

Infinite Networks and Random Walks

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§ 1. Introduction

Nash-Williams [7] carried out the method of associating a denumerable Markov chain with an electric circuit and gave a criterion for the Markov chain to be recurrent. This is a discrete analogue of the geometric criteria for a Riemann surface to have no Green function. Following his idea, several practical criteria for the Markov chain to be transient have been established by Griffeath and Liggett [1] and Lyons [4].

On the other hand, the study of an electric circuit has been developed as a theory of infinite networks in the last twenty years. On a locally finite infinite network, a flow problem and a classification problem for infinite networks have been studied with the aid of the reasoning in the theory of potentials on a Riemann surfaces (cf. [2], [12]).

Recently, Schlesinger [8] and Soardi [9] shed light on Lyons' results with the aid of flow problems on an infinite network which is not necessarily locally finite.

One of our aims is to notice that most of the results in the theory of locally finite networks developed in [2], [10], [11] and [12] remain valid under the condition studied in [6] that the network is p -almost locally finite. Every p -almost locally finite network can be classified to be either of parabolic type of order p or of hyperbolic type of order p . The parabolic index $\text{ind } N$ of a

network N can be defined as the infimum of $p > 1$ such that N is p -almost locally finite and of parabolic type of order p . Our theory has many counterparts in the theory of Markov chains in case $p = 2$. For 2-almost locally finite network N , the Green function of N exists if and only if it is of hyperbolic type (of order 2). By means of this fact, we shall show that the Markov chain $\mathbb{P}$ associated with the network N is transient if and only if N is of hyperbolic type. Consequently, many practical criteria for $\mathbb{P}$ to be transient follow from the theory of networks.

Geometric criteria for N to be of hyperbolic type of order p will be given by using the extremal length and the extremal width of N of order p (cf. [6]). They are related to the set of paths from a finite set to the ideal boundary of N and the set of cuts between a finite set and the ideal boundary of N respectively. By reviewing the Lyons' product of two flows, we obtain another proof of the fact that the parabolic index of the network defined by the d -dimensional lattice domain with unit arc resistance is equal to d (cf. [5]).

§ 2. Notation and terminology

Let X be a countable set of nodes, Y be a countable set of arcs and K be a node-arc incidence function, i.e.,

$$K : X \times Y \rightarrow \{-1, 0, 1\}.$$

We always assume that the graph $G = \{X, Y, K\}$ is connected and has no self-loop. For $y \in Y$ and $x \in X$, set

$$e(y) = \{x \in X; K(x, y) \neq 0\} \text{ (the set of extremities of arc } y),$$

$$Y(x) = \{y \in Y; K(x, y) \neq 0\} \text{ (the set of arcs incident to node } x),$$

$$X(x) = \cup \{e(y); y \in Y(x)\} \text{ (the set of neighboring nodes of } x),$$

$$U(x) = X(x) - \{x\}.$$

The pair $N = \{G, r\}$ of the graph $G = \{X, Y, K\}$ and a strictly positive real valued function r on Y is called an infinite network. For $1 < p < \infty$, we say that the network N is p -almost locally finite if the following condition holds:

$$(ALF)_p \quad \sum_{y \in Y(x)} r(y)^{p-1} < \infty \text{ for every } x \in X.$$

We say simply that N is almost locally finite if it is 2-almost locally finite. Clearly, N is p -almost locally finite if it is locally finite, i.e., $Y(x)$ is a finite set for every $x \in X$.

Let $L(X)$ be the set of real valued functions on X and let $L_0(X)$ be the set of all $u \in L(X)$ with finite support, i.e., $\{x \in X; u(x) \neq 0\}$ is a finite set. We define $L(Y)$ and $L_0(Y)$ similarly.

For $w, w' \in L(Y)$, define the inner product $\langle w, w' \rangle$ by

$$\langle w, w' \rangle = \sum_{y \in Y} r(y)w(y)w'(y)$$

if the sum is well-defined. For $w \in L(Y)$, its energy $H(w)$ is defined by

$$H(w) = \sum_{y \in Y} r(y)|w(y)|^2 = \langle w, w \rangle.$$

Denote by $L_2(Y; r)$ the set of all $w \in L(Y)$ with finite $H(w)$.

For $u \in L(X)$, its discrete derivative (weighted difference) $du \in L(Y)$ and its discrete Dirichlet integral (sum) $D(u)$ are defined as follows:

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x, y)u(x),$$

$$D(u) = H(du) = \sum_{y \in Y} r(y)[du(y)]^2.$$

Denote by $D(N)$ the set of all $u \in L(X)$ with finite Dirichlet integral and by ε_a ($a \in X$) the characteristic function of $\{a\}$:

$$\varepsilon_a(a) = 1 \text{ and } \varepsilon_a(x) = 0 \text{ for } x \neq a.$$

By definition, we have

$$d\varepsilon_a(y) = -r(y)^{-1}K(a, y)$$

for every $y \in Y$ and

$$D(\varepsilon_a) = \sum_{y \in Y} r(y)^{-1} |K(a, y)|.$$

Hence N is almost locally finite if and only if $\varepsilon_a \in D(N)$ for all $a \in X$, or equivalently $L_0(X) \subset D(N)$.

Hereafter always assume in this section that N is almost locally finite. In order to introduce the discrete Laplacian, we set

$$LS(X; r) = \{u \in L(X); \sum_{y \in Y} |K(x, y)du(y)| < \infty \text{ for all } x \in X\},$$

$$LS(Y; r) = \{w \in L(Y); \sum_{y \in Y} |K(x, y)w(y)| < \infty \text{ for all } x \in X\}.$$

The boundary operator $\partial: L(Y) \longrightarrow L(X)$ defined by

$$\partial w(x) = \sum_{y \in Y} K(x, y)w(y)$$

acts on $LS(Y; r)$. For $u \in LS(X; r)$, we define the (discrete) Laplacian $\Delta u \in L(X)$ by

$$\Delta u(x) = \sum_{y \in Y} K(x, y)du(y) = \partial[du].$$

For $w \in L_2(Y; r)$, we have

$$(2.1) \quad \begin{aligned} \sum_{y \in Y} |K(a, y)w(y)| &= \sum_{y \in Y} r(y) |[d\varepsilon_a(y)]w(y)| \\ &\leq [D(\varepsilon_a)]^{1/2} [H(w)]^{1/2} < \infty \end{aligned}$$

for all $a \in X$, so that $L_2(Y; r) \subset LS(Y; r)$. From the relation

$$\{du; u \in D(N)\} \subset L_2(Y; r),$$

it follows that $D(N) \subset LS(X; r)$.

To clarify the geometric meaning of Δu as in [11], we set

$$t(x) = \sum_{y \in Y} |K(x, y)|r(y)^{-1},$$

$$t(x, z) = - \sum_{y \in Y} K(x, y)K(z, y)r(y)^{-1} \text{ if } x \neq z,$$

$$t(x, x) = 0$$

for every $x, z \in X$.

Clearly, $t(x, z) = t(z, x)$ for all $x, z \in X$. Note that $t(x, z) > 0$ for all $x \in U(z)$ and

$$t(x) = \sum_{z \in X} t(x, z) > 0.$$

Let us put

$$S(N) = \{u \in L(X); \sum_{z \in X} t(x, z)|u(z)| < \infty \text{ for all } x \in X\}.$$

Proposition 2.1. $S(N) = LS(X; r)$ and

$$\Delta u(x) = -t(x)u(x) + \sum_{z \in X} t(x, z)u(z)$$

for all $u \in S(N)$.

Proof. Let $u \in S(N)$ and $a \in X$. Then, by Theorem A (the monotone convergence theorem) below

$$\begin{aligned} & \sum_{y \in Y} |K(a, y)du(y)| \\ & \leq \sum_{y \in Y} r(y)^{-1}|K(a, y)|[|K(a, y)u(a)| + \sum_{z \in U(a)} |K(z, y)u(z)|] \\ & = t(a)|u(a)| + \sum_{z \in U(a)} \sum_{y \in Y} |K(a, y)K(z, y)||u(z)| < \infty. \end{aligned}$$

Hence $S(N) \subset LS(X; r)$. Conversely, let $u \in LS(X; r)$ and $a \in X$.

In case u is non-negative, we have by Theorem A

$$\begin{aligned} \sum_{z \in U(a)} t(a, z)u(z) &= - \sum_{z \in X} \sum_{y \in Y} K(a, y)K(z, y)r(y)^{-1}u(z) \\ &\quad + \sum_{y \in Y} K(a, y)^2 r(y)^{-1}u(a) \\ &= - \sum_{y \in Y} \sum_{x \in X} K(a, y)K(z, y)r(y)^{-1}u(z) + t(a)u(a) \\ &= \sum_{y \in Y} K(a, y)du(y) + t(a)u(a) \\ &\leq \sum_{y \in Y} |K(a, y)du(y)| + t(a)u(a) < \infty, \end{aligned}$$

so that $u \in S(N)$. In the general case, we see that $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ belong to $LS(X; r)$, since $|du^+(y)| \leq |du(y)|$ and $|du^-(y)| \leq |du(y)|$. Thus $u^+, u^- \in S(N)$ by the above observation. Therefore $u = u^+ - u^- \in S(N)$ and $LS(X; r) \subset S(N)$.

As in the above proof, we have to pay a special attention to the pointwise summability (resp. convergence) of a function (resp. a sequence of functions on X or Y) if N is not locally finite. One of the following two well-known results (cf. [3; Theorems 1-44, 1-49]) will play a fundamental role in modifying the proofs of the results given for the case where N is locally finite:

Theorem A. (Monotone convergence theorem) Let Ω be a countable set and μ be a nonnegative function on Ω . If $\{f_n\}$ is a monotone increasing sequence of nonnegative functions on Ω and if it converges pointwise to f , then

$$\lim_{n \rightarrow \infty} \sum_{s \in \Omega} f_n(s) \mu(s) = \sum_{s \in \Omega} f(s) \mu(s).$$

Theorem B. (Dominated convergence theorem) Let Ω and μ be the same as above and let $\{f_n\}$ be a sequence of real valued functions on Ω which converges pointwise to f . If there exists $h \in L(\Omega)$ such that $|f_n| \leq h$ on Ω for all n and $\sum_{s \in \Omega} h(s) \mu(s) < \infty$, then

$$\lim_{n \rightarrow \infty} \sum_{s \in \Omega} f_n(s) \mu(s) = \sum_{s \in \Omega} f(s) \mu(s).$$

We say that a function $u \in S(N)$ is harmonic (resp. super-harmonic) on a subset A of X if $\Delta u(x) = 0$ (resp. $\Delta u(x) \leq 0$) on A .

Denote by $HD(N)$ the set of all harmonic Dirichlet finite functions, i.e.,

$$HD(N) = \{u \in D(N); \Delta u(x) = 0 \text{ on } X\}.$$

Notice that $D(N)$ is a Hilbert space with respect to the inner product

$$((u, v)) = \langle du, dv \rangle + u(x_0)v(x_0) \quad (x_0 \in X).$$

Denote by $D_0(N)$ the closure of $L_0(X)$ in $D(N)$ with respect to the norm $\|u\| = [((u, u))]^{1/2}$. This does not depend on the choice of x_0 . The norm convergence implies the pointwise convergence. In fact, we see by [6] that for every nonempty finite subset F of X , there exists a constant M_F such that

$$(2.2) \quad \sum_{x \in F} |u(x)| \leq M_F \|u\| \text{ for all } u \in D(N).$$

Let $\{u_n\}$ be a sequence $D(N)$ such that $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$.

For any $a \in X$, we see by (2.1) that $\{\Delta u_n\}$ converges pointwise to Δu . Thus $HD(N)$ is a closed subspace of $D(N)$ by the above observation.

Proposition 2.2. The Green's equality

$$(2.3) \quad \langle df, du \rangle = - \sum_{x \in X} f(x) [\Delta u(x)]$$

holds for every $f \in L_0(X)$ and $u \in S(N)$.

Proof. By definition,

$$\langle df, du \rangle = - \sum_{y \in Y} du(y) \sum_{x \in X} K(x, y) f(x).$$

Our relation follows if the change of the order of summation is assured. To prove this, let $S_f = \{x \in X; f(x) \neq 0\}$. Then S_f is a finite set, $c = \sum_{x \in S_f} \max\{|f(x)|; x \in X\} < \infty$ and

$$\begin{aligned} \sum_{x \in X} \sum_{y \in Y} |f(x) K(x, y) du(y)| &= \sum_{x \in S_f} \sum_{y \in Y} |f(x) K(x, y) du(y)| \\ &\leq c \sum_{y \in Y} |K(x, y) du(y)| < \infty. \end{aligned}$$

Corollary. $\langle dv, dh \rangle = 0$ for every $v \in D_0(N)$ and $h \in HD(N)$.

By the usual argument in the theory of Hilbert spaces, we obtain a discrete analogue of Royden's decomposition theorem:

Theorem 2.1. Every $u \in D(N)$ is uniquely decomposed in the form: $u = v + h$ with $v \in D_0(N)$ and $h \in HD(N)$.

§ 3. Parabolic and hyperbolic networks

We begin with

Lemma 3.1. Let N be almost locally finite and A be a nonempty finite subset of X . Then $1 \in D_0(N)$ holds if and only if the value of the following extremum problem vanishes:

$$(3.1) \quad d_2(A, \infty) = \inf\{D(u); u \in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

Proof. Assume that $d_2(A, \infty) = 0$. There exists a sequence

$\{u_n\}$ in $L_0(X)$ such that $u_n = 1$ on A and $D(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\|u_n - u_n(x_0)\| = [D(u_n)]^{1/2} \rightarrow 0$ as $n \rightarrow \infty$, we see by (2.2) that $u_n(x) - u_n(x_0)$ converges to 0 as $n \rightarrow \infty$ for each $x \in X$. Since $u_n = 1$ on A , $u_n(x_0) \rightarrow 1$ as $n \rightarrow \infty$, so that $\|u_n - 1\| = [D(u_n) + (u_n(x_0) - 1)^2]^{1/2} \rightarrow 0$ as $n \rightarrow \infty$. Thus $1 \in D_0(N)$. Next, we assume that $1 \in D_0(N)$. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|1 - f_n\| \rightarrow 0$ as $n \rightarrow \infty$. Note that $\{f_n\}$ converges to 1 by (2.2) and $D(f_n) \rightarrow 0$ as $n \rightarrow \infty$. By considering $\max\{0, \min\{f_n, 1\}\}$ for f_n , we may suppose that $0 \leq f_n \leq 1$ on X . Define $g_n \in L(X)$ by

$$g_n = 1 \text{ on } A \text{ and } g_n = f_n \text{ on } X - A.$$

Then $g_n \in L_0(X)$ and

$$D(f_n - g_n) = \sum_{y \in Y(A)} r(y) [df_n(y) - dg_n(y)]^2,$$

where $Y(A) = \bigcup \{Y(x); x \in A\}$. Put

$$w_n(y) = r(y)^{-1} [\sum_{x \in X} K(x, y) (f_n(x) - g_n(x))]^2.$$

Then

$$D(f_n - g_n) = \sum_{y \in Y(A)} w_n(y),$$

$$0 \leq w_n(y) \leq 2r(y)^{-1},$$

$$\sum_{y \in Y(A)} r(y)^{-1} = \sum_{x \in A} \sum_{y \in Y(x)} r(y)^{-1} < \infty \text{ by (ALF)}_2.$$

Since $w_n(y) \rightarrow 0$ as $n \rightarrow \infty$, it follows from Theorem B that $D(f_n - g_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus $D(g_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $d_2(A, \infty) \leq D(g_n)$, we have $d_2(A, \infty) = 0$.

Remark 3.1. It is easily seen that

$$(3.1)' \quad d_2(A, \infty) = \inf\{D(u); u \in D_0(N) \text{ and } u = 1 \text{ on } A\}.$$

On account of this lemma, we can introduce

Definition 3.1. We say that an almost locally finite network N is of parabolic type (of order 2) if there exists a nonempty finite subset A of X such that $d_2(A, \infty) = 0$. We say that an

almost locally finite network N is of hyperbolic type (of order 2) if it is not of parabolic type.

The Green function g_a of N with pole at $a \in X$ is defined (if it exists) by the condition:

$$(3.2) \quad g_a \in D_0(N) \text{ and } \Delta g_a(x) = -\varepsilon_a(x) \text{ on } X.$$

If there exist two functions f_1 and f_2 which satisfy (3.2), then $u = f_1 - f_2 \in HD(N) \cap D_0(N)$, so that $D(u) = 0$ by Proposition 2.2. Therefore u is a constant function. In case N is of hyperbolic type, $u = 0$ by Lemma 3.1, namely the uniqueness of the Green

function follows. Assume that N is of hyperbolic type. Then there exists a unique optimal solution u_a of the extremum problem:

$$(3.3) \quad d_2(\{a\}, \infty) = \inf\{D(u); u \in D_0(N) \text{ and } u(a) = 1\} > 0.$$

By the standard variational technique, we see that $g_a = u_a/D(u_a)$ satisfies (3.2). By the usual argument as in [11], we have

$$(3.4) \quad 0 < g_a(x) = g_x(a) \leq \min\{g_a(a), g_x(x)\};$$

$$(3.5) \quad \langle dg_a, dv \rangle = v(a) \quad \text{for all } v \in D_0(N).$$

In case N is of parabolic type, there is no function which satisfies (3.2) (cf. Remark 6.1).

We see by Proposition 2.1 that the minimum principle and Harnak's principle for superharmonic functions holds in our case (cf. [11; Lemmas 2.1 and 2.4]). In order to approximate the Green function of N by the sequence of Green functions of subnetworks, let $\{N_n\}$ ($N_n = \{X_n, Y_n, K, r\}$) be an exhaustion of N (cf. [6]). Note that each N_n is a finite subnetwork of N .

The Green function $g_a^{(n)}$ of N_n with pole at a is defined by the condition:

$$(3.6) \quad \Delta g_a^{(n)}(x) = -\varepsilon_a(x) \text{ on } X_n, \quad g_a^{(n)}(x) = 0 \text{ on } X - X_n.$$

The uniqueness of $g_a^{(n)}$ follows from the minimum principle. To show the existence of $g_a^{(n)}$, consider the following extremum problem:

$$(3.7) \quad \text{Minimize } D(u)$$

subject to $u \in L(X)$, $u(a) = 1$ and $u = 0$ on $X - X_n$.

Denote by $d_2(\{a\}, X - X_n)$ the value of problem (3.7). There exists a unique optimal solution $u_a^{(n)}$ of this problem. Then $g_a^{(n)} = u_a^{(n)} / D(u_a^{(n)})$ satisfies our requirement.

By the minimum principle,

$$0 \leq g_a^{(n)}(x) \leq g_a^{(n+1)}(x) \leq g_a^{(n+1)}(a) \text{ on } X.$$

Thus the pointwise limit of $\{g_a^{(n)}\}$ is either a real valued function on X or identically ∞ by Harnack's principle. Notice that

$$(3.8) \quad \lim_{n \rightarrow \infty} d_2(\{a\}, X - X_n) = d_2(\{a\}, \infty).$$

Lemma 3.2. Let $u_a^{(n)}$ be the unique optimal solution of problem (3.7). Then there exists $v_a \in D_0(N)$ such that $v_a(a) = 1$ and $\|u_a^{(n)} - v_a\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. By the standard argument, we have for $n < m$

$$D(u_a^{(n)} - u_a^{(m)}) = D(u_a^{(n)}) - D(u_a^{(m)}).$$

It follows that $\{u_a^{(n)}\}$ is a Cauchy sequence in $D_0(N)$. Thus there exists $v_a \in D_0(N)$ such that $\|u_a^{(n)} - v_a\| \rightarrow 0$ as $n \rightarrow \infty$.

Clearly, $v_a(a) = 1$.

Since $g_a^{(n)}(a) = d_2(\{a\}, X - X_n)$, we see by (3.8) that $\{g_a^{(n)}\}$ converges to a real valued function on X if and only if $d_2(\{a\}, \infty) > 0$, i.e., N is of hyperbolic type. If N is of hyperbolic type, then $\|g_a^{(n)} - g_a\| \rightarrow 0$ as $n \rightarrow \infty$ by the above observation.

By modifying the proofs of [10; Theorem 3.2] and [11; Theorems 3.1 and 3.2], we obtain

Theorem 3.1. An infinite network N is of hyperbolic type if any one of the following conditions is fulfilled:

$$(C.1) \quad 1 \notin D_0(N);$$

$$(C.2) \quad D_0(N) \neq D(N);$$

(C.3) The Green function g_a of N with pole at a exists;

(C.4) For any exhaustion $\{N_n\}$ of N , the sequence $\{g_a^{(n)}\}$ of the Green functions of N_n with pole at a converges to g_a .

(C.5) There exists a nonconstant positive superharmonic function on X .

We shall prove another criterion:

Theorem 3.2. An infinite network N is of hyperbolic type if and only if the following condition is fulfilled:

(C.6) For any nonempty finite subset F of X , there exists a constant $M(F)$ such that

$$(3.9) \quad \sum_{x \in F} |u(x)| \leq M(F)[D(u)]^{1/2}$$

for all $u \in D_0(N)$.

Proof. Assume that N is of hyperbolic type and let F be a nonempty finite subset of X . It suffices to show that there exists a constant $M(F)$ which satisfies (3.9) for all $u \in D_0(N)$ with $D(u) = 1$. Supposing the contrary, we can find a sequence $\{u_n\}$ in $D_0(N)$ such that $D(u_n) = 1$ and $\sum_{x \in F} |u_n(x)| \rightarrow \infty$ as $n \rightarrow \infty$. Since F is a finite set, we may assume that $|u_n(b)| \rightarrow \infty$ as $n \rightarrow \infty$ for some $b \in F$.

Let $v_n = u_n/u_n(b)$. Then $v_n \in D_0(N)$ and $v_n(b) = 1$, so that

$$d_2(\{b\}, \infty) \leq D(v_n) = D(u_n/u_n(b)) = [u_n(b)^2]^{-1} \rightarrow 0$$

as $n \rightarrow \infty$. Thus N is of parabolic type by Theorem 3.1, which

contradicts our assumption. Thus (C.6) holds. Next we assume

(C.6). If N is of parabolic type, then there exists a finite nonempty subset F of X such that $d_2(F, \infty) = 0$. Thus we can find a sequence $\{u_n\}$ in $L_0(X)$ such that $u_n = 1$ on F and $D(u_n) \rightarrow 0$ as $n \rightarrow \infty$. By (C.6), there exists a constant $M(F)$ such that

$$1 \leq \sum_{x \in F} |u_n(x)| \leq M(F)[D(u_n)]^{1/2}$$

for all n . This is a contradiction. Therefore N is of hyperbolic type.

Similarly to [11; Theorem 5.1], we have a discrete analogue of Riesz's decomposition theorem:

Theorem 3.3. Assume that N is of hyperbolic type and let u be non-negative and superharmonic on X . Then u is decomposed uniquely in the form: $u = G\mu + h$, where

$$G\mu(x) = \sum_{z \in X} g_z(x)\mu(z) \text{ with } \mu(z) = -\Delta u(z),$$

and h is the greatest harmonic minorant of u .

Corollary. Let $v \in L^+(X)$. If $\Delta v(x) \leq -\varepsilon_a(x)$ on X , then $g_a(x) \leq v(x)$ on X .

Remark 3.2. In order to relate our theory of networks to the theory of denumerable Markov chains in [3], we remark the following correspondence of the notation:

$$\alpha(x) = t(x) \quad \text{and} \quad c(x, z) = t(x, z)$$

for every $x, z \in X$. Note that

$$c(x, z) = c(z, x) \geq 0 \text{ for every } x, z \in X,$$

$$c(x, z) = 0 \text{ for } z \notin U(x),$$

$$\alpha(x) = \sum_{z \in X} c(x, z) = \sum_{z \in U(x)} c(x, z) > 0,$$

Thus we have

$$(3.10) \quad \Delta u(x) = -\alpha(x)u(x) + \sum_{z \in X} c(x, z)u(z).$$

In what follows, we use $\alpha(x)$ and $c(x, z)$ instead of $t(x)$ and $t(x, z)$.

§ 4. The Markov chain associated with a network

Let $N = \{X, Y, K, r\}$ be an infinite network which is almost locally finite. The Markov chain $P(N)$ associated with the network N is defined as a Markov chain $\{X, X_n, P\}$ with a state space X , a process $\{X_n\}$ and a transition matrix (function) P defined by

$$(4.1) \quad P(x, z) = c(x, z)/\alpha(x)$$

for every $x, z \in X$ (cf. [3; Proposition 9-125]).

Since $P(x, x) = 0$ for all $x \in X$, there is no absorbing state (cf. [3; p. 81]). By Remark 3.1 and (4.1), we have

$$(4.2) \quad \sum_{x \in X} \alpha(x)P(x, z) = \alpha(z) \text{ for all } z \in X,$$

i.e., the function α is a regular measure (cf. [3; p. 86]), or equivalently, α is P -regular (cf. [3; Proposition 9-127]). As for the α -dual matrix (function) $\hat{P}$ defined by

$$\hat{P}(x, z) = \alpha(z)P(z, x)/\alpha(x) \text{ (cf. [3; p. 132])},$$

we see by (3.1) that $\hat{P}(x, z) = P(x, z)$, so that the Markov chain $P(N)$ is α -reversible (cf. [3; p. 308]).

Since N is connected and $P(x, z) > 0$ for all $z \in U(x)$, we see that, for any $a, b \in X$ with $a \neq b$, there exists $n > 0$ such that

$$P^n(a, b) = \sum_{z \in X} [P(a, z)][P^{n-1}(z, b)] > 0,$$

where $P^0(z, b) = \varepsilon_b(z)$. Thus all states a and b communicate, i.e., $a \sim b$ (cf. [3; p. 98]). Hence there is a single equivalence class caused by this equivalence relation " $\sim$ ". Namely the Markov chain $P(N)$ is irreducible.

We have by [3; Proposition 9-128]

Theorem 4.1. Let $P = \{S, X_n, P\}$ be a Markov chain with a state space S (countable set), a process $\{X_n\}$ and a transition probability P such that $P(s, s) = 0$ for all $s \in S$. Then there exists a network $N = \{X, Y, K, r\}$ such that $X = S$ and $P = P(N)$ if and only if its states communicate and it has a positive regular measure α with respect to which it is α -reversible.

Remark 4.1. In case $P(x, z) > 0$ and $P(z, x) > 0$, then there exist positive numbers $\alpha(x)$ and $\alpha(z)$ which satisfy (4.3).

Remark 4.2. For $u \in L(X)$, define Pu by

$$Pu(x) = \sum_{z \in X} P(x, z)u(z),$$

if they are well defined. Let I be the identity mapping from $L(X)$ to $L(X)$, i.e. $Iu(x) = u(x)$ on X . We have by (4.10)

$$\Delta u(x) = -\alpha(x)[(I - P)u(x)].$$

§ 5. Transient Markov chains

Let $P = P(N) = \{X, X_n, P\}$ be the Markov chain associated the network $N = \{X, Y, K, r\}$. By [3; Definition 4-18], we recall

Definition 5.1. A state x is said to be recurrent if the probability $\bar{h}(x, x)$ starting in x of returning to x at least one time is equal to 1; it is said to be transient if $\bar{h}(x, x) < 1$.

Since the Markov chain P is irreducible, we say that P is recurrent if there exists a recurrent state and that P is transient if there exists a transient state (cf. [3; Proposition 4-24 and Definition 4-29]).

Let $N(x, z)$ be the expected number of visits to z starting at x . By [3; Propositions 4-12 and 4-20 and Corollary 4-21],

$$(5.1) \quad N(x, z) = \sum_{k=0}^{\infty} P^k(x, z);$$

$$(5.2) \quad N(x, x) = \sum_{k=1}^{\infty} [\bar{h}(x, x)]^{k-1}.$$

By [3; Proposition 4-20 and Corollary 4-21], we have

Theorem 5.1. The Markov chain P is transient if and only if $N(x, x) < \infty$. In this case, we have $N(x, z) < \infty$ for all $x, z \in X$,

$$(5.3) \quad N(x, x) = 1/[1 - \bar{h}(x, x)].$$

$$(5.4) \quad N(x, z) \leq N(z, z).$$

To combine the classification of networks with the theory of Markov chains, we shall prove

Theorem 5.2. The Markov chain $P = P(N)$ associated with the network N is transient if and only if N is of hyperbolic type.

Proof. Assume that P is transient and let $u(x) = N(x, a)$. Then u is nonnegative. By the monotone convergence theorem,

$$\begin{aligned} Pu(x) &= \sum_{n=0}^{\infty} \sum_{z \in X} P(x, z) P^n(z, a) = \sum_{n=0}^{\infty} P^{n+1}(x, a) \\ &= u(x) - \varepsilon_a(x), \end{aligned}$$

or equivalently,

$$(5.5) \quad N(x, a) = \varepsilon_a(x) + \sum_{z \in X} P(x, z) N(z, a)$$

for all $a, x \in X$ (cf. [3; Proposition 4-13]). By Remark 4.2,

$$\Delta u(x) = -\alpha(x)[u(x) - Pu(x)] = -\alpha(a)\varepsilon_a(x)$$

for all $x \in X$. Namely, u is a nonconstant positive superharmonic function on X . Therefore N is of hyperbolic type by Theorem 3.1.

We see by the corollary of Theorem 3.3 that $\alpha(a)g_a(x) \leq N(x, a)$ for all $a, x \in X$. Next assume that N is of hyperbolic type and let g_a be the Green function of N with pole at a . By Remark 4.2,

$$g_a(x) - Pg_a(x) = -\alpha(x)^{-1}\Delta g_a(x) = \alpha(a)^{-1}\varepsilon_a(x).$$

Since g_a is nonnegative, we have

$$\begin{aligned} g_a &= P^{n+1}g_a + (I + P + \dots + P^n)(g_a - Pg_a) \\ &= P^{n+1}g_a + \alpha(a)^{-1}(I + P + \dots + P^n)\varepsilon_a, \end{aligned}$$

and hence

$$g_a(x) \geq \alpha(a)^{-1} \sum_{k=0}^n P^k(x, a)$$

for all $n \in \mathbb{Z}^+$ and $x \in X$. Therefore $g_a(x) \geq \alpha(a)^{-1}N(x, a)$ and the Markov chain P is transient.

Corollary 1. If P is transient, then $N(x, a) = \alpha(a)g_a(x)$.

Corollary 2. $\alpha(z)N(z, x) = \alpha(x)N(x, z)$ for all $x, z \in X$.

We observe that some values of extremum problems which play important roles in network theory have stochastic interpretations (cf. [1]).

§ 6. Nonlinear network theory

Before studying more criteria for the recurrence of Markov chains and the parabolicity of infinite networks, we give some generalizations of the results concerning networks.

Let p and q be positive numbers such that

$$1/p + 1/q = 1 \quad \text{and} \quad 1 < p < \infty.$$

The energy $H(w)$ and the Dirichlet integral $D(u)$ defined in § 2 can be generalized as follows:

$$\begin{aligned} H_p(w) &= \sum_{y \in Y} r(y) |w(y)|^p, \\ D_p(u) &= \sum_{y \in Y} r(y) |du(y)|^p. \end{aligned}$$

The real valued function $\varphi_p(t)$ defined by

$$\varphi_p(t) = |t|^{p-1} \text{sign}(t) \quad \text{for } t \in \mathbb{R}$$

plays an important role in our study. Note that

$$t\varphi_p(t) = |t|^p, \quad |\varphi_p(t)|^q = |t|^p \quad \text{and} \quad \frac{d}{dt} |t|^p = p\varphi_p(t).$$

For $w \in L(Y)$, define $\varphi_p(w) \in L(Y)$ by $(\varphi_p(w))(y) = \varphi_p(w(y))$.

Let us put

$$L_p(Y; r) = \{w \in L(Y); H_p(w) < \infty\},$$

$$D^{(p)}(N) = \{u \in L(X); D_p(u) < \infty\}.$$

By the relation

$$D_p(\varepsilon_a) = \sum_{y \in Y} r(y)^{1-p} |K(a, y)|,$$

we see that $L_0(X) \subset D^{(p)}(N)$ if and only if N is p -almost locally finite. For $w \in L_q(Y; r)$, we have

$$\begin{aligned} \sum_{y \in Y} |K(a, y)w(y)| &= \sum_{y \in Y} r(y) |[d\varepsilon_a(y)]w(y)| \\ &\leq [D_p(\varepsilon_a)]^{1/p} [H_q(w)]^{1/q} < \infty \end{aligned}$$

by Hölder's inequality, so that the boundary operator $\partial: L_q(Y; r) \longrightarrow L(X)$ is defined by $\partial w(x) = \sum_{y \in Y} K(x, y)w(y)$.

For $u \in D^{(p)}(N)$, we see easily that $H_q(\varphi_p(du)) = D_p(u)$ and $\varphi_p(du) \in L_q(Y; r)$, so that the discrete p -Laplacian $\Delta_p u \in L(X)$ is defined by

$$\Delta_p u(x) = \sum_{y \in Y} K(x, y) \varphi_p(du(y)) = \partial[\varphi_p(du)].$$

Note that $\Delta_p u$ is nonlinear in u unless $p = 2$. Let $D_0^{(p)}(N)$ be the closure of $L_0(X)$ with respect to the norm:

$$\|u\|_p = [D_p(u) + |u(x_0)|^p]^{1/p} \quad (x_0 \in X),$$

and let $HD^{(p)}(N) = \{u \in D^{(p)}(N); \Delta_p u(x) = 0 \text{ on } X\}$.

As a generalization of Theorem 2.1, we can prove that every $u \in D^{(p)}(N)$ can be decomposed uniquely in the form: $u = v + h$, where $v \in D_0^{(p)}(N)$ and $h \in HD^{(p)}(N)$.

We say that N is parabolic type of order p if it is p -almost locally finite and if the value of the following extremum problem vanishes for a nonempty finite subset A of X :

$$(6.1) \quad d_p(A, \infty) = \inf\{D_p(u); u \in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

In case N is not parabolic type of order p but p -almost locally finite, N is called to be of hyperbolic type of order p .

Let A be a nonempty finite subset of X and denote by $P_{A,\infty}(N)$ the set of all paths from A to the ideal boundary ∞ of N and by $Q_{A,\infty}(N)$ the set of all cuts between A and the ideal boundary ∞ of N (cf. [6]). The extremal length $EL_p(A, \infty)$ and the extremal width $EW_q(A, \infty)$ of N between A and ∞ (of order p and q respectively) are defined by

$$(6.2) \quad EL_p(A, \infty)^{-1} = \inf\{H_p(W); W \in E(P_{A,\infty}(N))\},$$

where $E(P_{A,\infty}(N))$ is the set of all $W \in L^+(Y)$ such that

$$\sum_P r(y)W(y) \geq 1 \quad \text{for all } P \in P_{A,\infty}(N).$$

$$(6.3) \quad EW_q(A, \infty)^{-1} = \inf\{H_q(W); W \in E^*(Q_{A,\infty}(N))\},$$

where $E^*(Q_{A,\infty}(N))$ is the set of all $W \in L^+(Y)$ such that

$$\sum_Q W(y) \geq 1 \quad \text{for all } Q \in Q_{A,\infty}(N).$$

By using the limit property in [6], we have

$$(6.4) \quad d_p(A, \infty) = EL_p(A, \infty)^{-1}.$$

By this fact and by the inverse relation

$$(6.5) \quad [EL_p(A, \infty)]^{1/p} [EW_q(A, \infty)]^{1/q} = 1,$$

which was proved in [6] for p -almost locally finite networks, we obtain the following result:

Theorem 6.1. Let N be an infinite network which is p -almost locally finite. Then N is of hyperbolic type of order p if any one of the following conditions is fulfilled:

$$(C.1)_p \quad 1 \notin D_0^{(p)}(N);$$

$$(C.2)_p \quad D_0^{(p)}(N) \neq D^{(p)}(N);$$

(C.6)_p For any nonempty finite subset F of X , there exists a constant $M(F)$ such that

$$\sum_{x \in F} |u(x)| \leq M(F) [D_p(u)]^{1/p} \quad \text{for all } u \in D_0^{(p)}(N);$$

(C.7)_p There exist $a \in X$ and $w \in L_q(Y; r)$ such that

$$\sum_{y \in Y} K(x, y)w(y) = -\varepsilon_a(x) \quad \text{on } X;$$

$$(C.8)_p \quad EL_p(A, \infty) < \infty;$$

$$(C.9)_p \quad EW_q(A, \infty) > 0.$$

Note that $(C.7)_2$ is the criterion of the transience due to Lyons [4, Theorem, p. 394]. By the same argument as in [12; Theorems 4.2 and 3.2], we can prove

Lemma 6.1. Assume that N is of parabolic type of order p and let $u \in D^{(p)}(N)$. If $\sum_{x \in X} |\Delta_p u(x)| < \infty$, then $\sum_{x \in X} \Delta_p u(x) = 0$.

Remark 6.1. In case N is of parabolic type of order 2, any function does not satisfy condition (3.2) by this lemma.

Lemma 6.2. For any $w \in L_q(Y; r)$, there exists $u \in D^{(p)}(N)$ such that $\Delta_p u(x) = \partial w(x)$ on X .

By means of these lemmas, we have

Proposition 6.1. Let N be p -almost locally finite. Then N is of hyperbolic type of order p if and only if the following condition is fulfilled:

$$(C.10)_p \quad \text{There exists } w \in L_q(Y; r) \text{ such that } \sum_{x \in X} |\partial w(x)| < \infty \text{ and } \sum_{x \in X} [\partial w(x)] \neq 0.$$

Notice that $(C.10)_2$ is the better test for transience due to Lyons [4; Theorem, p. 398].

Let $1 < p_1 < p_2$. If N is p_1 -almost locally finite, then it is p_2 -almost locally finite. We see by the same reasoning as [10; Theorem 5.1] that if N is of parabolic type of order p_1 , then it is of parabolic type of order p_2 . Thus we can define the parabolic index $\text{ind } N$ of N as follows:

$\text{ind } N = \inf\{p > 1; N \text{ is of parabolic type of order } p\}.$

As for a geometric meaning of the parabolic index, we refer to [10].

Remark 6.1. In case N is of parabolic type of order 2, any function does not satisfy condition (3.2) by Lemma 6.1.

§ 7. Random walks

The symmetric random walk $RW^{(d)}$ in d -dimensions is defined to be a sums of independent random variables process on the lattice of integer points in d -dimensional Euclidean space. The transition probability from one lattice point to another is $(2d)^{-1}$ if the two points are a Euclidean distance of one unit apart; the transition probability is zero otherwise. Thus, from each point the process moves to one of $2d$ neighboring points with probability $(2d)^{-1}$. (cf. [3; p. 84]). It is well-known that $RW^{(d)}$ is one of the typical examples in the theory of Markov chains. Let $N^{(d)}$ be the associated network with this random walk $RW^{(d)}$.

Lyons [4] proved by using condition $(C.7)_2$ that $RW^{(d)}$ is transient, i.e., $\text{ind } N^{(d)} \geq 2$. By reviewing his construction of a flow for this proof, we shall prove that $\text{ind } N^{(d)} = d$.

More precisely, let $\mathbf{Z}$ be the set of integers, $X^{(d)} = \mathbf{Z}^d$ and let $e_1^{(d)}, \dots, e_d^{(d)}$ be the standard base of R^d , i.e., the k -th component of $e_j^{(d)}$ is 1 for $k = j$ and 0 for $k \neq j$. For $a, b \in R^d$, let $[a, b]$ denote the directed line segment from a to b . For each $j (= 1, \dots, d)$, set

$$\begin{aligned} S_{j,+}^{(d)} &= \{[x, x + e_j^{(d)}]; x \in X^{(d)} \cap R_+^d\}, \\ S_{j,-}^{(d)} &= \{[x, x - e_j^{(d)}]; x \in X^{(d)} \cap (-R_+^d)\}, \\ S_j^{(d)} &= S_{j,+}^{(d)} \cup S_{j,-}^{(d)}, \end{aligned}$$

where R_+^d is the non-negative orthant of R^d . We define $Y^{(d)}$ by

$$Y^{(d)} = \bigcup_{j=1}^d S_j^{(d)}.$$

For $x \in X^{(d)}$ and $y = [x_1, x_2] \in Y^{(d)}$, let $K_d(x_2, y) = 1$, $K_d(x_1, y) = -1$ and $K_d(x, y) = 0$ if $x \neq x_1$ and $x \neq x_2$. With $r(y) = 1$ on $Y^{(d)}$, $N^{(d)} = \{X^{(d)}, Y^{(d)}, K_d, r\}$ is a locally finite infinite network. It is easily seen that the Markov chain associated with this network is $RW^{(d)}$. Hereafter, we omit the superscript (d) in the notation in case no confusion occurs from the context.

It was shown in [5] that $N^{(d)}$ is of parabolic type of order d and $\text{ind } N^{(d)} \leq d$ by using a geometric criterion in [10]. In case $d = 2$, this proof is the same as the proof of the fact in [4] that $RW^{(d)}$ is recurrent. We shall give another proof of the inequality $\text{ind } N^{(d)} \geq d$.

For $u \in L(X^{(d)} \times X^{(d)})$ and $x \in X^{(d)}$, put

$$J_d(u; x) = \sum_{j=1}^d \{u(x, x + e_j^{(d)}) + u(x, x - e_j^{(d)})\}.$$

We begin with a remark on our notation.

Lemma 7.1. Let $X = X^{(d)}$ and $Y = Y^{(d)}$ and suppose that there exists a function u with the following properties:

$$(7.1) \quad u \in L(X \times X) \text{ and } u(x, z) = -u(z, x) \text{ for all } x, z \in X.$$

Then there exists $w \in L(Y)$ such that

$$\sum_{y \in Y} K(x, y)w(y) = -J_d(u; x).$$

Proof. Let us identify each $(x, z) \in X \times X$ with the directed arc $[x, z]$. For $y \in Y$, there exists a unique j such that $y \in S_j$, so we define $w(y)$ by

$$w(y) = u(x, x + e_j) \text{ if } y = [x, x + e_j];$$

$$w(y) = u(x, x - e_j) \text{ if } y = [x, x - e_j].$$

Let $x = (v_1, \dots, v_d)$. If $v_j > 0$, then $Y(x) \cap S_j = \{y_1, y_2\} \subset S_{j,+}$ with $y_1 = [x, x + e_j]$ and $y_2 = [x - e_j, x]$, so that

$$\begin{aligned}\sum_{y \in S_j} K(x, y)w(y) &= w(y_2) - w(y_1) \\ &= -u(x, x - e_j) - u(x, x + e_j).\end{aligned}$$

If $v_j < 0$, then $Y(x) \cap S_j = \{y_1, y_2\} \subset S_{j,-}$ with $y_1 = [x, x - e_j]$ and $y_2 = [x + e_j, x]$, so that

$$\begin{aligned}\sum_{y \in S_j} K(x, y)w(y) &= w(y_2) - w(y_1) \\ &= -u(x, x + e_j) - u(x, x - e_j).\end{aligned}$$

If $v_j = 0$, then $Y(x) \cap S_j = \{y_1, y_2\}$ with $y_1 = [x, x + e_j] \in S_{j,+}$ and $y_2 = [x, x - e_j] \in S_{j,-}$, so that

$$\begin{aligned}\sum_{y \in S_j} K(x, y)w(y) &= -w(y_1) - w(y_2). \\ &= -u(x, x + e_j) - u(x, x - e_j).\end{aligned}$$

Thus we have

$$\sum_{y \in S_j} K(x, y)w(y) = -u(x, x + e_j) - u(x, x - e_j),$$

and hence

$$\sum_{y \in Y} K(x, y)w(y) = -J_d(u; x).$$

Now we recall the product of flows due to Lyons. Consider transformations $A_1 = A_1^{(k+1)}$ and $A_2 = A_2^{(k+1)}$ from $X^{(k+1)}$ into $X^{(2)}$ and $X^{(k)}$ defined by

$$\begin{aligned}A_1 x &= A_1(v_1, \dots, v_k, v_{k+1}) = (v_1, v_{k+1}); \\ A_2 x &= A_2(v_1, \dots, v_k, v_{k+1}) = (v_2, \dots, v_k, v_{k+1}).\end{aligned}$$

The Lyons product $f = u * v$ of $u \in L(X^{(2)} \times X^{(2)})$ and $v \in L(X^{(k)} \times X^{(k)})$ ($k \geq 2$) is given by:

$$\begin{aligned}f(x, x \pm e_1^{(k+1)}) &= u(A_1 x, A_1 x \pm e_1^{(2)}) \times \\ &\quad [v(A_2 x, A_2 x + e_k^{(k)}) - v(A_2 x, A_2 x - e_k^{(k)})]; \\ f(x, x \pm e_j^{(k+1)}) &= v(A_2 x, A_2 x \pm e_{j-1}^{(k)}) \times \\ &\quad [u(A_1 x, A_1 x + e_2^{(2)}) - u(A_1 x, A_1 x - e_2^{(2)})]\end{aligned}$$

for $2 \leq j \leq k$;

$$f(x, x \pm e_{k+1}^{(k+1)}) = \pm 2 u(A_1 x, A_1 x \pm e_2^{(2)}) \times \\ v(A_2 x, A_2 x \pm e_k^{(k)}).$$

If u and v satisfy (7.1), then f does it.

Lemma 7.2. Let u , v and f be the same as above and assume that $J_2(u; a) = 0$ and $J_k(v; \xi) = 0$ for all $a \in X^{(2)}$, $a \neq 0$ and $\xi \in X^{(k)}$, $\xi \neq 0$. Then $J_{k+1}(f; x) = 0$ for all $x \in X^{(k+1)}$ such that $A_1 x \neq 0$ and $A_2 x \neq 0$.

Proof. Let us put

$$\alpha_j = \alpha_j(u; x) = u(A_1 x, A_1 x + e_j^{(2)}), \\ \alpha_{j+2} = \alpha_{j+2}(u; x) = u(A_1 x, A_1 x - e_j^{(2)})$$

for $j = 1, 2$;

$$\beta_j = \beta_j(v; x) = v(A_2 x, A_2 x + e_j^{(k)}), \\ \beta_{j+k} = \beta_{j+k}(v; x) = v(A_2 x, A_2 x - e_j^{(k)})$$

for $j = 1, \dots, k$. Then

$$(7.2) \quad J_{k+1}(f; x) = (\alpha_1 + \alpha_3)(\beta_k - \beta_{2k}) + \\ \sum_{j=2}^k (\beta_{j-1} + \beta_{j-1+k})(\alpha_2 - \alpha_4) + 2\alpha_2\beta_k - 2\alpha_4\beta_{2k}.$$

By our assumption, we have

$$\sum_{j=1}^4 \alpha_j(u; x) = 0 \text{ and } \sum_{j=1}^{2k} \beta_j(v; x) = 0 \\ \text{for } x \in X^{(k+1)} \text{ such that } A_1 x \neq 0 \text{ and } A_2 x \neq 0. \text{ Thus by (7.2)}$$

$$J_{k+1}(f; x) = (\alpha_2 - \alpha_4) \sum_{j=1}^k (\beta_j + \beta_{j+k}) = 0.$$

To construct a flow from 0 to the ideal boundary of $N^{(k)}$, let us put for $k \geq 2$

$$C_n^{(k)} = \{(v_1, \dots, v_{k-1}, n) \in \mathbb{Z}^k; |v_i| \leq n \ (i = 1, \dots, k-1)\}, \\ \Omega_k = \bigcup_{n=0}^{\infty} C_n^{(k)}.$$

Let $u_2 \in L(X^{(2)} \times X^{(2)})$ be a function which satisfies

(8.1) and let for $n \geq 0$

$$u_2((\mu, n), (\mu, n+1)) = \begin{cases} 1/(2n+1) & \text{for } |\mu| \leq n \\ 0 & \text{for } |\mu| > n \end{cases}$$

$$u_2((\mu, n), (\mu+1, n)) = (2\mu+1)[(2n-1)(2n+1)]^{-1} \quad \text{for}$$

$$0 \leq \mu \leq n-1;$$

$$u_2((\mu, n), (\mu-1, n)) = (-2\mu+1)[(2n-1)(2n+1)]^{-1}$$

$$\text{for } -n+1 \leq \mu \leq 0;$$

$$u_2((\mu, n), (\mu \pm 1, n)) = 0 \quad \text{for } |\mu| > n;$$

For $n < 0$, let

$$u_2((\mu, n), (\mu, n+1)) = u_2((\mu, n), (\mu \pm 1, n)) = 0$$

for all μ . It is easily seen that $J_2(u_2; a) = 0$ for all $a \in X^{(2)}$, $a \neq 0$ and $J_2(u_2; 0) = 1$, i.e., $J_2(u; a) = \varepsilon_0(a)$. Let w_2 be the function on $Y^{(2)}$ defined by u_2 in Lemma 7.1. Then w_2 is a flow from 0 to the ideal boundary of $N^{(2)}$.

Let $u_3 = u_2 * u_2$ be the Lyons product of u_2 and u_2 . Then $J_3(u_3; x) = 0$ for all $x \in X^{(3)}$ such that $A_1 x \neq 0$ and $A_2 x \neq 0$ by Lemma 7.2. In case $A_1 x = 0$ and $A_2 x \neq 0$,

$$\alpha_1 = \alpha_3 = \alpha_4 = 0, \alpha_2 = 1; \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$$

by our construction, and hence $J_3(u_3; x) = 0$. This is true in case $A_1 x \neq 0$ and $A_2 x = 0$. In case $A_1 x = A_2 x = 0$, we see that $x = 0$ and $J_3(u_3; 0) = 2$ by (7.2). Therefore $J_3(u_3; x) = 2\varepsilon_0(x)$.

By our construction, we have

$$u_3(x, x + e_3^{(3)}) = 2(2n+1)^{-2} \quad \text{if } x \in C_n^{(3)} \quad (n \geq 0);$$

$$|u_3(x, z)| \leq B_3(2n+1)^{-2} \quad \text{if } x, z \in C_n^{(3)} \text{ and } |x - z|_3 = 1;$$

Here B_3 is a constant independent of n and $|x|_k$ denotes the Euclidean distance between x and 0 in R^k .

$$u_3(x, z) = 0 \text{ unless } \{x, z\} \subset \Omega_3.$$

Let $u_k = u_2 * u_{k-1}$ for $k = 4, \dots, d$. Then we see by induction

$$u_d(x, x + e_d^{(d)}) = 2^{d-2}(2n+1)^{-d+1} \quad \text{if } x \in C_n^{(d)};$$

$|u_d(x, z)| \leq B_d(2n + 1)^{-d+1}$ if $x, z \in C_n^{(d)}$ and $|x - z|_d = 1$, where B_d is a constant independent of n ;

$$u_d(x, z) = 0 \text{ unless } \{x, z\} \subset \Omega_d.$$

By Lemma 7.2 and the fact that $u_k(x, z) = 0$ unless $\{x, z\} \subset \Omega_k$, we see that $J_d(u_d; x) = 0$ for all $x \in X^{(d)}$, $x \neq 0$. By our construction, $J_d(u_d; 0) = 2^{d-2}$. Let w_d be the function on $Y^{(d)}$ defined by u_d in Lemma 7.1. Then

$$\sum_{y \in Y} K(x, y)w_d(y) = -2^{d-2}\epsilon_0(x)$$

for all $x \in X^{(d)}$, i.e., w_d is a flow from 0 to the ideal boundary of $N^{(d)}$. Put

$$\begin{aligned} E_1(n) &= \{y \in Y^{(d)}; e(y) \cap C_n^{(d)} \neq \emptyset \text{ and } e(y) \cap C_{n+1}^{(d)} \neq \emptyset\}, \\ E_2(n) &= \{y \in Y^{(d)}; e(y) \subset C_n^{(d)}\}. \end{aligned}$$

Then

$$\text{Card } E_1(n) = (2n + 1)^{d-1} \text{ and } \text{Card } E_2(n) = 2n(2n + 1)^{d-2}.$$

(Here, Card stands for the cardinal.) Let $1 < p < d$ and $1/p + 1/q = 1$. Since $w_d(y) = 0$ unless $e(y) \subset \Omega_d$, we have

$$\begin{aligned} H_q(w_d) &= \sum_{n=0}^{\infty} [\sum_{y \in E_1(n)} |w_d(y)|^q + \sum_{y \in E_2(n)} |w_d(y)|^q] \\ &\leq \sum_{n=0}^{\infty} [2^{d-2}(2n + 1)^{-d+1}]^q (2n + 1)^{d-1} \\ &\quad + \sum_{n=0}^{\infty} [B_d(2n + 1)^{-d+1}]^q [2n(2n + 1)^{d-2}] \\ &\leq [2^{(d-2)q} + (B_d)^q] \sum_{n=0}^{\infty} (2n + 1)^{(d-1)(1-q)} < \infty, \end{aligned}$$

since $p < d$ implies $(d - 1)(1 - q) < -1$. Therefore $N^{(d)}$ is of hyperbolic of order p , $1 < p < d$. Thus $\text{ind } N^{(d)} \geq d$.

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