

On T -algebra homomorphisms between rational function semifields of tropical curves

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1. Introduction

1. Background

$\mathcal{T} := (\mathcal{R} \cup \{-\infty\}, \max, +)$: the tropical semifield

tropical geometry = algebraic geometry/ $\mathcal{T}$

algebraic geometry $\xrightarrow{\text{tropicalization}}$ tropical geometry

tropical curves = tropicalizations of algebraic curves

1. Question

Fact

Category of nonsingular projective curves

$\xleftrightarrow{\text{equiv.}}$ Category of function fields of $\dim = 1$

Γ : a tropical curve

$\text{Rat}(\Gamma)$: the rational function semifield of Γ

- (1) $\text{Rat}(\Gamma)$: finitely generated as a semifield/ $\mathcal{T}$? $\dim(\text{Rat}(\Gamma)) = 1$?
- (2) What kind of semifields/ $\mathcal{T}$ determines a tropical curve? How?
- (3) Does a $\mathcal{T}$ -algebra homomorphism $\text{Rat}(\Gamma) \rightarrow \text{Rat}(\Gamma')$ induce a morphism $\Gamma' \rightarrow \Gamma$?

2. Preliminaries

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(abstract) **tropical curve** = metric graph = the underlying metric space of (G, l)

G : a graph

$l : E(G) \rightarrow \mathbf{R}_{>0}$

* “graph” = unweighted, undirected, finite, connected nonempty multigraph that may have loops

Γ : the tropical curve obtained from (G, l)

(G, l) : a **model** for Γ

(G, l) : **loopless** $\stackrel{\text{def}}{\iff} G$: loopless

2. Preliminaries

Γ : a tropical curve

$f : \Gamma \rightarrow \mathbf{R} \cup \{-\infty\}$: a **rational function** on $\Gamma \stackrel{\text{def}}{\iff}$

f : continuous piecewise $\mathbf{Z}$ -affine or $f \equiv -\infty$

$\text{Rat}(\Gamma)$:= {rational functions on Γ }

$(\text{Rat}(\Gamma), \max, +)$: a **semifield**, a **$\mathbf{T}$ -algebra**

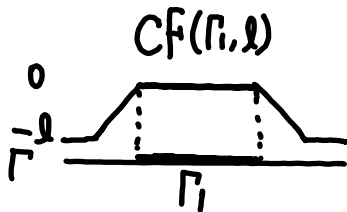
2. Preliminaries

Γ : a tropical curve

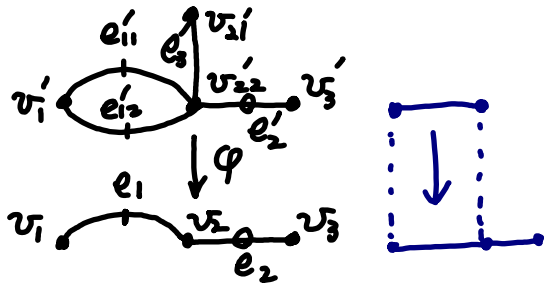
$\Gamma_1 \subset \Gamma$: closed, $\#$ connected components $< \infty$

$l \in \mathbf{R}_{>0}$

$\text{CF}(\Gamma_1, l)(x) := -\min\{\text{dist}(\Gamma_1, x), l\} \quad (x \in \Gamma)$



2. Preliminaries



Γ, Γ' : tropical curves

$\varphi : \Gamma' \rightarrow \Gamma$: a continuous map

φ : a **morphism**² $\stackrel{\text{def}}{\iff}$

$\exists (G, l)$ (resp. (G', l')) : a loopless model for Γ (resp. Γ') s.t.

$\varphi : V(G') \cup E(G') \rightarrow V(G) \cup E(G)$:

(1) $\varphi(V(G')) \subset V(G)$,

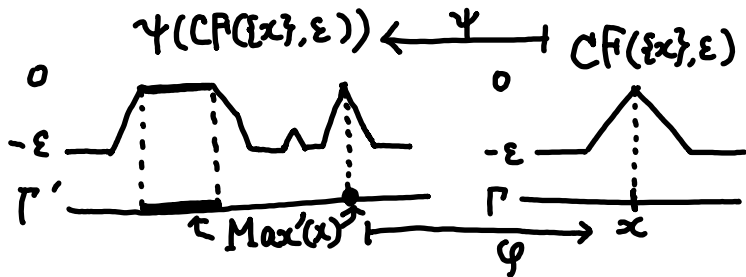
(2) $e' = x'y' \in E(G'), \varphi(e') \in V(G) \implies \varphi(e') = \varphi(x') = \varphi(y')$, and

(3)
 $e' = x'y' \in E(G'), \varphi(e') \in E(G) \implies \varphi(e') = \varphi(x')\varphi(y'), l(\varphi(e'))/l'(e') \in \mathbf{Z}_{>0}$.

²M. Chan, *Tropical hyperelliptic curves*, J. Algebr. Comb. **37**(2):331–359, 2013.

3. Main results

3. Main theorem



$\Gamma, \Gamma' : \text{tropical curves}$

$\psi : \text{Rat}(\Gamma) \rightarrow \text{Rat}(\Gamma') : \text{an injective } T\text{-algebra homomorphism}$

$\forall x \in \Gamma, \forall \varepsilon > 0, \text{Max}'(x) := \{x' \in \Gamma' \mid \psi(CF(\{x\}, \varepsilon))(x') = 0\}$

Theorem 1. (S.)

ψ induces a surjective morphism $\varphi : \Gamma' \rightarrow \Gamma; \text{Max}'(x) \mapsto x$.

unique

3. Contrary

Proposition 2. (S.)

$\Gamma, \Gamma' : \text{tropical curves}$

$\varphi : \Gamma' \rightarrow \Gamma : \text{a surjective morphism}$

$\implies \varphi^* : \text{Rat}(\Gamma) \rightarrow \text{Rat}(\Gamma'); f \mapsto f \circ \varphi$ is an injective $\mathcal{T}$ -algebra homomorphism.

3. Corollary

Corollary 3. (S.)

The following categories $\mathcal{C}, \mathcal{D}$ are isomorphic:

(1) $\text{Ob}(\mathcal{C}) := \{\text{tropical curves}\}$

$\text{Hom}_{\mathcal{C}}(\Gamma, \Gamma') := \{\text{injective } \mathbf{T}\text{-algebra homomorphisms } \text{Rat}(\Gamma) \rightarrow \text{Rat}(\Gamma')\}$

(2) $\text{Ob}(\mathcal{D}) := \{\text{tropical curves}\}$

$\text{Hom}_{\mathcal{D}}(\Gamma, \Gamma') := \{\text{surjective morphisms } \Gamma \rightarrow \Gamma'\}$