

Subordination relations for certain analytic functions missing some coefficients

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Abstract

For a positive integer n , applying Schwarz's lemma related to analytic functions $w(z) = c_n z^n + \cdots$ in the open unit disk $\mathbb{U}$, some properties concerning with the subordinations for functions $f(z) = a + a_1 z + \cdots$ and $g(z) = a + b_n z^n + \cdots$ which are analytic in $\mathbb{U}$ are discussed, and an extension of some subordination relation which was proven by T. J. Suffridge (Duke Math. J. **37** (1970), 775–777) is given.

1 Introduction and preliminaries

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For a positive integer n and a complex number a , let $\mathcal{H}[a, n]$ be the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

Also, let $\mathcal{A}$ denote the class of functions $f(z) \in \mathcal{H}$ normalized by $f(0) = 0$ and $f'(0) = 1$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. An important member of the class $\mathcal{S}$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k.$$

A function $f(z) \in \mathcal{H}$ is said to be convex in $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a convex domain (A domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex if the line segment joining any two points of $\mathbb{D}$ lies entirely in $\mathbb{D}$). It is well-known that the function $f(z)$ is convex in $\mathbb{U}$ if and only if $f'(0) \neq 0$ and

$$(1.1) \quad \operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

The normalized class of convex functions denoted by $\mathcal{K}$ consists of the set of all functions $f(z) \in \mathcal{S}$ for which $f(\mathbb{U})$ is convex.

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Furthermore, a function $f(z) \in \mathcal{H}$ is said to be starlike in $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike domain (A domain $\mathbb{D} \subset \mathbb{C}$ is said to be starlike with respect to the origin if $0 \in \mathbb{D}$ and the line segment joining 0 and any point of $\mathbb{D}$ lies entirely in $\mathbb{D}$). It is well-known that the function $f(z)$ is starlike in $\mathbb{U}$ if and only if $f(0) = 0$, $f'(0) \neq 0$ and

$$(1.2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

The class of starlike functions denoted by $\mathcal{S}^*$ consists of the set of all functions $f(z) \in \mathcal{S}$ for which $f(\mathbb{U})$ is starlike.

The equivalent analytic descriptions of $\mathcal{K}$ and $\mathcal{S}^*$ are given respectively as follows.

Remark 1.1 A necessary and sufficient condition for $f(z) \in \mathcal{K}$ is that $f(z) \in \mathcal{A}$ satisfies the inequality (1.1). Also, $f(z) \in \mathcal{S}^*$ if and only if $f(z) \in \mathcal{A}$ satisfies the inequality (1.2).

From the definitions of $\mathcal{K}$ and $\mathcal{S}^*$, we know that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

We next introduce the familiar principle of differential subordinations between analytic functions. Let $f(z)$ and $g(z)$ be members of the class $\mathcal{H}$. Then the function $g(z)$ is said to be subordinate to $f(z)$ in $\mathbb{U}$, written by

$$(1.3) \quad g(z) \prec f(z) \quad (z \in \mathbb{U}),$$

if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$), and such that $g(z) = f(w(z))$ ($z \in \mathbb{U}$). From the definition of the subordinations, it is easy to show that the subordination (1.3) implies that

$$(1.4) \quad g(0) = f(0) \quad \text{and} \quad g(\mathbb{U}) \subset f(\mathbb{U}).$$

In particular, if $f(z)$ is univalent in $\mathbb{U}$, then the subordination (1.3) is equivalent to the condition (1.4).

In order to discuss our main results, we need the following lemma which is well-known as Jack's lemma [3] proven by Miller and Mocanu [5] (see also [4]). For $0 < r_0 < 1$, we let

$$\mathbb{U}_{r_0} = \{z : z \in \mathbb{C} \text{ and } |z| < r_0\}, \quad \partial\mathbb{U}_{r_0} = \{z : z \in \mathbb{C} \text{ and } |z| = r_0\}$$

and $\overline{\mathbb{U}_{r_0}} = \mathbb{U}_{r_0} \cup \partial\mathbb{U}_{r_0}$. Jack's lemma is contained in Lemma 1.2.

Lemma 1.2 *Let n be a positive integer, and let $z_0 \in \mathbb{U}$ with $|z_0| = r$ and $0 < r < 1$. Also, let*

$$(1.5) \quad w(z) = \sum_{k=n}^{\infty} c_k z^k = c_n z^n + c_{n+1} z^{n+1} + \dots$$

be continuous on $\overline{\mathbb{U}_r}$ and analytic on $\mathbb{U}_r \cup \{z_0\}$ with $w(z) \neq 0$. If

$$|w(z_0)| = \max_{z \in \overline{\mathbb{U}_r}} |w(z)|,$$

then there exists a number k with $k \geq n$, and such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k.$$

Suffridge [6] independently discovered some particular case of Jack's lemma from a result of Julia [1], and deduced the following subordination relation for convex functions by making good use of it.

Lemma 1.3 *Let $f(z) \in \mathcal{K}$ and $g(z) \in \mathcal{H}[0, 1]$. If $zg'(z) \prec zf'(z)$ ($z \in \mathbb{U}$), then $g(z) \prec f(z)$ ($z \in \mathbb{U}$).*

In the present paper, applying Schwarz's lemma related to analytic function $w(z)$ which has the form (1.5), we discuss some subordination properties containing the Lindelöf (or subordination) principle (cf. [2, Vol. I, Theorem 10]) concerning with the subordination (1.3) for $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$. Further, by making use of those properties and Lemma 1.2, we deduce the following subordination relation

$$(1.6) \quad zg'(z) \prec nzf'(z) \quad (z \in \mathbb{U}) \quad \text{implies} \quad g(z) \prec f(z) \quad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$, where $f(z)$ is convex in $\mathbb{U}$.

2 Some properties for certain subordination

To considering some subordination properties, we need Schwarz's lemma related to $w(z) \in \mathcal{H}[0, n]$ (see [2, Vol. I, Theorem 11]) as follows.

Lemma 2.1 *Let $w(z) = \sum_{k=n}^{\infty} c_k z^k \in \mathcal{H}[0, n]$. If $w(z)$ satisfies $|w(z)| < 1$ ($z \in \mathbb{U}$), then*

$$(2.1) \quad |w(z)| \leq |z|^n$$

for each point $z \in \mathbb{U}$. Further, if equality occurs in the inequality (2.1) for one point $z_0 \in \mathbb{U} \setminus \{0\}$, then

$$(2.2) \quad w(z) = xz^n$$

for some complex number x with $|x| = 1$, and the equality in the inequality (2.1) holds for all $z \in \mathbb{U}$. Finally, we have

$$|c_n| = \frac{1}{n!} |w^{(n)}(0)| \leq 1,$$

and $|c_n| = 1$ if and only if $w(z)$ is given by the equation (2.2).

Proof. If we define the function $\phi(z)$ by

$$\phi(z) = \frac{w(z)}{z^n} \quad (z \in \mathbb{U}),$$

then $\phi(z)$ is analytic in $\mathbb{U}$ with $\phi(0) = c_n$. Suppose that $0 < \rho < 1$. Then the modulus $|\phi(z)|$ for $z \in \overline{\mathbb{U}_\rho}$ takes the maximum value at some point $z_\rho \in \partial\mathbb{U}_\rho$. Hence this fact combined with $|w(z)| < 1$ ($z \in \mathbb{U}$) yields that

$$|\phi(z)| \leq |\phi(z_\rho)| = \frac{|w(z_\rho)|}{|z_\rho|^n} < \frac{1}{\rho^n} \quad (z \in \overline{\mathbb{U}_\rho}),$$

which is satisfied for every ρ with $0 < \rho < 1$. Thus, by letting $\rho \rightarrow 1^-$, we obtain

$$(2.3) \quad |\phi(z)| \leq 1 \quad (z \in \mathbb{U}),$$

which implies that the inequality (2.1). Then since the inequality (2.3) is satisfied for all interior points of $\mathbb{U}$, we may conclude that

$$|\phi(0)| = |c_n| \leq 1.$$

Moreover, if there is a point $z_0 \in \mathbb{U} \setminus \{0\}$ such that $|w(z_0)| = |z_0|^n$, then we have $|\phi(z_0)| = 1$. Also, it is clear that $|c_n| = 1$ if and only if $|\phi(0)| = 1$. Thus, from the inequality (2.3), by applying the **maximum modulus principle** to analytic function $\phi(z)$, we see that $\phi(z)$ is constant and $|\phi(z)| = 1$, and hence we must have $\phi(z) = x$ for some complex number x with $|x| = 1$, which implies that equation (2.2). $\square$

Applying Lemma 2.1, we can obtain the following coefficient estimation by the subordination (1.3) for $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$.

Theorem 2.2 Let $f(z) = a + \sum_{k=1}^{\infty} a_k z^k \in \mathcal{H}[a, 1]$ and $g(z) = a + \sum_{k=n}^{\infty} b_k z^k \in \mathcal{H}[a, n]$. If $g(z) \prec f(z)$ ($z \in \mathbb{U}$), then

$$|b_n| \leq |a_1|,$$

and equality occurs if and only if $g(z) = f(xz^n)$ for some complex number x with $|x| = 1$.

Proof. For $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$, since $g(z) \prec f(z)$ ($z \in \mathbb{U}$), there exists an analytic function $w(z) \in \mathcal{H}[0, n]$ which has the form (1.5), with $|w(z)| < 1$ ($z \in \mathbb{U}$), and such that $g(z) = f(w(z))$ ($z \in \mathbb{U}$), where

$$f(w(z)) = a + a_1 c_n z^n + \cdots + a_1 c_{2n-1} z^{2n-1} + (a_1 c_{2n} + a_2 c_n^2) z^{2n} + \cdots.$$

Then, by comparing the coefficient of z^n in the both sides of equality $g(z) = f(w(z))$, we obtain $b_n = a_1 c_n$, and hence by Lemma 2.1, we have

$$|b_n| = |a_1 c_n| \leq |a_1|.$$

Further, since $|c_n| = 1$ if and only if equation (2.2) holds true, it is clear that $|b_n| = |a_1|$ if and only if $g(z) = f(xz^n)$ for some complex number x with $|x| = 1$. $\square$

Remark 2.3 Letting $n = 1$ in Theorem 2.2, we know that $g(z) \prec f(z)$ ($z \in \mathbb{U}$) which implies that $|g'(0)| \leq |f'(0)|$ with equality for $g(z) = f(xz)$, where $|x| = 1$.

Moreover, Lemma 2.1 provides a slight extension of the Lindelöf principle below.

Theorem 2.4 *Let $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$. If $g(z) \prec f(z)$ ($z \in \mathbb{U}$), then*

$$g(\mathbb{U}_r) \subset f(\mathbb{U}_{r^n})$$

for each r with $0 < r < 1$. Further, if $g(z_0)$ is on the boundary of $f(\mathbb{U}_{r^n})$ for one point $z_0 \in \partial\mathbb{U}_r$, then there is a complex number x with $|x| = 1$ such that $g(z) = f(xz^n)$, and $g(z)$ is on the boundary of $f(\mathbb{U}_{r^n})$ for every point $z \in \partial\mathbb{U}_r$.

Proof. If $g(z) \prec f(z)$ ($z \in \mathbb{U}$), then there exists an analytic function $w(z) \in \mathcal{H}[0, n]$ such that $g(z) = f(w(z))$ ($z \in \mathbb{U}$) and $w(z)$ satisfies the conditions of Lemma 2.1. Hence by Lemma 2.1, since $w(\mathbb{U}_r) \subset \mathbb{U}_{r^n}$ for each r with $0 < r < 1$, we see that

$$g(\mathbb{U}_r) = f(w(\mathbb{U}_r)) \subset f(\mathbb{U}_{r^n})$$

for each r with $0 < r < 1$.

Further, if there is a point $z_0 \in \partial\mathbb{U}_r$ such that $g(z_0)$ is a boundary point of $f(\mathbb{U}_{r^n})$, then since $w(z_0)$ is on the boundary of $\mathbb{U}_{r^n}$, we have

$$|w(z_0)| = r^n,$$

where $|z_0| = r$, and this means that the equality for (2.1) holds for a point $z_0 \in \partial\mathbb{U}_r$. Thus, by Lemma 2.1, we have $w(z) = xz^n$ for some complex number x with $|x| = 1$, which implies that

$$g(z) = f(xz^n),$$

where $|x| = 1$. And, from the above equation, it is easy to see that $g(z)$ is the boundary point of $f(\mathbb{U}_{r^n})$ for every point $z \in \partial\mathbb{U}_r$. $\square$

Remark 2.5 Since $f(\mathbb{U}_{r^n}) \subset f(\mathbb{U}_r)$ for each r with $0 < r < 1$, we know that $g(z) \prec f(z)$ ($z \in \mathbb{U}$) implies

$$g(\mathbb{U}_r) \subset f(\mathbb{U}_r)$$

for each r with $0 < r < 1$ by Theorem 2.4. This is the well-known property as the Lindelöf principle (cf. [2]). Also, Theorem 2.4 for $n = 1$ is the Lindelöf principle.

As an example of Theorem 2.4, we give the following.

Example 2.6 For a complex number a such that $|a| < M$ with $M > 0$, let us consider two functions $f(z)$ and $g(z)$ given respectively by

$$(2.4) \quad f(z) = M \frac{a + Mz}{M + \bar{a}z} \in \mathcal{H}[a, 1] \quad \text{and} \quad g(z) = M \frac{a + Mz^n}{M + \bar{a}z^n} \in \mathcal{H}[a, n].$$

Then, since $f(z^n) \prec f(z)$ ($z \in \mathbb{U}$), we know that

$$g(z) = f(z^n) \prec f(z) \quad (z \in \mathbb{U}).$$

For a radius r with $0 < r < 1$, a simple check shows that $f(z)$ given by (2.4) maps $\mathbb{U}_r$ onto the interior of the circle with radius R and center at C , where

$$(2.5) \quad R = \frac{r^n M(M^2 - |a|^2)}{M^2 - r^{2n}|a|^2} \quad \text{and} \quad C = \frac{(1 - r^{2n})M^2 a}{M^2 - r^{2n}|a|^2} \quad (|a| < M).$$

Further the calculation provides that $g(z)$ given by (2.4) maps $\mathbb{U}_r$ onto the open disk with radius R and center at C , where R and C are defined by (2.5), and hence we must have

$$g(\mathbb{U}_r) = f(\mathbb{U}_{r^n})$$

for each r with $0 < r < 1$.

3 An extension of some subordination relation

Applying some subordination properties which were discussed in the previous section, and by using Lemma 1.2, we will give the proof of the subordination relation (1.6).

Theorem 3.1 *Let $f(z) \in \mathcal{H}[a, 1]$ and $g(z) \in \mathcal{H}[a, n]$, and suppose that $f(z)$ is convex in $\mathbb{U}$. If $zg'(z) \prec nf'(z)$ ($z \in \mathbb{U}$), then $g(z) \prec f(z)$ ($z \in \mathbb{U}$).*

Proof. If we let

$$F(z) = nzf'(z) \quad \text{and} \quad G(z) = zg'(z),$$

then the assumption $zg'(z) \prec nf'(z)$ ($z \in \mathbb{U}$) can be rewritten by

$$(3.1) \quad G(z) \prec F(z) \quad (z \in \mathbb{U}).$$

Moreover, if we set

$$f(z) = a + \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = a + \sum_{k=n}^{\infty} b_k z^k,$$

then

$$F(z) = n \sum_{k=1}^{\infty} k a_k z^k \quad \text{and} \quad G(z) = \sum_{k=n}^{\infty} k b_k z^k.$$

It follows that the subordination (3.1) implies $|b_n| \leq |a_1|$ by Theorem 2.2. Since $|b_n| = |a_1|$ if and only if

$$(3.2) \quad G(z) = F(xz^n),$$

where $|x| = 1$, we have $zg'(z) = nxz^n f'(xz^n)$ which implies that $g(z) = f(xz^n)$, where $|x| = 1$, and this means that $g(z) \prec f(z)$ ($z \in \mathbb{U}$). Therefore, we may continue the argument assuming that $|b_n| < |a_1|$.

Suppose that $g(z)$ is not subordinate to $f(z)$ in $\mathbb{U}$. If we let

$$f_n(z) = f(z^n) = a + \sum_{k=1}^{\infty} a_k z^{nk} = a + a_1 z^n + a_2 z^{2n} + \cdots,$$

then since $|b_n| < |a_1|$ implies that $g(\mathbb{U}_\varepsilon) \subset f_n(\mathbb{U}_\varepsilon) = f(\mathbb{U}_{\varepsilon^n})$ for all sufficiently small values of ε , there exists a radius r with $0 < r < 1$ such that $g(re^{i\theta}) = f_n(re^{i\varphi}) = f(r^n e^{in\varphi})$ for some real θ and φ , and that $g(\overline{\mathbb{U}_r}) \subset f_n(\overline{\mathbb{U}_r}) = f(\overline{\mathbb{U}_{r^n}})$. Since $f(z) \in \mathcal{H}[a, 1]$ is convex in $\mathbb{U}$ which implies that $f(z)$ is univalent in $\mathbb{U}$, we know that the inverse f^{-1} is analytic in a domain $\mathbb{D} = f(\mathbb{U})$ and maps $\mathbb{D}$ onto $\mathbb{U}$ with $f^{-1}(a) = 0$. Then from $g(\overline{\mathbb{U}_r}) \subset f(\overline{\mathbb{U}_{r^n}}) \subset f(\mathbb{U})$, we see that f^{-1} is analytic on $g(\overline{\mathbb{U}_r})$. Also, it follows from $g(z) \in \mathcal{H}[a, n]$ that $g(z)$ is analytic on $\overline{\mathbb{U}_r}$. Therefore, we may define a function $w(z)$ by

$$(3.3) \quad w(z) = f^{-1}(g(z)) \quad (z \in \overline{\mathbb{U}_r})$$

which has the form (1.5). Then $w(z)$ is analytic on $\overline{\mathbb{U}_r}$ with $w(0) = f^{-1}(g(0)) = f^{-1}(a) = 0$. Since $w(\overline{\mathbb{U}_r}) = f^{-1}(g(\overline{\mathbb{U}_r})) \subset \overline{\mathbb{U}_{r^n}}$, we have

$$|w(z)| \leq r^n \quad (z \in \overline{\mathbb{U}_r}).$$

Further, noting that $w(re^{i\theta}) = f^{-1}(g(re^{i\theta})) = f^{-1}(f(r^n e^{in\varphi})) = r^n e^{in\varphi}$, we find that

$$|w(re^{i\theta})| = |r^n e^{in\varphi}| = r^n = \max_{z \in \overline{\mathbb{U}_r}} |w(z)|.$$

That is, the modulus $|w(z)|$ takes the maximum value r^n at a point $z = re^{i\theta} \in \overline{\mathbb{U}_r}$. Thus, according to Lemma 1.2, there is a real number k so that $k \geq n \geq 1$ and

$$(3.4) \quad \frac{z_0 w'(z_0)}{w(z_0)} = k,$$

where $z_0 = re^{i\theta}$. Equation (3.3) implies that $g(z) = f(w(z))$ and $g'(z) = w'(z)f'(w(z))$. If we use these relations at $z = z_0$ and equation (3.4), then from $w(z_0) = w(re^{i\theta}) = r^n e^{in\varphi}$, we see that

$$(3.5) \quad k = \frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0 g'(z_0)}{w(z_0) f'(w(z_0))} = \frac{re^{i\theta} g'(re^{i\theta})}{r^n e^{in\varphi} f'(r^n e^{in\varphi})} \geq n.$$

Since $F(z) = nzf'(z)$ and $G(z) = zg'(z)$, the inequality (3.5) is the same as

$$(3.6) \quad \frac{G(re^{i\theta})}{F(r^n e^{in\varphi})} \geq 1.$$

In addition, it follows from the inequality (3.6) that

$$(3.7) \quad \arg(G(re^{i\theta})) = \arg(F(r^n e^{in\varphi})).$$

Now, as $f(z) = a + \sum_{k=1}^{\infty} a_k z^k$ is convex in $\mathbb{U}$, we have $\frac{f(z) - a}{a_1} \in \mathcal{K}$. Moreover, since

$$\frac{f(z) - a}{a_1} \in \mathcal{K} \quad \text{if and only if} \quad z \left(\frac{f(z) - a}{a_1} \right)' \in \mathcal{S}^*,$$

it is clear that $F(z) = nzf'(z)$ is starlike and univalent in $\mathbb{U}$. By Theorem 2.4, we see that $G(z) \prec F(z)$ ($z \in \mathbb{U}$) implies

$$(3.8) \quad G(\mathbb{U}_r) \subset F(\mathbb{U}_{r^n}) \quad (0 < r < 1).$$

Then, since $G(0) = F(0) = 0$ and $F(\mathbb{U})$ is starlike with respect to the origin, we find that

$$(3.9) \quad |G(re^{i\theta})| \leq |F(r^n e^{in\varphi})|$$

for some real θ and φ which satisfy the equality (3.7). Here, if $G(z_0)$ is on the boundary of $F(\mathbb{U}_{r^n})$ for one point $z_0 \in \partial\mathbb{U}_r$, then by Theorem 2.4, we have $G(z) = F(xz^n)$, where $|x| = 1$, and $G(z)$ is on the boundary of $F(\mathbb{U}_{r^n})$ for every point $z \in \partial\mathbb{U}_r$. From this fact and the relation (3.8), we see that the equality in the inequality (3.9) occurs for $G(z) = F(xz^n)$, where $|x| = 1$. But we now continue the argument assuming that $|b_n| < |a_1|$, which is same as that $G(z)$ does not have the form (3.2). Therefore, we obtain that $G(z) \prec F(z)$ ($z \in \mathbb{U}$) which implies that

$$(3.10) \quad |G(re^{i\theta})| < |F(r^n e^{in\varphi})|$$

for some real θ and φ which satisfy the equality (3.7). Moreover, the inequality (3.10) combined with the equality (3.7) yields that

$$(3.11) \quad \frac{G(re^{i\theta})}{F(r^n e^{in\varphi})} < 1.$$

From the above-mentioned, since the inequality (3.6) contradicts the inequality (3.11), we see that the inequality (3.6) contradicts the assumption (3.1) of the theorem, and hence we must have $g(z) \prec f(z)$ ($z \in \mathbb{U}$). Therefore, we conclude that $zg'(z) \prec nzf'(z)$ ($z \in \mathbb{U}$) implies $g(z) \prec f(z)$ ($z \in \mathbb{U}$), which completes the proof of Theorem 3.1. $\square$

Remark 3.2 Letting $n = 1$, $a = 0$ and $a_1 = 1$ in Theorem 3.1, we obtain Lemma 1.3 which was shown by Suffridge [6].

As an example of Theorem 3.1, we introduce the following.

Example 3.3 For a complex number a such that $\operatorname{Re} a \neq 0$, let us consider two functions $f(z)$ and $g(z)$ given respectively by

$$(3.12) \quad f(z) = \frac{a + \bar{a}z}{1 - z} \in \mathcal{H}[a, 1] \quad \text{and} \quad g(z) = \frac{a + \bar{a}z^n}{1 - z^n} \in \mathcal{H}[a, n].$$

Then, it is easy to see that the above function $f(z)$ is convex and univalent in $\mathbb{U}$. A simple calculation yields that

$$zf'(z) = \frac{2(\operatorname{Re} a)z}{(1 - z)^2} = 2(\operatorname{Re} a)k(z)$$

and

$$zg'(z) = \frac{2n(\operatorname{Re} a)z^n}{(1 - z^n)^2} = 2n(\operatorname{Re} a)k(z^n),$$

where $k(z)$ is the Koebe function. Since $k(z^n) \prec k(z)$ ($z \in \mathbb{U}$), we have

$$zg'(z) = 2n(\operatorname{Re} a)k(z^n) \prec 2n(\operatorname{Re} a)k(z) = nzf'(z) \quad (z \in \mathbb{U}),$$

which implies that all the assumptions of Theorem 3.1 are satisfied. Hence by Theorem 3.1, we obtain the subordination relation (1.6). Actually, it is clear that

$$g(z) = f(z^n) \prec f(z) \quad (z \in \mathbb{U})$$

for $f(z)$ and $g(z)$ given by (3.12).

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