

On compact composition operators acting between Bergman spaces

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Abstract

In this note we consider the compact composition operator acting different weighted Bergman spaces of the unit ball of $\mathbb{C}^N$. We will give an estimate for the essential norm of the composition operator. As a corollary, we can characterize the compactness of this operator in terms of the boundary behavior of the symbol.

1 Introduction

For a fixed integer $N > 1$, let $\mathbb{C}^N$ denote the complex N -dimensional Euclidean space and B denote the open unit ball of $\mathbb{C}^N$. For each p , $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A_\alpha^p(B)$ is the space of all holomorphic functions f on B for which

$$\|f\|_\alpha^p = \int_B |f(z)|^p (1 - |z|^2)^\alpha dV(z) < \infty.$$

Here dV denotes the normalized Lebesgue volume measure on B . When $1 \leq p < \infty$ the space $A_\alpha^p(B)$ is a Banach space. In particular, the space $A_\alpha^2(B)$ is a functional Hilbert space with inner product

$$\langle f, g \rangle_\alpha = \int_B f(z) \overline{g(z)} (1 - |z|^2)^\alpha dV(z).$$

Since each point evaluation is a bounded linear functional, $A_\alpha^2(B)$ has the reproducing kernel function which is given by

$$K_w^\alpha(z) = \frac{c_\alpha}{(1 - \langle z, w \rangle)^{\alpha+N+1}},$$

where $c_\alpha = 1 / \int_B (1 - |z|^2)^\alpha dV(z)$.

Let φ be a holomorphic self-map of B , that is

$$\varphi = (\varphi_1, \dots, \varphi_N) : B \rightarrow B,$$

where each φ_j is a holomorphic function on B . Then φ induces the composition operator C_φ , defined on the space of all holomorphic functions on B by

$$C_\varphi f = f \circ \varphi.$$

Many authors have studied these operators on various holomorphic function spaces. For these studies, see the monograph [3]. In this note, we discuss this operator on $A_\alpha^p(B)$. In the one variable case, Littlewood's subordination principle shows that every holomorphic function φ on the unit disk $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ induces the bounded composition operator C_φ on the weighted Bergman space $A_\alpha^p(\mathbb{D})$. Thus the concern with the compactness of C_φ had been growing since the end of the last century. In 1986 B.D. MacCluer and J.H. Shapiro [5] gave a characterization for the symbol φ which induces the compact composition operator on $A_\alpha^p(\mathbb{D})$ as follows.

Theorem 1. *Let $0 < p < \infty$, $\alpha > -1$ and φ be a holomorphic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the composition operator C_φ is the compact operator on $A_\alpha^p(\mathbb{D})$ if and only if φ satisfies the condition*

$$\lim_{|z| \rightarrow 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0. \quad (1)$$

By Julia-Carathéodory's theorem we see that the above condition (1) is equivalent to φ has no finite angular derivative at any point of the boundary of $\mathbb{D}$.

The several variables (unit ball) case have some difficulties on the property of the composition operator C_φ . For instance, there is a holomorphic self-map of B such that the composition operator is not bounded on $A_\alpha^p(B)$. It is easy to construct the example. For the sake of the simplicity, we consider the case $N = 2$ and $p = 2$. We put $\varphi(z) = (2z_1z_2, 0)$ and consider the test function $f_k(z)$ defined by

$$f_k(z) = \sqrt{\frac{\Gamma(k + \alpha + 3)}{k! \Gamma(\alpha + 3)}} z_1^k \quad (z = (z_1, z_2) \in B),$$

for $k \geq 1$ positive integer. Then $\{f_k\}$ is bounded in $A_\alpha^2(B)$ with $\sup_{k \geq 1} \|f_k\|_\alpha = 1$ and

$$f_k(\varphi(z)) = \sqrt{\frac{\Gamma(k + \alpha + 3)}{k! \Gamma(\alpha + 3)}} 2^k z_1^k z_2^k.$$

This implies that $\|C_\varphi f_k\|_\alpha \sim k^{\frac{1}{2}}$, and so C_φ is not bounded on $A_\alpha^2(B)$. When we study on the compact composition operator in the case $N \geq 2$, hence, we will need some assumptions which verify the boundedness of C_φ . For an univalent holomorphic self-map of B , the following sufficient condition for the boundedness of C_φ is known.

Theorem 2. *Suppose that an univalent holomorphic self-map of B which satisfies*

$$\sup_{z \in B} \frac{\|\varphi'(z)\|^2}{|J_\varphi(z)|^2} < \infty. \quad (2)$$

Then C_φ is bounded on $A_\alpha^p(B)$.

However it is also known that the condition (2) is not a necessary condition for the boundedness of C_φ . See [3, p.247]. Hence many authors have tried to characterize the compactness of C_φ on $A_\alpha^p(B)$ under some assumptions.

2 Well-Known Results

In [5], B.D. MacCluer and J.H. Shapiro also gave the following characterization.

Theorem 3. *Suppose that φ is an univalent holomorphic self-map of B which satisfy the condition (2) in Theorem 2. Then C_φ is compact on $A_\alpha^p(B)$ if and only if φ has no finite angular derivative at any point of the boundary of B .*

This result is the higher dimensional case of Theorem 1.

D.D. Clahane [2] proved the following result.

Theorem 4. *Let $p > 0$ and $\alpha \geq 0$. Suppose that φ is a holomorphic self-map of B such that C_φ is bounded on $A_\alpha^p(B)$ and φ satisfies the following condition*

$$\lim_{|z| \rightarrow 1^-} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \|\varphi'(z)\|^2 = 0.$$

Then C_φ is compact on $A_\beta^p(B)$ for all $\beta \geq \alpha$.

Clahane's result does not require the assumption φ is univalent but the relation between the compactness of C_φ and the boundary behavior of φ became unclear. Furthermore the spaces $A_\alpha^p(B)$ is restricted to the case $\alpha \geq 0$.

Recently, K. Zhu [8] have given the following characterization.

Theorem 5. *Let $p > 0$ and $\alpha > -1$. Suppose that C_φ is bounded on $A_\beta^q(B)$ for some $q > 0$ and $-1 < \beta < \alpha$. Then C_φ is compact on $A_\alpha^p(B)$ if and only if φ satisfies*

$$\lim_{|z| \rightarrow 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Note that Julia-Carathéodory's theorem for the unit ball case implies that the above condition is equivalent to φ has no finite angular derivative at any point of the boundary of B . Zhu's result does not also require the univalence of φ . Since he gave the characterization for the compactness of C_φ in terms of the angular derivative condition, we can consider that this result is the improved version of Theorem 3 or the higher dimensional case of Theorem 1.

In Theorem 3, Theorem 4 or Theorem 5, their results need some hypotheses on the symbol φ . The reason to need these assumptions on φ seems to be a technical request in their proof. Since every holomorphic self-map φ of B does not induce the bounded composition operator on $A_\alpha^p(B)$, the assumption that C_φ is bounded on $A_\alpha^p(B)$ is very natural condition for the unit ball case.

3 Main Result

Under the condition C_φ is bounded on $A_\alpha^p(B)$, we will consider the compactness problem. Recall that the essential norm of the bounded operator on Banach spaces. Let X and Y be Banach spaces. For a bounded operator $T : X \rightarrow Y$, the essential norm $\|T\|_{e, X \rightarrow Y}$ of T is defined to be the distance from T to the set of compact operators, namely $\|T\|_{e, X \rightarrow Y}$ is defined by

$$\|T\|_{e, X \rightarrow Y} = \inf \{ \|T - K\| : K \text{ is compact from } X \text{ to } Y \}.$$

Here $\|\cdot\|$ denotes the usual operator norm. By this definition, we see that $T : X \rightarrow Y$ is a compact operator if and only if $\|T\|_{e, X \rightarrow Y} = 0$. Thus the essential norm is closely related to the compactness problem of concrete operators. In Theorem 3, Theorem 4 and Theorem 5, they have not mentioned the essential norm of C_φ . In this note we give an estimate for the essential norm of $C_\varphi : A_\alpha^2(B) \rightarrow A_\beta^2(B)$ ($-1 < \alpha \leq \beta$).

Theorem 6. *Let $\alpha > -1$ and $\beta \geq \alpha$. Suppose that φ is a holomorphic self-map of B such that $C_\varphi : A_\alpha^2(B) \rightarrow A_\beta^2(B)$ is bounded. Then the essential norm of C_φ is comparable to*

$$\limsup_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\beta+N+1}}{(1 - |\varphi(z)|^2)^{\alpha+N+1}}.$$

So $C_\varphi : A_\alpha^2(B) \rightarrow A_\beta^2(B)$ is compact if and only if φ satisfies

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\beta+N+1}}{(1 - |\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

In the previous our works [6, 7], we have the following characterization for the boundedness and compactness of $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$.

Theorem 7. *Let $0 < p < \infty$ and $-1 < \alpha, \beta < \infty$. Suppose that φ is a holomorphic self-map of B . Then the following conditions are equivalent.*

- (a) $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$ is a bounded operator,
- (b) φ satisfies the condition

$$\sup_{z \in B} \int_B \left\{ \frac{1 - |z|^2}{|1 - \langle \varphi(w), z \rangle|^2} \right\}^{\alpha+N+1} dV_\beta(w) < \infty.$$

Here dV_β denotes the weighted measure $dV_\beta(w) = (1 - |w|^2)^\beta dV(w)$. Moreover,

- (c) $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$ is a compact operator,
- (d) φ satisfies the condition

$$\sup_{|z| \rightarrow 1^-} \int_B \left\{ \frac{1 - |z|^2}{|1 - \langle \varphi(w), z \rangle|^2} \right\}^{\alpha+N+1} dV_\beta(w) = 0.$$

This theorem shows the following result.

Corollary 1. *The boundedness and compactness of the composition operator $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$ are independent of the exponent p .*

Combining Theorem 6 with Corollary 1, we have the following characterization.

Corollary 2. *Let $0 < p < \infty$ and $-1 < \alpha \leq \beta$. Suppose that φ is a holomorphic self-map of B which induces the bounded composition operator $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$. Then $C_\varphi : A_\alpha^p(B) \rightarrow A_\beta^p(B)$ is compact if and only if*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\beta+N+1}}{(1 - |\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

According to the result due to J.A. Cima and P.R. Mercer [1], every holomorphic self-map φ of B induces the bounded composition operator $C_\varphi : A_\alpha^p(B) \rightarrow A_{\alpha+N-1}^p(B)$. Hence it would be very interesting to know the compactness criteria for this situation. Indeed, H. Koo has proposed the following problem in [4].

Characterize the compactness of the composition operator

$$C_\varphi : A_\alpha^p(B) \rightarrow A_{\alpha+N-1}^p(B).$$

Since we see that $\alpha + N - 1 > \alpha$ for $\alpha > -1$, this situation suits the assumption in Theorem 6. Thus we can give an answer to Koo's question as follows.

Corollary 3. *Let $\alpha > -1$, $0 < p < \infty$ and φ be a holomorphic self-map of B . Then $C_\varphi : A_\alpha^p(B) \rightarrow A_{\alpha+N-1}^p(B)$ is compact if and only if φ satisfies*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\alpha+2N}}{(1 - |\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

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