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On multidimensional inverse scattering in time-dependent electric fields†

Atsuhide ISHIDA

Faculty of Economics, Otemon Gakuin University
1-15 Nishiai 2-chome, Ibaraki, Osaka 567-8502, Japan
E-mail: a-ishida@res.otemon.ac.jp

1 Introduction

Throughout this paper, we assume the spatial dimension $d \geqslant 2$. We report one of the inverse scattering problems for quantum systems in a time-dependent electric field $E(t) \in \mathbb{R}^d$, which was obtained in Adachi-Fujiwara-I [1]. By Enss-Weder time-dependent method [5], we can show that the high speed limit of the scattering operator determines uniquely the potential V belonging to the wider class than the classes given by the previous work in Adachi-Maehara [3], Adachi-Kamada-Kazuno-Toratani [2], Nicoleau [10] and Fujiwara [6].

The free and full Hamiltonians under the consideration are given by

$$H_0(t) = p^2/2 - E(t) \cdot x, \quad H(t) = H_0(t) + V \tag{1.1}$$

acting as the self-adjoint operators on $L^2(\mathbb{R}^d)$, where $p = -i\nabla_x$ is the momentum, $E(t)$ is the time-dependent electric field and the interaction potential V is real-valued multiplicative operator. $E(t)$ and $V = V^{\mathrm{vs}} + V^{\mathrm{s}} + V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu} + \mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$ satisfy following assumptions.

Assumption 1.1. *The time-dependent electric field $E(t) \in \mathbb{R}^d$ is represented as*

$$E(t) = E_0(1 + |t|)^{-\mu} + E_1(t), \tag{1.2}$$

where $0 \leqslant \mu < 1$, $E_0 \in \mathbb{R}^d \setminus \{0\}$ and $E_1(t) \in C(\mathbb{R}, \mathbb{R}^d)$ such that

$$\left| \int_0^t \int_0^s E_1(\tau) d\tau ds \right| \leqslant C \max\{|t|, |t|^{2-\mu_1}\} \tag{1.3}$$

with $\mu < \mu_1 \leqslant 1$.

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Roughly speaking about the perturbation part $E_1(t)$, we assume that $|E_1(t)| \leqslant C(1+|t|)^{-\mu_2}$ for some $\mu_2 > \mu$ and take μ_1 as follows:

$$\begin{cases} \mu_1 = \mu_2 & \mu < \mu_2 < 1 \\ \mu < \mu_1 < \mu_2 & \mu_2 = 1 \\ \mu_1 = 1 & \mu_2 > 1. \end{cases} \tag{1.4}$$

Such $E(t)$ was first dealt with in Adachi-Kamada-Kazuno-Toratani [2]. For brevity's sake, we suppose that $E_0 = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$.

Assumption 1.2. *$\mathscr{V}^{\mathrm{vs}}$ is the class of real-valued multiplicative operators V^{vs} is satisfying that V^{vs} is decomposed into the sum of a singular part V_1^{vs} and a regular part V_2^{vs}. V_1^{vs} is compactly supported, belongs to $L^{q_1}(\mathbb{R}^d)$ and satisfies $|\nabla V_1^{\mathrm{vs}}| \in L^{q_2}(\mathbb{R}^d)$. $V_2^{\mathrm{vs}} \in C^1(\mathbb{R}^d)$ satisfies that V_2^{vs} and its first derivatives are all bounded in $\mathbb{R}^d$ and that*

$$\int_0^\infty \|F(|x| \geqslant R) V_2^{\mathrm{vs}}(x)\|_{\mathscr{B}(L^2)} dR < \infty. \tag{1.5}$$

Here q_1 satisfies that $q_1 > d/2$ and $q_1 \geqslant 2$, q_2 satisfies

$$\begin{cases} 1/q_2 = 1/(2q_1) + 2/d & d \geqslant 5 \\ 1/q_2 < 1/(2q_1) + 1/2 & d = 4 \\ 1/q_2 = 1/(2q_1) + 1/2 & d \leqslant 3, \end{cases} \tag{1.6}$$

and $F(|x| \geqslant R)$ is the characteristic function of $\{x \in \mathbb{R}^d \mid |x| \geqslant R\}$.

$\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$ with some $\alpha_\mu > 0$ is the class of real-valued multiplicative operators V^{s} is satisfying that V^{s} belongs to $C^1(\mathbb{R}^d)$ and satisfies

$$|V^{\mathrm{s}}(x)| \leqslant C\langle x\rangle^{-\gamma}, \quad |\partial_x^\beta V^{\mathrm{s}}(x)| \leqslant C_\beta \langle x\rangle^{-1-\alpha}, \quad |\beta| = 1 \tag{1.7}$$

with some γ and α such that $1/(2-\mu) < \gamma \leqslant 1$ and $\alpha_\mu < \alpha \leqslant \gamma$.

Finally, $\mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$ with some $\gamma_\mu \geqslant 1/(2(2-\mu))$ is the class of real-valued multiplicative operators V^{l} is satisfying that V^{l} belongs to $C^2(\mathbb{R}^d)$ and satisfies

$$|\partial_x^\beta V^{\mathrm{l}}(x)| \leqslant C\langle x\rangle^{-\gamma_D - |\beta|/(2-\mu)}, \quad |\beta| \leqslant 2, \tag{1.8}$$

with some γ_D such that $\gamma_\mu < \gamma_D \leqslant 1/(2-\mu)$.

We note that one can obtain

$$\int_0^\infty \|F(|x| \geqslant R) V^{\mathrm{vs}}(x)\langle p\rangle^{-2}\|_{\mathscr{B}(L^2)} dR < \infty \tag{1.9}$$

by this assumption and it is equivalent to

$$\int_0^\infty \|V^{\mathrm{vs}}(x)\langle p\rangle^{-2} F(|x| \geqslant R)\|_{\mathscr{B}(L^2)} dR < \infty \tag{1.10}$$

because V^{vs} is a multiplicative operator (see e.g. Reed-Simon [11]).

As for the class $\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$, we also note that by virtue of $\alpha \leqslant \gamma$, we can treat an oscillation part. For example, the following function belongs to $\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$:

$$V^{\mathrm{s}}(x) = \langle x\rangle^{-\gamma}\cos\langle x\rangle^{\gamma-\alpha}. \tag{1.11}$$

In fact, we can verify easily that $|\nabla_x V^{\mathrm{s}}(x)| \leqslant C(\langle x\rangle^{-1-\gamma} + \langle x\rangle^{-1-\alpha}) \leqslant C\langle x\rangle^{-1-\alpha}$ holds with some $C > 0$.

2 Results

We first state the case where $V^{\mathrm{l}} = 0$. Then we can see the wave operators

$$W^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{2.1}$$

exist as this fact was shown in Adachi-Kamada-Kazuno-Toratani [2], where we denote the propagators generated by $H_0(t)$ and $H(t)$ as $U_0(t,0)$ and $U(t,0)$. The existence and uniqueness of these propagators are guaranteed by virtue of Yajima [14]. The scattering operator $S = S(V)$ is defined by

$$S = (W^+)^* W^-. \tag{2.2}$$

The following obtained in [1] is one of those which we would like to report in this paper.

Theorem 2.1. (Adachi-Fujiwara-I [1]) *Put*

$$\tilde{\alpha}_\mu = \begin{cases} \dfrac{7-3\mu-\sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)} & 0 \leqslant \mu \leqslant 1/2 \\[2ex] \dfrac{1+\mu}{2(2-\mu)} & 1/2 < \mu < 1. \end{cases} \tag{2.3}$$

Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}$. *If* $S(V_1) = S(V_2)$, *then* $V_1 = V_2$.

In the case where $E(t) \equiv E_0$, that is, the case of the Stark effect, this theorem was first proved by Weder [12] under the condition $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{0,0}$ and the additional assumption $\gamma > 3/4$. However, as it is well-known, the short-range condition on V under the Stark effect is $\gamma > 1/2$. Later Nicoleau [9] proved this theorem for real-valued $V \in C^\infty(\mathbb{R}^d)$ satisfying $|\partial_x^\beta V(x)| \leqslant C_\beta\langle x\rangle^{-\gamma-|\beta|}$ with $\gamma > 1/2$, under the spatial dimension $d \geqslant 3$. After that, this theorem was obtained by Adachi-Maehara [3] for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{0,1/2}$. In our case where $\mu = 0$, substitute $\mu = 0$ in $\tilde{\alpha}_\mu$. We have

$$\tilde{\alpha}_0 = \left.\frac{7-3\mu-\sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)}\right|_{\mu=0} = \frac{7-\sqrt{17}}{8} < \frac{1}{2}. \tag{2.4}$$

If $a < b$, then $\mathscr{V}^{\mathrm{s}}_{\mu,b} \subsetneq \mathscr{V}^{\mathrm{s}}_{\mu,a}$. Therefore this implies

$$\mathscr{V}^{\mathrm{s}}_{0,1/2} \subsetneq \mathscr{V}^{\mathrm{s}}_{0,\tilde{\alpha}_0}. \tag{2.5}$$

In the time-dependent case where $0 < \mu < 1$ and $E_1(t) \not\equiv 0$, the result corresponding to Theorem 2.1 was also obtained by Adachi-Kamada-Kazuno-Toratani [2] under the assumption that $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,1/(2-\mu)}$. We can verify that $\tilde{\alpha}_\mu < 1/(2-\mu)$ and this implies that above result is finer than the previous one:

$$\mathscr{V}^{\mathrm{s}}_{\mu,1/(2-\mu)} \subsetneqq \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}. \tag{2.6}$$

We next state the case where $V^{\mathrm{l}} \neq 0$. If $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$, the Dollard-type modified wave operators due to White [13] (see also Adachi-Tamura [4] and Jensen-Yajima [8])

$$W_D^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) M_D(t), \quad M_D(t) = e^{-i\int_0^t V^{\mathrm{l}}(p\tau + c(\tau))d\tau} \tag{2.7}$$

can exist by virtue of the condition $\gamma_D > 1/(2(2-\mu))$ (see [2]), where we put

$$c(t) = \int_0^t b(\tau)d\tau, \quad b(t) = \int_0^t E(\tau)d\tau. \tag{2.8}$$

Then the Dollard-type modified scattering operator $S_D = S_D(V^{\mathrm{l}}, V^{\mathrm{vs}} + V^{\mathrm{s}})$ is defined by

$$S_D = (W_D^+)^* W_D^-. \tag{2.9}$$

Then we also report the following result.

Theorem 2.2. (Adachi-Fujiwara-I [1]) *Suppose that a given V^{l} satisfies $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,\tilde{\gamma}_\mu}$ with*

$$\tilde{\gamma}_\mu = \frac{1}{2(2-\mu)} + \frac{1-\mu}{4(2-\mu)}. \tag{2.10}$$

Put

$$\tilde{\alpha}_{\mu,D} = \begin{cases} \dfrac{13-5\mu-\sqrt{(1-\mu)(41-25\mu)}}{8(2-\mu)} & 0 \leqslant \mu \leqslant 5/7 \\ \dfrac{1+\mu}{2(2-\mu)} & 5/7 < \mu < 1. \end{cases} \tag{2.11}$$

Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_{\mu,D}}$. If $S_D(V^{\mathrm{l}}, V_1) = S(V^{\mathrm{l}}, V_2)$, then $V_1 = V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential V.

In the case where $0 < \mu < 1$ and $E_1(t) \not\equiv 0$, Adachi-Kamada-Kazuno-Toratani [2] proved this theorem under the condition that

$$V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,\hat{\gamma}_\mu}, \quad V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,1/(2-\mu)} \tag{2.12}$$

with $\hat{\mu} = (7-\sqrt{3}-\sqrt{60-22\sqrt{3}})/4$ and

$$\hat{\gamma}_\mu = \begin{cases} \dfrac{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2(1-\mu)^2}{1+\mu}}}{2-\mu} & 0 < \mu \leqslant \hat{\mu} \\ -\dfrac{3-\mu}{8} + \sqrt{\dfrac{(3-\mu)^2}{64} + \dfrac{2\mu^2-7\mu+7}{4(2-\mu)^2}} & \hat{\mu} < \mu < 1. \end{cases} \tag{2.13}$$

Computing straightforwardly, we can see $\tilde{\alpha}_{\mu,D} < 1/(2-\mu)$ and $\tilde{\gamma}_\mu < \hat{\gamma}_\mu$. We thus obtain

$$\mathscr{V}^{\mathrm{s}}_{\mu,1/(2-\mu)} \subsetneq \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_{\mu,D}}, \quad \mathscr{V}^{\mathrm{l}}_{\mu,\hat{\gamma}_\mu} \subsetneq \mathscr{V}^{\mathrm{l}}_{\mu,\tilde{\gamma}_\mu}. \tag{2.14}$$

In particular, there was no result for the case where $\mu = 0$. Here we emphasize that if $5/7 \leqslant \mu < 1$, then $\tilde{\alpha}_{\mu,D} = \tilde{\alpha}_\mu$ holds, although if $0 \leqslant \mu < 5/7$, then $\tilde{\alpha}_{\mu,D} > \tilde{\alpha}_\mu$ holds.

Remark 2.3. *We assume that $E(t) \in C(\mathbb{R}, \mathbb{R}^d)$ is T-periodic in time with non-zero mean E_0, that is,*

$$E_0 = \int_0^T E(\tau)d\tau/T \neq 0, \tag{2.15}$$

which was treated by Nicoleau [10] and Fujiwara [6]. In this case, the method in the proofs of Theorems 2.1 and 2.2 does work well also, because we have

$$|b(t) - tE_0| \leqslant \int_0^T |E(\tau) - E_0|d\tau, \tag{2.16}$$

$$|c(t) - t^2E_0/2| \leqslant \int_0^{|t|} |b(\tau) - \tau E_0|d\tau \leqslant C|t|, \tag{2.17}$$

with $C = \int_0^T |E(\tau) - E_0|d\tau$ by the periodicity of $E(t)$. (2.17) implies $\mu = 0$ in (1.2) and $\mu_1 = 1$ in (1.3).

By virtue of this fact, we can obtain an improvement of the results of [10] and [6].

Theorem 2.4. *Suppose that $E(t) \in C(\mathbb{R}, \mathbb{R}^d)$ is T-periodic in time with non-zero mean E_0. Then the followings hold.*

1. *Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{0,\tilde{\alpha}_0}$. If $S(V_1) = S(V_2)$, then $V_1 = V_2$.*
2. *Suppose that a given V^{l} satisfies $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{0,\tilde{\gamma}_0}$. Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{0,\tilde{\alpha}_{0,D}}$. If $S_D(V^{\mathrm{l}}, V_1) = S_D(V^{\mathrm{l}}, V_2)$, then $V_1 = V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential V.*

Nicoleau [10] proved the uniqueness assuming that $|\partial_x^\beta V(x)| \leqslant C_\beta \langle x\rangle^{-\gamma-|\beta|}$ with $\gamma > 1/2$ for $V \in C^\infty(\mathbb{R})^d$ and the additional condition $d \geqslant 3$. Fujiwara [6] assumed that $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{0,1/2}$. These two results did not treat the long-range potentials.

3 Short-range Case

By virtue of Theorem 3.1 below and the Plancherel formula associated with the Radon transform (see Helgason [7]), Theorem 2.1 can be shown in the quite same way as in the proof of Theorem 1.2 in [12] (see also Enss-Weder [5]).

Theorem 3.1. (Reconstruction Formula [1]) *Let $\hat{v} \in \mathbb{R}^d$ be given such that $|\hat{v} \cdot e_1| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in \mathscr{S}(\mathbb{R}^d)$ be such that $\mathscr{F}\Phi_0, \mathscr{F}\Psi_0 \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\mathscr{F}\Phi_0, \operatorname{supp}\mathscr{F}\Psi_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. Put $\Phi_v = e^{iv\cdot x}\Phi_0, \Psi_v = e^{iv\cdot x}\Psi_0$. Let $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}$, where $\tilde{\alpha}_\mu$ is the same as in Theorem 2.1 and $\mathscr{F}$ is the Fourier transformation. Then*

$$\begin{aligned}|v|(i[S,p_j]\Phi_v,\Psi_v) = \int_{-\infty}^{\infty} &((V^{\mathrm{vs}}(x+\hat{v}t)p_j\Phi_0,\Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}t)\Phi_0,p_j\Psi_0)\\ &+(i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}t)\Phi_0,\Psi_0))dt + o(1)\end{aligned} \tag{3.1}$$

holds as $|v| \to \infty$ for $1 \leqslant j \leqslant d$.

To prove Theorem 3.1, the following estimate is the key.

Proposition 3.2. *Let v and Φ_v be as in Theorem 3.1 and $\epsilon > 0$. Put*

$$\Theta(\alpha) = \begin{cases} \alpha + \dfrac{(\alpha-\mu)(1-\alpha)}{(1-\mu)(2-\alpha)} & \alpha > \mu \\ \alpha - \dfrac{\mu-\alpha}{1-\mu} & \mu/(2-\mu) < \alpha \leqslant \mu. \end{cases} \tag{3.2}$$

Then

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+c(t)))U_0(t,0)\Phi_v\|dt = O(|v|^{-\Theta(\alpha)+\epsilon}) \tag{3.3}$$

holds as $|v| \to \infty$ for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\mu/(2-\mu)}$.

In Adachi-Maehara [3], the corresponding estimate to this proposition was

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+c(t)))U_0(t,0)\Phi_v\|dt = O(|v|^{-\alpha}) \tag{3.4}$$

(see Lemma 2.2 in [3]). When we denote the error term of (3.1) by $R(v)$, $\lim_{|v|\to\infty} R(v) = 0$ is equivalent to $2(-\alpha)+1 < 0$. Therefore $\alpha > 1/2$ was required. On the other hand, in Adachi-Kamada-Kazuno-Toratani [2], the corresponding one was

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+c(t)))U_0(t,0)\Phi_v\|dt = O(|v|^{-\rho}), \tag{3.5}$$

where

$$\rho = \frac{(2-\mu)\alpha - \mu}{2(1-\mu)} \tag{3.6}$$

(see Lemma 3.4 in [2]) and $\lim_{|v|\to\infty} R(v) = 0$ is equivalent to $2(-\rho)+1 < 0$. Solving this inequality for α, we see that $\alpha > 1/(2-\mu)$ was required. In our estimate, $\alpha > \tilde{\alpha}_\mu$ comes from the inequality $2(-\Theta(\alpha))+1 < 0$ which is equivalent to $\lim_{|v|\to\infty} R(v) = 0$.

4 Long-range Case

In the case where $V^{\mathrm{l}} \neq 0$, the reconstruction formula is represented as follows, which also yields the proof of Theorem 2.2.

Theorem 4.1. (Reconstruction Formula [1]) *Let $\hat{v} \in \mathbb{R}^d$ be given such that $|\hat{v} \cdot e_1| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in \mathscr{S}(\mathbb{R}^d)$ be such that $\mathscr{F}\Phi_0, \mathscr{F}\Psi_0 \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\mathscr{F}\Phi_0, \operatorname{supp}\mathscr{F}\Psi_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. Put $\Phi_v = e^{iv\cdot x}\Phi_0, \Psi_v = e^{iv\cdot x}\Psi_0$. Let $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_{\mu,D}}$ and $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,\tilde{\gamma}_\mu}$, where $\tilde{\alpha}_{\mu,D}$ and $\tilde{\gamma}_\mu$ are the same as in Theorem 2.2. Then*

$$\begin{aligned}|v|(i[S_D,p_j]\Phi_v,\Psi_v) = \int_{-\infty}^{\infty}&((V^{\mathrm{vs}}(x+\hat{v}t)p_j\Phi_0,\Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}t)\Phi_0,p_j\Psi_0)\\ &+(i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}t)\Phi_0,\Psi_0) + (i(\partial_{x_j}V^{\mathrm{l}})(x+\hat{v}t)\Phi_0,\Psi_0))dt + o(1)\end{aligned} \tag{4.1}$$

holds as $|v| \to \infty$ for $1 \leqslant j \leqslant d$.

To prove Theorem 4.1, the following estimate is the key.

Proposition 4.2. *Let v and Φ_v be as in Theorem 4.1, $\epsilon > 0$ and $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,1/(2(2-\mu))}$. Put*

$$\Theta_D(\gamma_D) = \begin{cases} 1 & \gamma_D > 1/2 \\ \dfrac{2\gamma_D(2-\mu)-1}{1-\mu} & \gamma_D \leqslant 1/2. \end{cases} \tag{4.2}$$

Then

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-b(t)) + c(t)))U_D(t)\Phi_v\|dt = O(|v|^{-\Theta_D(\gamma_D)+\epsilon}) \tag{4.3}$$

holds as $|v| \to \infty$, where $U_D(t) = U_0(t,0)M_D(t)$ and $M_D(t)$ is the same as in (2.7).

In Adachi-Kamada-Kazuno-Toratani [2], the corresponding estimate to this proposition was

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-b(t)) + c(t)))U_D(t)\Phi_v\|dt = O(|v|^{-\rho_{\mathrm{l}}}), \tag{4.4}$$

where

$$\rho_{\mathrm{l}} = \frac{(1-\sigma_\kappa)(\tilde{\gamma}_D + 2 - \mu)}{(1-\mu)(\sigma_\kappa(\tilde{\gamma}_D + 2 - \mu) - 1 + \tilde{\gamma}_D)} \tag{4.5}$$

with $\tilde{\gamma}_D = (2-\mu)\gamma_D$ and $\sigma_\kappa = 1 - \kappa(1-\mu)/(2-\mu)$ for $0 < \kappa < 1$ (see Lema 4.5 in [2]). When we denote the error term of (4.1) by $R_D(v)$, $\lim_{|v|\to\infty} R_D(v) = 0$ is equivalent to $2(-\rho_{\mathrm{l}}) + 1 < 0$. Therefore $\gamma_D > \hat{\gamma}_\mu$ was required. In our case, $\lim_{|v|\to\infty} R_D(v) = 0$ is equivalent to $2(-\Theta_D(\gamma_D)) + 1 < 0$ and $\gamma_D > \tilde{\gamma}_\mu$ comes from this inequality.

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