

ON SOME PROPERTIES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in $|z| < 1$, $p(0) = 1$, $p(z) \neq 0$ in $|z| < 1$ and suppose that $|p(z)|$ takes its maximum and minimum value at points $z = z_0$ and $z = z_1$ respectively on the closed disc $|z| = |z_0| = |z_1| \leq r$ ($0 < r < 1$). Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = m \geq 0$$

and

$$\frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0.$$

1. INTRODUCTION.

In [2], Jack proved the following theorem.

Theorem A. Let $w(z)$ be analytic in $\mathbb{E} = \{z : |z| < 1\}$ and suppose that $w(0) = 0$. If $|w(z)|$ takes its maximum value on the circle $|z| = r < 1$ at a point $z = z_0$, then we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

where k is real number and $k \geq 1$.

Fukui and Sakaguchi [1] generalized Theorem A and gave a simple and geometrical proof.

Furthermore, Miller and Mocanu [3] generalized Theorem A and obtained the following theorem.

Theorem B. Let $w(z) = \sum_{k=n}^{\infty} a_k z^k$ be analytic in $\mathbb{E}$, $n \in \mathbb{N}$, $w(z) \neq 0$. If $z_0 = r_0 e^{i\theta_0}$ ($0 < r_0 < 1$) and

$$|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$$

then

$$\frac{z_0 w'(z_0)}{w(z_0)} = m$$

and

$$1 + \operatorname{Re} \frac{z_0 w''(z_0)}{w'(z_0)} \geq m$$

where $1 \leq n \leq m$.

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It is a purpose of the present paper to obtain some similar but a little different results from Theorem A and B.

2. MAIN RESULT.

Theorem. Let $p(z)$ be analytic in $\mathbb{E}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that

$$\max_{|z| \leq r} |p(z)| = |p(z_0)|$$

and

$$\min_{|z| \leq r} |p(z)| = |p(z_1)|$$

where $0 < r < 1$ and $|z_0| = |z_1| = r$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = m \geq 0,$$

$$\frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0,$$

$$1 + \operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} \geq m$$

and

$$1 + \operatorname{Re} \frac{z_1 p''(z_1)}{p'(z_1)} \geq n$$

where m and n are real, $0 \leq m$ and $n \leq 0$.

Proof. Let us put

$$g(z) = \left(\frac{R+1}{R-1} \right) \left(\frac{R-p(z)}{R+p(z)} \right) = A \left(\frac{R-p(z)}{R+p(z)} \right), \quad g(0) = 1$$

where $R = |p(z_0)|$ and $A = (R+1)/(R-1)$. Then we have

$$p(z) = R \left(\frac{A-g(z)}{A+g(z)} \right).$$

When $|p(z)|$ takes its maximum value at a point $z = z_0$, then we have $\operatorname{Re} g(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} g(z_0) = 0$. Putting $|z| = |z_0|$, $z = |z_0|e^{i\theta}$ and $0 \leq \theta \leq 2\pi$, we have

$$\begin{aligned} \frac{zp'(z)}{p(z)} &= \frac{d \arg p(z)}{d\theta} - i \frac{d \log |p(z)|}{d\theta} \\ &= -\frac{zg'(z)}{A-g(z)} - \frac{zg'(z)}{A+g(z)} \\ &= -\frac{2zg'(z)}{A^2 - g(z)^2}. \end{aligned} \tag{1}$$

Therefore we have

$$\begin{aligned}\frac{z_0 p'(z_0)}{p(z_0)} &= -\frac{2z_0 g'(z_0)}{A^2 - g(z_0)^2} \\ &= \left(\frac{d \arg p(z)}{d\theta} \right)_{z=z_0} - i \left(\frac{1}{|p(z)|} \right) \left(\frac{d|p(z)|}{d\theta} \right)_{z=z_0} \\ &= \left(\frac{d \arg p(z)}{d\theta} \right)_{z=z_0} = m \geq 0\end{aligned}$$

where m is a real and $0 \leq m$. By logarithmic differentiation of (1), we have

$$1 + \frac{z p''(z)}{p'(z)} = \frac{z p'(z)}{p(z)} + 1 + \frac{z g''(z)}{g'(z)} - \frac{2z g'(z) g(z)}{A^2 - g(z)^2}.$$

Then we have

$$\begin{aligned}1 + \operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} &= \operatorname{Re} \frac{z_0 p'(z_0)}{p(z_0)} + 1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} - \operatorname{Re} \frac{2z_0 g'(z_0) g(z_0)}{A^2 - g(z_0)^2} \\ &= m + 1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} - m \operatorname{Re} g(z_0) \\ &= m + 1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)}.\end{aligned}$$

From the hypothesis, we have

$$\operatorname{Re} g(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} g(z_0) = 0,$$

then from the geometrical property of $g(z)$, we have

$$1 + \operatorname{Re} \frac{z_0 g''(z_0)}{g'(z_0)} \geq 0.$$

This shows that

$$1 + \operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} \geq m.$$

On the other hand, let us put

$$h(z) = \left(\frac{p(z) - l}{p(z) + l} \right) \left(\frac{1 + l}{1 - l} \right), \quad h(0) = 1$$

where $0 < l = \min_{|z| \leq |z_1|} |p(z)| < 1$, then we have

$$p(z) = l \left(\frac{B + h(z)}{B - h(z)} \right) \quad (2)$$

where $B = (1 + l)/(1 - l)$. From the hypothesis of the theorem, we have

$$\operatorname{Re} h(z) > 0 \quad \text{for } |z| < |z_1|$$

and

$$\operatorname{Re} h(z_1) = 0.$$

From (2), we have

$$\frac{zp'(z)}{p(z)} = \frac{2zh'(z)}{B^2 - h(z)^2}. \quad (3)$$

By the same reason as the above, we have on the circle $|z| = |z_1|e^{i\theta}$ and $0 \leq \theta \leq 2\pi$

$$\left(\frac{d|p(z)|}{d\theta} \right)_{z=z_1} = 0$$

and from the geometrical property, we have

$$\left(\frac{d \arg p(z)}{d\theta} \right)_{z=z_1} \leq 0.$$

This shows that

$$\frac{z_1 p'(z_1)}{p(z_1)} = \operatorname{Re} \frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0.$$

where n is a real number. From (3), we have

$$1 + \frac{zp''(z)}{p'(z)} = \frac{zp'(z)}{p(z)} + 1 + \frac{zh''(z)}{h'(z)} + \frac{2zh'(z)h(z)}{B^2 - h(z)^2}$$

and therefore we have

$$\begin{aligned} 1 + \operatorname{Re} \frac{z_1 p''(z_1)}{p'(z_1)} &= n + 1 + \operatorname{Re} \frac{z_1 h''(z_1)}{h'(z_1)} - \operatorname{Re} \frac{2z_1 h'(z_1)h(z_1)}{B^2 - h(z_1)^2} \\ &= n + 1 + \operatorname{Re} \frac{z_1 h''(z_1)}{h'(z_1)} - n \operatorname{Re} h(z_1) \\ &= n + 1 + \operatorname{Re} \frac{z_1 h''(z_1)}{h'(z_1)}. \end{aligned}$$

Applying the same reason as the above, we have

$$1 + \operatorname{Re} \frac{z_1 h''(z_1)}{h'(z_1)} \geq 0$$

and this shows that

$$1 + \operatorname{Re} \frac{z_1 p''(z_1)}{p'(z_1)} \geq n$$

where n is a real number and $n \leq 0$. This completes the proof. $\square$

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