar{\Omega}$ is a blow up point of $(\bar{u}_\varepsilon)_{\varepsilon > 0}$, then by passing to a subsequence,

we see $|\nabla \bar{u}_\varepsilon|^2 \stackrel{*}{\rightharpoonup} S^{\frac{N}{2}} \delta_{a_\infty}$ as $\varepsilon \rightarrow 0$, and by construction, $(\bar{u}_\varepsilon)$ is a minimizing sequence for the best Sobolev constant. So from the result of Han and Rey, we know that $a_\infty \in \Omega$ (interior point) and a_∞ is a critical point of the Robin function on Ω .

Our main result is to further locate the blow up point a_∞ of the least energy solutions on a general bounded domain Ω in $\mathbf{R}^N$, $N \geq 4$.

Theorem 1. *Let a_∞ be a blow up point of the least energy solutions $(\bar{u}_\varepsilon)$ obtained by the method of Brezis and Nirenberg. Then a_∞ is a minimum point of the Robin function of Ω :*

$$H(a_\infty, a_\infty) = \inf_{a \in \Omega} H(a, a).$$

To prove Theorem 1, we will make a precise asymptotic expansion of the value S_ε as $\varepsilon \rightarrow 0$. For this purpose, we combine the method developed by Isobe [10] [11] and technical calculations in Rey [12] [13]. As a by-product of our method, we prove that the blow up point is the interior point of Ω by using only an energy comparison argument. Also we can give another explanation of the exact blow up rate of L^∞ -norm of $\bar{u}_\varepsilon$ along the line of our context.

Wei [15] treated the subcritical problem:

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\varepsilon > 0$, and he proved that as $\varepsilon \rightarrow 0$, the least energy solutions to this problem blow up at exactly one point, and the blow up point is a minimum point of the Robin function. His method is the usual blow-up (rescaling) technique and he obtained a second order expansion of the rescaled function, which leads to an asymptotic expansion as $\varepsilon \rightarrow 0$ of the value

$$\inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^{p+1-\varepsilon}(\Omega)}=1}} \left\{ \int_{\Omega} |\nabla u|^2 dx \right\}.$$

In the course of the proof, he used the result of Han and Rey, and a crucial pointwise estimate obtained by Han for the rescaled function.

We might follow the method of Wei to study the problem (P_ε) when $N \geq 5$, but even in this case, I believe that our method is more consistent and somewhat simpler because we do not need any use of Pohozaev identity, Kelvin transformation and Gidas-Ni-Nirenberg theory. See also [6].

2. Asymptotic behavior of S_ε . In this section, we obtain an asymptotic formula of the value S_ε as $\varepsilon \rightarrow 0$ and derive the suitable upper bound for S_ε . See Lemma 2.5 and Lemma 2.7.

For $\varepsilon \in (0, \lambda_1)$, let $v_\varepsilon^0 \in H_0^1(\Omega)$ be a solution to the minimization problem (1.1).

Define

$$v_\varepsilon := S^{\frac{N-2}{4}} v_\varepsilon^0. \quad (2.1)$$

Then (1.2), (1.3) and $S_\varepsilon = S + o(1)$ as $\varepsilon \rightarrow 0$ imply

$$\|\nabla(v_\varepsilon - \alpha_N P U_{\lambda_\varepsilon, a_\varepsilon})\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (2.2)$$

$$\int_{\Omega} v_\varepsilon^{p+1} dx = S^{\frac{N}{2}}. \quad (2.3)$$

Define for $\eta > 0$,

$$M(\eta) := \left\{ v \in H_0^1(\Omega) : \begin{array}{l} \exists \alpha > 0, |\alpha - \alpha_N| < \eta, \exists a \in \Omega, \exists \lambda > 0 \\ \text{with } \lambda/d(a, \partial\Omega) < \eta \\ \text{such that } \|\nabla(v - \alpha P U_{\lambda, a})\|_{L^2(\Omega)} < \eta. \end{array} \right\}$$

where $d(a, \partial\Omega) = \text{dist}(a, \partial\Omega)$.

It is proved in [1]: Proposition 7, that for $v \in M(\eta)$ and $\eta > 0$ small enough, the minimization problem:

$$\text{Minimize } \left\{ \|\nabla(v - \alpha P U_{\lambda, a})\|_{L^2(\Omega)} : \begin{array}{l} \alpha \in (\alpha_N - 2\eta, \alpha_N + 2\eta), \\ \lambda > 0, a \in \Omega, \\ \lambda/d(a, \partial\Omega) < 2\eta \end{array} \right\} \quad (2.4)$$

has a unique solution $(\alpha^0, \lambda^0, a^0) \in (\alpha_N - 2\eta, \alpha_N + 2\eta) \times \mathbf{R}_+ \times \Omega$.

Let $a_\infty \in \overline{\Omega}$ be a blow up point of $(\bar{u}_\varepsilon)_{\varepsilon>0}$. By definition of the blow up point, there exist $\varepsilon_n \rightarrow 0, \lambda_n \rightarrow 0, \Omega \ni a_n \rightarrow a_\infty$ such that $(v_n := v_{\varepsilon_n}, d_n := \text{dist}(a_n, \partial\Omega))$

$$\|\nabla(v_n - \alpha_N P U_{\lambda_n, a_n})\|_{L^2(\Omega)} \rightarrow 0, \quad \lambda_n/d_n \rightarrow 0 \quad (n \rightarrow \infty). \quad (2.5)$$

(2.5) implies there exists $\eta_n \rightarrow 0$ such that $v_n \in M(\eta_n)$. We denote the unique solution $(\alpha_n^0, \lambda_n^0, a_n^0)$ to (2.4) for $v = v_n, \eta = \eta_n$ again by $(\alpha_n, \lambda_n, a_n)$.

Then by our choice of $(\alpha_n, \lambda_n, a_n)$, if we write

$$v_n = \alpha_n P U_{\lambda_n, a_n} + w_n, \quad w_n \in H_0^1(\Omega), \quad (2.6)$$

it follows that

$$\begin{aligned}\alpha_n &\rightarrow \alpha_N = (N(N-2))^{\frac{N-2}{4}}, \quad a_n \rightarrow a_\infty, \\ \frac{\lambda_n}{d_n} &\rightarrow 0 \quad \text{where } d_n = \text{dist}(a_n, \partial\Omega), \\ w_n &\in E_{\lambda_n, a_n}, \quad w_n \rightarrow 0 \text{ in } H_0^1(\Omega)\end{aligned}\tag{2.7}$$

as $n \rightarrow \infty$. Here for $\lambda > 0$ and $a \in \Omega$,

$$\begin{aligned}E_{\lambda, a} &:= \{w \in H_0^1(\Omega) : 0 = \int_{\Omega} \nabla w \cdot \nabla P U_{\lambda, a} dx \\ &= \int_{\Omega} \nabla w \cdot \nabla \left(\frac{\partial}{\partial a_i} P U_{\lambda, a} \right) dx \quad (i = 1, \dots, N) \\ &= \int_{\Omega} \nabla w \cdot \nabla \left(\frac{\partial}{\partial \lambda} P U_{\lambda, a} \right) dx\}.\end{aligned}\tag{2.8}$$

In the following, we estimate

$$J_n := \int_{\Omega} |\nabla v_n|^2 dx - \varepsilon_n \int_{\Omega} v_n^2 dx\tag{2.9}$$

by using the expression (2.6).

Lemma 2.1. (Asymptotic behavior of H_0^1 norm of the main part)

As $n \rightarrow \infty$, we have

$$\begin{aligned}\int_{\Omega} |\nabla P U_{\lambda_n, a_n}|^2 dx &= N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} \\ &+ O\left(\frac{\lambda_n^N}{d_n^N} \left| \log\left(\frac{\lambda_n}{d_n}\right) \right| \right),\end{aligned}$$

where

$$A = \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx = \frac{\Gamma(N/2)}{\Gamma(N)} \pi^{N/2}.$$

Proof. We have

$$\begin{aligned}\int_{\Omega} |\nabla P U_{\lambda_n, a_n}|^2 dx &= \int_{\Omega} -\Delta P U_{\lambda_n, a_n} \cdot P U_{\lambda_n, a_n} dx \\ &= N(N-2) \int_{\Omega} U_{\lambda_n, a_n}^p \cdot (U_{\lambda_n, a_n} - \varphi_{\lambda_n, a_n}) dx \\ &= N(N-2) \int_{\Omega} U_{\lambda_n, a_n}^{p+1} dx - N(N-2) \int_{\Omega} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx \\ &=: N(N-2)I_1 - N(N-2)I_2.\end{aligned}\tag{2.10}$$

Here we have used the fact that $PU_{\lambda_n, a_n} \in H_0^1(\Omega)$ satisfies the equation

$$-\Delta PU_{\lambda_n, a_n} = N(N-2)U_{\lambda_n, a_n}^p \quad \text{in } \Omega. \quad (2.11)$$

Now,

$$\begin{aligned} I_1 &= \int_{\Omega} U_{\lambda_n, a_n}^{p+1} dx = \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx - \int_{\mathbf{R}^N \setminus \Omega} U_{\lambda_n, a_n}^{p+1} dx \\ &= A + O\left(\int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p+1} dx\right) \\ &= A + O\left(\lambda_n^N \int_{r=d_n}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2 + r^2)^N} dr\right) \quad (r = |x - a_n|) \\ &= A + O\left(\frac{\lambda_n^N}{d_n^N}\right). \end{aligned} \quad (2.12)$$

We divide I_2 in the second term of (2.10) as

$$\begin{aligned} I_2 &= \int_{\Omega} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx \\ &= \int_{\Omega \setminus B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx + \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx \\ &=: I_2^1 + I_2^2. \end{aligned} \quad (2.13)$$

Now,

$$\begin{aligned} I_2^1 &= \int_{\Omega \setminus B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx \\ &= O\left(\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)} \int_{\Omega \setminus B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p dx\right) \\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right) \cdot \lambda_n^{\frac{N+2}{2}} \int_{r=d_n/2}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2 + r^2)^{\frac{N+2}{2}}} dr\right) \\ &= O\left(\frac{\lambda_n^N}{d_n^N}\right). \end{aligned} \quad (2.14)$$

Here, we have used the estimate

$$\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)} = O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right), \quad (2.15)$$

which is a consequence of (1.5) and the maximum principle of harmonic functions.

In calculating I_2^2 , we make a Taylor expansion of φ_{λ_n, a_n} on $B_{d_n/2}(a_n)$:

$$\begin{aligned}\varphi_{\lambda_n, a_n} &= \varphi_{\lambda_n, a_n}(a_n) + \nabla \varphi_{\lambda_n, a_n}(a_n) \cdot (x - a_n) \\ &+ O\left(\|\nabla^2 \varphi_{\lambda_n, a_n}\|_{L^\infty(B_{d_n/2}(a_n))} |x - a_n|^2\right).\end{aligned}$$

Note that we have

$$\varphi_{\lambda_n, a_n}(a_n) = (N-2)\omega_N \lambda_n^{\frac{N-2}{2}} H(a_n, a_n) + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^N}\right) \quad (2.16)$$

by [13]: Proposition 1, and

$$\|\nabla^2 \varphi_{\lambda_n, a_n}\|_{L^\infty(B_{d_n/2}(a_n))} = O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^N}\right) \quad (2.17)$$

by the elliptic estimate $d_n^k \|\nabla^k \varphi_{\lambda_n, a_n}\|_{L^\infty(B_{d_n/2}(a_n))} \leq \|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)}$ ($k \in \mathbb{N}$) for a harmonic function φ_{λ_n, a_n} .

Then by (2.16), (2.17) and the oddness of the integral, we calculate:

$$\begin{aligned}I_2^2 &= \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} dx \\ &= \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n}(a_n) dx \\ &+ \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \nabla \varphi_{\lambda_n, a_n}(a_n) \cdot (x - a_n) dx \\ &+ \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p \cdot O\left(\|\nabla^2 \varphi_{\lambda_n, a_n}\|_{L^\infty(B_{d_n/2}(a_n))} |x - a_n|^2\right) dx \\ &= \left\{ (N-2)\omega_N \lambda_n^{\frac{N-2}{2}} H(a_n, a_n) + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^N}\right) \right\} \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p dx + 0 \\ &+ O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^N} \int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p |x - a_n|^2 dx\right) \\ &= \left(\frac{N-2}{N}\right) \omega_N^2 \lambda_n^{N-2} H(a_n, a_n) + O\left(\frac{\lambda_n^N}{d_n^N}\right) + O\left(\frac{\lambda_n^N}{d_n^N} \left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right). \quad (2.18)\end{aligned}$$

Here in the last equality, we have used the estimates

$$\int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p dx = \omega_N \int_0^{d_n/2} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^{\frac{N+2}{2}} r^{N-1} dr$$

$$\begin{aligned}
&= \omega_N \lambda_n^{\frac{N-2}{2}} \int_0^{d_n/2\lambda_n} \frac{s^{N-1}}{(1+s^2)^{\frac{N+2}{2}}} ds = \omega_N \lambda_n^{\frac{N-2}{2}} \left(\int_0^\infty - \int_{d_n/2\lambda_n}^\infty \right) \\
&= \frac{\omega_N}{N} \lambda_n^{\frac{N-2}{2}} + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^2}\right), \tag{2.19}
\end{aligned}$$

$$\begin{aligned}
&\int_{B_{d_n/2}(a_n)} U_{\lambda_n, a_n}^p O(|x - a_n|^2) dx = O\left(\lambda_n^{\frac{N+2}{2}} \int_0^{d_n/2\lambda_n} \frac{s^{N+1}}{(1+s^2)^{\frac{N+2}{2}}} ds\right) \\
&= O\left(\lambda_n^{\frac{N+2}{2}} \left|\log\left(\frac{\lambda_n}{d_n}\right)\right|\right), \tag{2.20}
\end{aligned}$$

and the estimate of the Robin function:

$$H(a_n, a_n) = \frac{1}{(N-2)\omega_N} \left(\frac{1}{2d_n}\right)^{N-2} + o\left(\frac{1}{d_n^{N-2}}\right) \quad \text{as } d_n \rightarrow 0 \tag{2.21}$$

(see [13]:(2.8)).

(2.19) is a consequence of

$$\int_0^\infty \frac{s^{N-1}}{(1+s^2)^{\frac{N+2}{2}}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(1)}{2\Gamma(\frac{N+2}{2})} = \frac{1}{N},$$

where we used the formula

$$\int_0^\infty \frac{s^\alpha}{(1+s^2)^\beta} ds = \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{2\beta-\alpha-1}{2})}{2\Gamma(\beta)} \tag{2.22}$$

for $\alpha > 0, \beta > 0$ and $2\beta - \alpha - 1 > 0$.

From (2.10)-(2.18), we obtain the conclusion of Lemma 2.1. $\square$

Lemma 2.2.(Asymptotic behavior of L^2 norm of the main part)

When $N \geq 5$, we have

$$\int_{\Omega} PU_{\lambda_n, a_n}^2 dx = \omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad \text{as } n \rightarrow \infty,$$

where

$$C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)}.$$

When $N = 4$, we have

$$\begin{aligned}
\int_{\Omega} PU_{\lambda_n, a_n}^2 dx &= \omega_4 \lambda_n^2 |\log \lambda_n| + o(\lambda_n^2 |\log \lambda_n|) \\
&+ O\left(\frac{\lambda_n^2}{d_n} |\log \lambda_n|^{1/2}\right) + O\left(\frac{\lambda_n^2}{d_n^2}\right) \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

Proof ($N \geq 5$). We extend PU_{λ_n, a_n} and φ_{λ_n, a_n} to $\mathbf{R}^N$ by setting $PU_{\lambda_n, a_n} = 0$ in $\mathbf{R}^N \setminus \Omega$ and $\varphi_{\lambda_n, a_n} = U_{\lambda_n, a_n}$ in $\mathbf{R}^N \setminus \Omega$. We denote them again by PU_{λ_n, a_n} and φ_{λ_n, a_n} respectively.

Since $PU_{\lambda_n, a_n} = U_{\lambda_n, a_n} - \varphi_{\lambda_n, a_n}$, we have

$$\begin{aligned} \int_{\Omega} PU_{\lambda_n, a_n}^2 dx &= \int_{\Omega} U_{\lambda_n, a_n}^2 dx + \int_{\Omega} \varphi_{\lambda_n, a_n}^2 dx \\ &\quad + O\left(\left(\int_{\Omega} U_{\lambda_n, a_n}^2 dx\right)^{1/2} \left(\int_{\Omega} \varphi_{\lambda_n, a_n}^2 dx\right)^{1/2}\right). \end{aligned} \quad (2.23)$$

We estimate the first term in (2.23) as follows: By monotonicity of the integral, we have

$$\int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^2 dx \leq \int_{\Omega} U_{\lambda_n, a_n}^2 dx \leq \int_{B_R(a_n)} U_{\lambda_n, a_n}^2 dx, \quad (2.24)$$

where $R = \text{diam}(\Omega)$.

Calculation shows

$$\begin{aligned} \int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^2 dx &= \omega_N \int_0^{d_n} \left(\frac{\lambda_n}{\lambda_n^2 + r^2} \right)^{N-2} r^{N-1} dr \\ &= \omega_N \lambda_n^2 \int_0^{d_n/\lambda_n} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \\ &= \omega_N \lambda_n^2 \left(\int_0^{\infty} - \int_{d_n/\lambda_n}^{\infty} \right) \\ &= \omega_N \lambda_n^2 \left(C_N + O\left(\int_{d_n/\lambda_n}^{\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \right) \right) \\ &= \omega_N C_N \lambda_n^2 + O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}} \right), \end{aligned}$$

here we have used the assumption $N \geq 5$.

The same calculation shows

$$\int_{B_R(a_n)} U_{\lambda_n, a_n}^2 dx = \omega_N C_N \lambda_n^2 + O(\lambda_n^{N-2}).$$

So dividing both the integrals of (2.24) by $\omega_N C_N \lambda_n^2$ and noting $(\lambda_n/d_n) = o(1)$ (see (2.7)), we obtain

$$\lim_{n \rightarrow \infty} \frac{\int_{\Omega} U_{\lambda_n, a_n}^2 dx}{\omega_N C_N \lambda_n^2} = 1,$$

$$\int_{\Omega} U_{\lambda_n, a_n}^2 dx = \omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad (n \rightarrow \infty). \quad (2.25)$$

To estimate the second term in (2.23), we divide the integral in two parts:

$$\int_{\Omega} \varphi_{\lambda_n, a_n}^2 dx = \int_{B_{d_n}(a_n)} \varphi_{\lambda_n, a_n}^2 dx + \int_{\Omega \setminus B_{d_n}(a_n)} \varphi_{\lambda_n, a_n}^2 dx.$$

Then:

$$\begin{aligned} \int_{B_{d_n}(a_n)} \varphi_{\lambda_n, a_n}^2 dx &= O\left(\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)}^2 \cdot \text{vol}(B_{d_n}(a_n))\right) \\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right)^2 \cdot d_n^N\right) = O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right) \end{aligned}$$

by (2.15), and

$$\begin{aligned} \int_{\Omega \setminus B_{d_n}(a_n)} \varphi_{\lambda_n, a_n}^2 dx &= O\left(\int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^2 dx\right) \\ &= O\left(\int_{d_n}^{\infty} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^{N-2} r^{N-1} dr\right) \\ &= O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right), \end{aligned}$$

since $0 < \varphi_{\lambda_n, a_n} < U_{\lambda_n, a_n}$ in Ω and $\varphi_{\lambda_n, a_n} = U_{\lambda_n, a_n}$ on $\mathbf{R}^N \setminus \Omega$.

In conclusion, we have

$$\int_{\Omega} \varphi_{\lambda_n, a_n}^2 dx = O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right) = o(\lambda_n^2) \quad \text{as } n \rightarrow \infty. \quad (2.26)$$

By (2.23), (2.25) and (2.26), we have the conclusion of Lemma 2.2. $\square$

From Lemma 2.1, Lemma 2.2 and the fact that

$$\int_{\Omega} |\nabla v_n|^2 dx = \alpha_n^2 \int_{\Omega} |\nabla P U_{\lambda_n, a_n}|^2 dx + \int_{\Omega} |\nabla w_n|^2 dx$$

(which follows since $w_n \in E_{\lambda_n, a_n}$; see (2.8)), we have the following lemma, for example when $N \geq 5$.

Lemma 2.3.(Asymptotic behavior of J_n) When $N \geq 5$, we have

$$\begin{aligned}
J_n &:= \int_{\Omega} |\nabla v_n|^2 dx - \varepsilon_n \int_{\Omega} v_n^2 dx \\
&= \alpha_n^2 \left\{ N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} \right\} - \varepsilon_n \alpha_n^2 \omega_N C_N \lambda_n^2 \\
&+ \|\nabla w_n\|_{L^2(\Omega)}^2 - \varepsilon_n \|w_n\|_{L^2(\Omega)}^2 + O\left(\frac{\lambda_n^N}{d_n^N} \left| \log\left(\frac{\lambda_n}{d_n}\right) \right| \right) + o(\varepsilon_n \lambda_n^2) \\
&+ O(\varepsilon_n \lambda_n \|w_n\|_{L^2(\Omega)}) \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

To proceed further, we need the precise asymptotic behavior of α_n as $n \rightarrow \infty$. This is given by the next lemma.

Lemma 2.4.(Asymptotic behavior of α_n)

When $N \geq 4$, we have

$$\alpha_n^2 = \alpha_N^2 + \alpha_N^2 \left(\frac{N-2}{N} \right) \left(\frac{2\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} + O\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)$$

as $n \rightarrow \infty$, where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$.

Proof. After extending v_n , PU_{λ_n, a_n} , and w_n by 0 outside Ω , we have

$$S^{N/2} = \int_{\Omega} v_n^{p+1} dx = \int_{\mathbf{R}^N} |\alpha_n PU_{\lambda_n, a_n} + w_n|^{p+1} dx \quad (2.27)$$

by (2.3). We set $W_n := -\alpha_n \varphi_{\lambda_n, a_n} + w_n$, here as before, φ_{λ_n, a_n} is extended to $\mathbf{R}^N$ by U_{λ_n, a_n} on $\mathbf{R}^N \setminus \Omega$.

By expanding the right hand side of (2.27), we have

$$\begin{aligned}
S^{N/2} &= \int_{\mathbf{R}^N} (\alpha_n U_{\lambda_n, a_n} + W_n)^{p+1} dx \\
&= \alpha_n^{p+1} \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx + (p+1) \alpha_n^p \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^p W_n dx \\
&+ O\left(\int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p-1} W_n^2 dx + \int_{\mathbf{R}^N} |W_n|^{p+1} dx\right). \quad (2.28)
\end{aligned}$$

First, we know

$$\alpha_n^{p+1} \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx = \alpha_n^{p+1} A. \quad (2.29)$$

Next, by using the equation $-\Delta U_{\lambda_n, a_n} = N(N-2)U_{\lambda_n, a_n}^p$ in $\mathbf{R}^N$, we calculate

$$\begin{aligned}
& (p+1)\alpha_n^p \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^p W_n dx = \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbf{R}^N} (-\Delta U_{\lambda_n, a_n}) W_n dx \\
&= \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbf{R}^N} \nabla U_{\lambda_n, a_n} \cdot \nabla W_n dx \\
&= \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbf{R}^N} (\nabla P U_{\lambda_n, a_n} + \nabla \varphi_{\lambda_n, a_n}) \cdot (-\alpha_n \nabla \varphi_{\lambda_n, a_n} + \nabla w_n) dx \\
&= \frac{-2\alpha_n^{p+1}}{(N-2)^2} \int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n, a_n}|^2 dx \\
&= \frac{-2\alpha_n^{p+1}}{(N-2)^2} \left\{ (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N} \left| \log\left(\frac{\lambda_n}{d_n}\right) \right| \right) \right\} \\
&= -2\alpha_n^{p+1} \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N} \left| \log\left(\frac{\lambda_n}{d_n}\right) \right| \right). \tag{2.30}
\end{aligned}$$

Here we have used the fact that φ_{λ_n, a_n} is a harmonic function on Ω , $w_n \in E_{\lambda_n, a_n}$ and

$$\begin{aligned}
& \int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n, a_n}|^2 dx = \int_{\mathbf{R}^N} |\nabla U_{\lambda_n, a_n}|^2 dx - \int_{\mathbf{R}^N} |\nabla P U_{\lambda_n, a_n}|^2 dx \\
&= (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N} \left| \log\left(\frac{\lambda_n}{d_n}\right) \right| \right) \tag{2.31}
\end{aligned}$$

by Lemma 2.1.

Now, we claim that the error term in (2.28) can be estimated as

$$O\left(\int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p-1} W_n^2 dx + \int_{\mathbf{R}^N} |W_n|^{p+1} dx\right) = O\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.32}$$

Indeed, we divide the integral as

$$\int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p-1} W_n^2 dx = \int_{\mathbf{R}^N \setminus \Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx + \int_{\Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx. \tag{2.33}$$

Since $W_n = -\alpha_n U_{\lambda_n, a_n}$ on $\mathbf{R}^N \setminus \Omega$, the first term in (2.33) is estimated as

$$\int_{\mathbf{R}^N \setminus \Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx = \alpha_n^2 \int_{\mathbf{R}^N \setminus \Omega} U_{\lambda_n, a_n}^{p+1} dx = O\left(\int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p+1} dx\right).$$

Now we compute

$$\int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p+1} dx = \omega_N \int_{d_n}^{\infty} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^N r^{N-1} dr = O\left(\frac{\lambda_n^N}{d_n^N}\right),$$

so we have

$$\int_{\mathbf{R}^N \setminus \Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx = O\left(\frac{\lambda_n^N}{d_n^N}\right). \quad (2.34)$$

Substituting W_n by $-\alpha_n \varphi_{\lambda_n, a_n} + w_n$ in the second term in (2.33), we have

$$\begin{aligned} \int_{\Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx &= \alpha_n^2 \int_{\Omega} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx + \int_{\Omega} U_{\lambda_n, a_n}^{p-1} w_n^2 dx \\ &+ O\left(\left(\int_{\Omega} U_{\lambda_n, a_n}^{p-1} w_n^2 dx\right)^{1/2} \left(\int_{\Omega} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx\right)^{1/2}\right). \end{aligned} \quad (2.35)$$

Now by Hölder and Sobolev inequality, we find

$$\begin{aligned} \int_{\Omega} U_{\lambda_n, a_n}^{p-1} w_n^2 dx &= O\left(\left(\int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx\right)^{\frac{p-1}{p+1}} \left(\int_{\Omega} w_n^{p+1} dx\right)^{\frac{2}{p+1}}\right) \\ &= O(\|\nabla w_n\|_{L^2(\Omega)}^2). \end{aligned} \quad (2.36)$$

On the other hand, when we estimate the first term in (2.35), we divide the integral as

$$\int_{\Omega} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx = \int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx + \int_{\Omega \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx. \quad (2.37)$$

First term in (2.37) is estimated as

$$\begin{aligned} \int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx &= O\left(\|\varphi_{\lambda_n, a_n}\|_{L^\infty(\Omega)}^2 \cdot \int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} dx\right) \\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right)^2 \cdot \lambda_n^2 d_n^{N-4}\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right). \end{aligned} \quad (2.38)$$

Here we have used the fact

$$\int_{B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} dx = \omega_N \int_0^{d_n} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^2 r^{N-1} dr = O(\lambda_n^2 d_n^{N-4}),$$

since $N \geq 5$.

Second term in (2.37) is estimated as before:

$$\int_{\Omega \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx = O\left(\int_{\mathbf{R}^N \setminus B_{d_n}(a_n)} U_{\lambda_n, a_n}^{p+1} dx\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right). \quad (2.39)$$

By (2.37)-(2.39), we have

$$\int_{\Omega} U_{\lambda_n, a_n}^{p-1} \varphi_{\lambda_n, a_n}^2 dx = O\left(\frac{\lambda_n^N}{d_n^N}\right). \quad (2.40)$$

Combining (2.35),(2.36) and (2.40), we obtain

$$\int_{\Omega} U_{\lambda_n, a_n}^{p-1} W_n^2 dx = O\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + O\left(\frac{\lambda_n^N}{d_n^N}\right). \quad (2.41)$$

Finally, by Sobolev inequality and convex inequality $(a+b)^t \leq C(a^t + b^t)$ for some $C > 0$ ($a, b > 0, t > 1$), we have

$$\begin{aligned} \int_{\mathbf{R}^N} |W_n|^{p+1} dx &= O\left(\left(\int_{\mathbf{R}^N} |\nabla W_n|^2 dx\right)^{\frac{p+1}{2}}\right) \\ &= O\left(\left(\int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n, a_n}|^2 dx + \int_{\mathbf{R}^N} |\nabla w_n|^2 dx\right)^{\frac{p+1}{2}}\right) \\ &= O\left(\left(\int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n, a_n}|^2 dx\right)^{\frac{p+1}{2}}\right) + O\left(\left(\int_{\mathbf{R}^N} |\nabla w_n|^2 dx\right)^{\frac{p+1}{2}}\right). \end{aligned} \quad (2.42)$$

(Recall we extend φ_{λ_n, a_n} to $\mathbf{R}^N \setminus \Omega$ by U_{λ_n, a_n}). So by (2.42), (2.31) and the estimate $H(a_n, a_n) = O(\frac{1}{d_n^{N-2}})$ (see (2.21)), we obtain

$$\begin{aligned} \int_{\mathbf{R}^N} |W_n|^{p+1} dx &= O\left(\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)^{\frac{N}{N-2}}\right) + O\left(\|\nabla w_n\|_{L^2(\Omega)}^{\frac{2N}{N-2}}\right) \\ &= O\left(\frac{\lambda_n^N}{d_n^N}\right) + o\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right). \end{aligned} \quad (2.43)$$

Combining (2.33),(2.34),(2.41) and (2.43), we conclude the claim (2.32).

Returning to (2.28) and using (2.29),(2.30) and (2.32), we obtain

$$S^{N/2} = \alpha_n^{p+1} A - 2\alpha_n^{p+1} \cdot \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

Dividing the both sides by A and noting that $\frac{S^{N/2}}{A} = \alpha_N^{p+1}$, we have

$$\alpha_N^{p+1} = \alpha_n^{p+1} - \alpha_n^{p+1} \left(\frac{2\omega_N^2}{A}\right) H(a_n, a_n) \lambda_n^{N-2} + O\left(\|\nabla w_n\|_{L^2(\Omega)}^2\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

From this we can derive the conclusion. □

Combining Lemma 2.3 and Lemma 2.4, we obtain:

Lemma 2.5.(Asymptotic behavior of S_{ε_n})

As $n \rightarrow \infty$,

$$\begin{aligned}
S_{\varepsilon_n} &:= \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_{\Omega} |\nabla v|^2 dx - \varepsilon_n \int_{\Omega} v^2 dx \right\} \\
&= S \cdot S^{-\frac{N}{2}} J_n \\
&= S + S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} - \varepsilon_n \left(\frac{S\omega_N C_N}{N(N-2)A} \right) \lambda_n^2 \\
&\quad + O(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\varepsilon_n \lambda_n^2). \quad (N \geq 5)
\end{aligned}$$

$$\begin{aligned}
S_{\varepsilon_n} &= S + \frac{S}{2} \left(\frac{\omega_4^2}{A} \right) H(a_n, a_n) \lambda_n^2 - \varepsilon_n \left(\frac{S\omega_4}{8A} \right) \lambda_n^2 |\log \lambda_n| \\
&\quad + O(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left(\frac{\lambda_n^2}{d_n^2}\right) + o(\varepsilon_n \lambda_n^2 |\log \lambda_n|). \quad (N = 4)
\end{aligned}$$

As for the “ w -part” of v_n , we have the following estimate due to Rey [13](Appendix C:(C.1)).

Lemma 2.6. *As $n \rightarrow \infty$, we have*

$$\|\nabla w_n\|_{L^2(\Omega)}^2 = o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\varepsilon_n \lambda_n^2).$$

Now, we need the appropriate bound of the value S_{ε_n} from the above. The restriction that we consider only least energy solutions is essential in the next lemma.

Lemma 2.7.(Upper bound of S_{ε})

For any $a \in \Omega$ and $\rho > 0$, there exists $\varepsilon_0 = \varepsilon_0(a, \rho)$ such that if $\varepsilon \in (0, \varepsilon_0)$, then the following holds:

$$\begin{aligned}
S_{\varepsilon} &= \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_{\Omega} |\nabla v|^2 dx - \varepsilon \int_{\Omega} v^2 dx \right\} \\
&\leq S - \left(\frac{N-4}{N-2} \right) \varepsilon \left\{ \frac{S\omega_N C_N}{N(N-2)A} - \rho \right\} \left[\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a, a)} \right]^{\frac{2}{N-4}}
\end{aligned}$$

when $N \geq 5$.

$$S_\varepsilon \leq S - \frac{S\varepsilon\omega_4}{16Ae} \exp\left(-\frac{8\omega_4 H(a, a) + \varepsilon/e + 2\rho}{\varepsilon}\right)$$

when $N = 4$.

Proof ($N \geq 5$). For $a \in \Omega$ and $\varepsilon > 0$, define $\psi_{\varepsilon, a} \in H_0^1(\Omega)$ as

$$\psi_{\varepsilon, a} := S^{-\frac{(N-2)}{4}} \alpha_N P U_{\lambda_a(\varepsilon), a}, \quad (2.44)$$

where

$$\lambda_a(\varepsilon) := \left[\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a, a)} \right]^{\frac{1}{N-4}}. \quad (2.45)$$

Note that $\lambda_a(\varepsilon)$ is the unique minimum point of the function

$$f(\lambda) = K_1 H(a, a) \lambda^{N-2} - K_2 \varepsilon \lambda^2 \quad \text{for } \lambda > 0,$$

and it gives the minimum value

$$\begin{aligned} \min_{\lambda > 0} f(\lambda) &= f(\lambda_a(\varepsilon)) = - \left(\frac{N-4}{N-2} \right) K_2 \varepsilon \left(\frac{2K_2 \varepsilon}{(N-2)K_1 H(a, a)} \right)^{\frac{2}{N-4}} \\ &= - \left(\frac{N-4}{N-2} \right) \varepsilon \left(\frac{S\omega_N C_N}{N(N-2)A} \right) \left(\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a, a)} \right)^{\frac{2}{N-4}}. \end{aligned} \quad (2.46)$$

Here, we denote

$$K_1 = S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right), \quad K_2 = \frac{S\omega_N C_N}{N(N-2)A}. \quad (2.47)$$

Define

$$J_\varepsilon(\psi) := \frac{\int_\Omega |\nabla \psi|^2 dx - \varepsilon \int_\Omega \psi^2 dx}{\left(\int_\Omega |\psi|^{p+1} dx \right)^{\frac{2}{p+1}}} \quad (2.48)$$

for $\psi \in H_0^1(\Omega) \setminus \{0\}$.

Now, we claim that:

$$\begin{aligned} J_\varepsilon(\psi_{\varepsilon, a}) &= S - \left(\frac{N-4}{N-2} \right) \varepsilon \left\{ \frac{S\omega_N C_N}{N(N-2)A} \right\} \left[\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a, a)} \right]^{\frac{2}{N-4}} \\ &\quad + o(\varepsilon^{\frac{N-2}{N-4}}) \end{aligned} \quad (2.49)$$

Indeed, as in the calculation in the proof of Lemma 2.1, Lemma 2.2 (note now $d(a, \partial\Omega)$ is a constant independent of ε), we have

$$\begin{aligned} \int_{\Omega} |\nabla \psi_{\varepsilon,a}|^2 dx &= S \cdot S^{-\frac{N}{2}} \alpha_N^2 \int_{\Omega} |\nabla P U_{\lambda_a(\varepsilon),a}|^2 dx \\ &= S - S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a, a) \lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon)), \end{aligned} \quad (2.50)$$

$$\begin{aligned} \int_{\Omega} \psi_{\varepsilon,a}^2 dx &= S \cdot S^{-\frac{N}{2}} \alpha_N^2 \int_{\Omega} P U_{\lambda_a(\varepsilon),a}^2 dx \\ &= \frac{S \omega_N C_N}{N(N-2)A} \lambda_a^2(\varepsilon) + o(\lambda_a^2(\varepsilon)) \end{aligned} \quad (2.51)$$

as $\varepsilon \rightarrow 0$.

Also by an argument similar to the one in the proof of Lemma 2.4, we have

$$\begin{aligned} \int_{\Omega} |\psi_{\varepsilon,a}|^{p+1} dx &= S^{-\frac{N}{2}} \alpha_N^{p+1} \int_{\Omega} |P U_{\lambda_a(\varepsilon),a}|^{p+1} dx \\ &= \frac{1}{A} \left\{ \int_{\Omega} U_{\lambda_a(\varepsilon),a}^{p+1} dx + (p+1) \int_{\Omega} U_{\lambda_a(\varepsilon),a}^p \varphi_{\lambda_a(\varepsilon),a} dx \right. \\ &\quad \left. + O \left(\int_{\Omega} U_{\lambda_a(\varepsilon),a}^{p-1} \varphi_{\lambda_a(\varepsilon),a}^2 dx + \int_{\Omega} |\varphi_{\lambda_a(\varepsilon),a}|^{p+1} dx \right) \right\} \\ &= \frac{1}{A} \left\{ A - 2\omega_N^2 \lambda_a^{N-2}(\varepsilon) H(a, a) + o(\lambda_a^{N-2}(\varepsilon)) \right\} \\ &= 1 - \left(\frac{2\omega_N^2}{A} \right) \lambda_a^{N-2}(\varepsilon) H(a, a) + o(\lambda_a^{N-2}(\varepsilon)). \end{aligned} \quad (2.52)$$

Note that $S^{N/2} = \alpha_N^2 N(N-2)A = \alpha_N^{p+1} A$.

So, by (2.50)-(2.52) and $(1+x)^{-\frac{2}{p+1}} = 1 - \frac{2}{p+1}x + o(x)$ as $x \rightarrow 0$, we obtain

$$\begin{aligned} &J_{\varepsilon}(\psi_{\varepsilon,a}) \\ &= \left\{ S - S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a, a) \lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon)) \right. \\ &\quad \left. - \varepsilon \left(\frac{S \omega_N C_N}{N(N-2)A} \right) \lambda_a^2(\varepsilon) + o(\varepsilon \lambda_a^2(\varepsilon)) \right\} \\ &\times \left\{ 1 + \frac{2}{p+1} \left(\frac{2\omega_N^2}{A} \right) H(a, a) \lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon)) \right\} \\ &= S + S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a, a) \lambda_a^{N-2}(\varepsilon) - \varepsilon \left(\frac{S \omega_N C_N}{N(N-2)A} \right) \lambda_a^2(\varepsilon) \end{aligned}$$

$$\begin{aligned}
& + o(\varepsilon \lambda_a^2(\varepsilon)) + o(\lambda_a^{N-2}(\varepsilon)) \\
& = S - \left(\frac{N-4}{N-2} \right) \varepsilon \left\{ \frac{S \omega_N C_N}{N(N-2)A} \right\} \left[\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a, a)} \right]^{\frac{2}{N-4}} \\
& + o(\varepsilon^{\frac{N-2}{N-4}})
\end{aligned} \tag{2.53}$$

as $\varepsilon \rightarrow 0$.

This proves the claim. The last equality in (2.53) follows from our choice of $\lambda_a(\varepsilon)$ (see (2.46)) and the fact

$$\varepsilon \lambda_a^2(\varepsilon) = C_1 \lambda_a^{N-2}(\varepsilon) = C_2 \varepsilon^{\frac{N-2}{N-4}}$$

by the definition of $\lambda_a(\varepsilon)$ (see (2.45)), where C_1, C_2 are constants independent of ε .

From (2.49) and the definition of S_ε , we obtain the conclusion of Lemma 2.7. $\square$

3. Proof of Theorem. In this section, we prove Theorem 1 by using lemmas we prepared in the previous section.

First, we will show that the blow up point a_∞ is in the interior of Ω .

Indeed, suppose the contrary. Then $a_\infty \in \partial\Omega$ and $d_n = d(a_n, \partial\Omega) \rightarrow 0$ as $n \rightarrow \infty$. Then by Lemma 2.5, Lemma 2.6 and the estimate (2.21), we can find constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}
S_{\varepsilon_n} & = S + S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} - \varepsilon_n \left(\frac{S \omega_N C_N}{N(N-2)A} \right) \lambda_n^2 \\
& + O \left(\|\nabla w_n\|_{L^2(\Omega)}^2 \right) + o \left(\frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) + o \left(\varepsilon_n \lambda_n^2 \right) \\
& \geq S + C_1 \left(\frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) - C_2 \varepsilon_n \lambda_n^2 \\
& \geq S - \left(\frac{N-4}{N-2} \right) C_2 \varepsilon_n \left\{ \frac{2C_2 \varepsilon_n}{(N-2)C_1 \left(\frac{1}{d_n^{N-2}} \right)} \right\}^{\frac{2}{N-4}} \\
& = S - C_3 \varepsilon_n^{\frac{N-2}{N-4}} d_n^{\frac{2(N-2)}{N-4}} = S + o(\varepsilon_n^{\frac{N-2}{N-4}}),
\end{aligned} \tag{3.1}$$

since we assume $d_n \rightarrow 0$.

Here as in the proof of Lemma 2.7, we have used the fact that $f(\lambda) = C_4 \lambda^{N-2} - C_5 \lambda^2$ has the unique global minimum value $-\left(\frac{N-4}{N-2}\right) C_5 \left(\frac{2C_5}{(N-2)C_4}\right)^{\frac{2}{N-4}}$ for $\lambda > 0$, where $C_4 = C_1 \left(\frac{1}{d_n^{N-2}}\right)$, $C_5 = C_2 \varepsilon_n$.

On the other hand, we know that $S_{\varepsilon_n} \leq S - C\varepsilon_n^{\frac{N-2}{N-4}} + o(\varepsilon_n^{\frac{N-2}{N-4}})$ for some $C > 0$ (see Lemma 2.7 (2.49)). This contradicts (3.1), so we conclude that a_∞ is in the interior of Ω .

Now, since we have proved that $d_n \geq C$ for some constant $C > 0$ uniformly in n , we may drop d_n in the asymptotic formulas Lemma 2.5 and Lemma 2.6.

Therefore, we can find $p_n > 0, p_n \rightarrow 0$ and $q_n > 0, q_n \rightarrow 0$ such that

$$\begin{aligned} S_{\varepsilon_n} &= S + S \left(\frac{N-2}{N} \right) \left(\frac{\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} - \varepsilon_n \left(\frac{S\omega_N C_N}{N(N-2)A} \right) \lambda_n^2 \\ &\quad + o(\lambda_n^{N-2}) + o(\varepsilon_n \lambda_n^2) \\ &\geq S + (K_1 H(a_n, a_n) - p_n) \lambda_n^{N-2} - (K_2 + q_n) \varepsilon_n \lambda_n^2 \\ &\geq S - \left(\frac{N-4}{N-2} \right) (K_2 + q_n) \varepsilon_n \left[\frac{2(K_2 + q_n) \varepsilon_n}{(N-2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}} \end{aligned} \quad (3.2)$$

where K_1, K_2 are defined in (2.47). The last inequality of (3.2) follows again by the property of the function $f(\lambda) = C_4 \lambda^{N-2} - C_5 \lambda^2$.

Combine (3.2) with Lemma 2.7, we have

$$\begin{aligned} S - \left(\frac{N-4}{N-2} \right) (K_2 + q_n) \varepsilon_n \left[\frac{2(K_2 + q_n) \varepsilon_n}{(N-2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}} \\ \leq S_{\varepsilon_n} \leq \\ S - \left(\frac{N-4}{N-2} \right) (K_2 - \rho) \varepsilon_n \left[\frac{2K_2 \varepsilon_n}{(N-2)K_1 H(a, a)} \right]^{\frac{2}{N-4}} \end{aligned}$$

for any $a \in \Omega$ and $\rho > 0$, if n sufficiently large.

From this we obtain

$$(K_2 + q_n) \varepsilon_n \left[\frac{2(K_2 + q_n) \varepsilon_n}{(N-2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}} \geq (K_2 - \rho) \varepsilon_n \left[\frac{2K_2 \varepsilon_n}{(N-2)K_1 H(a, a)} \right]^{\frac{2}{N-4}}.$$

Dividing both sides by $\varepsilon_n^{\frac{N-2}{N-4}}$ and letting $n \rightarrow \infty$, we have

$$K_2 \left[\frac{2K_2}{(N-2)K_1 H(a_\infty, a_\infty)} \right]^{\frac{2}{N-4}} \geq (K_2 - \rho) \left[\frac{2K_2}{(N-2)K_1 H(a, a)} \right]^{\frac{2}{N-4}}. \quad (3.3)$$

For $\rho > 0$ can be arbitrary small, (3.3) implies

$$H(a_\infty, a_\infty) \leq H(a, a)$$

for any $a \in \Omega$.

Therefore we conclude that a_∞ minimizes the Robin function $H(a, a)$. This completes the proof of Theorem. $\square$

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