

Nonplanar traveling waves of a bistable reaction-diffusion equation in the multi-dimensional space

By

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Abstract

We survey the existence result for nonplanar traveling waves of a bistable reaction-diffusion equation in the space $\mathbb{R}^{n+1}$, which gives a heteroclinic connection of the trivial solution and the unstable standing wave in $\mathbb{R}^n$. Since the problem can be formulated in a monostable case, the equation allows a continuous family of the traveling waves with the speed in a semi-infinite interval.

§ 1. Introduction

In this article we consider the following scalar reaction-diffusion equation in the multi-dimensional space $\mathbb{R}^{n+1}$:

$$(1.1) \quad u_t = \Delta u + u_{yy} + f(u),$$

where

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad y \in \mathbb{R}, \quad u_t = \frac{\partial u}{\partial t}, \quad \Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

We assume that $f(u)$ is C^2 on an open interval containing $[0, 1]$ and that $f(u)$ satisfies

$$(1.2) \quad f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0,$$

$$(1.3) \quad f(u) \neq 0 \quad \text{for } u \in (0, a) \cup (a, 1),$$

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where $0 < a < 1$, and

$$(1.4) \quad f''(u) \geq 0 \quad (0 \leq u \leq \beta), \quad f''(u) \leq 0 \quad (\beta \leq u \leq 1)$$

for a number $\beta \in (0, 1)$. The condition (1.2) implies that constant solutions $u = 0, 1$ are asymptotically stable equilibria of the diffusion free equation of (1.1) while $u = a$ is an unstable one. The equation (1.1) satisfying (1.2) and (1.3) is called a bistable reaction-diffusion equation. It is known that (1.1) with (1.2)-(1.3) allows a planar traveling wave, that is, there is a solution $u = \phi(y - ct)$ satisfying

$$\begin{cases} \phi_{zz} + c\phi_z + f(\phi) = 0, & \phi(z) > 0 \quad (z \in \mathbb{R}), \\ \lim_{z \rightarrow -\infty} \phi(z) = 1, & \lim_{z \rightarrow \infty} \phi(z) = 0. \end{cases}$$

where $z = y - ct$ (for instance, see [7]). Since $\phi'(z) < 0$ holds for the solution $\phi(z)$, it is called a monotone planar traveling wave (solution). Henceforth we assume

$$\int_0^1 f(u) du > 0,$$

which implies that the speed c is positive.

We note that the stationary equation

$$\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 0$$

allows a one hump solution $v(x)$ for $n = 1$ or for $n \geq 2$ a radially symmetric solution $v(x) = \Phi(r)$ ($r = |x|$)

$$(1.5) \quad \begin{cases} \Phi_{rr} + \frac{n-1}{r} \Phi_r + f(\Phi) = 0, & \Phi(r) > 0 \quad (0 < r < \infty), \\ \Phi_r(0) = 0, & \lim_{r \rightarrow \infty} \Phi(r) = 0 \end{cases}$$

with the monotone profile $\Phi_r(r) < 0$ ($0 < r < \infty$). These solutions $v(x)$ are called standing wave solutions and they are unique up to translation. Moreover, those are unstable as equilibrium solutions to

$$v_t = \Delta v + f(v) \quad (x \in \mathbb{R}^n)$$

for $n = 1$ and $n \geq 2$ respectively.

We survey the results for the traveling wave connecting $u = 1$ (at $y = -\infty$) with $u = v(x)$ (at $y = \infty$). We look for the solution with the form $u = U(x, y - ct)$, namely, we consider the equation

$$(1.6) \quad \begin{cases} \Delta U + U_{zz} + cU_z + f(U) = 0 & ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \rightarrow -\infty} U(x, z) = 1, & \lim_{z \rightarrow \infty} U(x, z) = v(x). \end{cases}$$

Similarly, we can also consider the solution of

$$\begin{cases} \Delta U + U_{zz} + cU_z + f(U) = 0 & ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \rightarrow -\infty} U(x, z) = 0, & \lim_{z \rightarrow \infty} U(x, z) = v(x). \end{cases}$$

The same argument can apply to the latter case, so we omit the statement for this case, though the profile of $U(x, z)$ turns to be monotone increasing in z while decreasing in (1.6).

The problem of (1.6) reminds us of the monostable case as in [15] and [1, 2]. Therefore, we can expect that there is a family of traveling waves with the speed c in a semi-infinite interval. We show that this is certainly true in §3.

§ 2. Stationary problem in $\mathbb{R}^n$

Assume (1.2) and (1.3) and consider

$$(2.1) \quad \Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 0.$$

For $n = 1$, the equation

$$(2.2) \quad v_{xx} + f(v) = 0$$

is converted to a Hamilton system and we can easily obtain a unique solution of (2.1) up to translation by a homoclinic orbit of the Hamiltonian system. Moreover, the linearized eigenvalue problem for such a solution $v(x)$

$$\varphi_{xx} + f'(v(x))\varphi = \mu\varphi, \quad \lim_{|x| \rightarrow \infty} \varphi(x) = 0$$

tells that the first eigenvalue is simple and positive while the second one is zero with the corresponding eigenfunction $\varphi = v_x(x)$.

For $n \geq 2$ we further assume (1.4). Given a positive R , we first consider the following problem in the ball with the radius R :

$$(2.3) \quad \Delta v + f(v) = 0 \quad (|x| < R), \quad v = 0 \quad (|x| = R)$$

By the famous result [8] all positive solutions to (2.3) is radially symmetric, so put $v = v(r)$, $r = |x|$, and they satisfy

$$(2.4) \quad v'' + \frac{n-1}{r}v' + f(v) = 0 \quad (0 < r < R), \quad v'(0) = v(R) = 0.$$

Then the solution satisfies $v'(r) < 0$ ($0 < r < R$). In fact, using the shooting method, we can classify the positive solutions to (2.4). In addition, the local bifurcation theorem applied to this case tells the following result:

Proposition 2.1. ([3], [19]) *There is a number $R^* > 0$ such that (2.4) with (1.2)~(1.4) has no positive solution for $R < R^*$, exactly one positive solution for $R = R^*$ and for $R > R^*$ exactly two positive solutions, denoted by $v = v_R^+, v_R^-$, satisfying $0 < v_R^-(r) < v_R^+(r) < 1$.*

We note that for the linearized eigenvalue problem of a positive solution v_R to (2.4), defined by

$$(2.5) \quad \mathcal{L}_R[w] := \Delta w + f'(v_R)w = \mu w \quad (|x| < R), \quad w = 0 \quad (|x| = R),$$

the eigenfunction w corresponding to a positive eigenvalue $\mu (> 0)$ is radially symmetric. This implies that the Morse index for the solution v_R in (2.3) coincides with that in (2.4). Consequently, we have the next result.

Proposition 2.2. ([3], [19]) *Assume $R > R^*$. Then the spectrum of the linearized operator $\mathcal{L}_R$ for $v = v_R^+$ consists of negative eigenvalues while for $v = v_R^-$ it consists of one positive eigenvalue and negative eigenvalues.*

Now we consider the problem in the whole space.

$$(2.6) \quad \Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 0.$$

By the argument in [9] we see that a positive solution to (2.6) must be radially symmetric. In fact, we have more properties for the solution to (2.6).

Theorem 2.3. ([3], [19]) *Assume (1.2)~(1.4). Then there is a positive solution to (2.6) which is radially symmetric and unique up to translation. For the solution denoted by $\Phi(|x|)$, consider the spectrum $\sigma(\mathcal{L})$ of*

$$(2.7) \quad \mathcal{L} := \Delta + f'(\Phi) : H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).$$

Then the following properties hold:

- (i) $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L}) \cup \sigma_e(\mathcal{L})$, where $\sigma_p(\mathcal{L})$ is the point spectrum and $\sigma_e(\mathcal{L})$ is the essential spectrum.
- (ii) $\sigma_e(\mathcal{L}) = (-\infty, f'(0)]$, $\sigma_p(\mathcal{L}) \subset (f'(0), \infty)$.
- (iii) For $\mu \in \sigma_p(\mathcal{L})$ the corresponding eigenfunction ψ satisfies

$$|\psi(x)| \leq C_\varepsilon e^{-\sqrt{(-f'(0)+\mu+\varepsilon)/2}|x|}$$

for any $\varepsilon > 0$ and a constant $C_\varepsilon > 0$ depending on ε .

- (iv) For $\mu \in (0, \infty) \cap \sigma_p(\mathcal{L})$ the corresponding eigenfunction is radially symmetric.
- (v) The principal eigenvalue $\lambda_1 > 0$ is simple and the corresponding eigenfunction can be chosen positive.
- (vi) The second eigenvalue is $\lambda_2 = 0$ and the eigenspace for $\lambda_2 = 0$ is spanned by

$$\frac{\partial \Phi}{\partial x_i} \quad (i = 1, 2, \dots, n).$$

We remark that as $R \rightarrow \infty$, the solution v_R^+ to (2.4) converges to $v = 1$ uniformly in any compact set while v_R^- converges to the solution Φ of (1.5).

§ 3. Nonplanar traveling waves

By [6], [10], [11, 12, 13] and [17, 18] the existence and the stability of nonplanar traveling waves, called V -shaped waves and conical waves, have been extensively studied. Moreover, the existence of pyramidal shaped fronts is also given in [20, 21]. Here we introduce another type of traveling waves studied in [14] and [16].

For $n = 1$, as mentioned in the previous section, the stationary problem (2.2) is converted to a Hamiltonian system, thus we see that it has a family of periodic solutions $\{v_p(x; T)\}_{\omega < T < \infty}$ ($\omega := 2\pi\sqrt{f'(a)}$), where T is the period of the solution, and a homoclinic solution $v_h(x)$. Then $v_p(\cdot; T) \rightarrow a$ as $T \rightarrow \omega + 0$, while as $T \rightarrow \infty$, $v_p(\cdot; T) \rightarrow v_h(\cdot)$ in any compact set.

The authors of [14] prove that (1.1) with (1.2) and (1.3) allows a heteroclinic connection between $u = 1$ ($y \rightarrow -\infty$) and $u = v_p(\cdot; T)$ ($y \rightarrow \infty$). More precisely, for each $T \in (\omega, \infty)$, there is a positive number $c_{min}(T)$ such that the equation allows a traveling wave solution given by solving (1.6) with $v = v_p(\cdot; T)$ if and only if $c \in [c_{min}(T), \infty)$. Moreover, the solution is unique up to translation.

They also prove that $c_{min}(T) \rightarrow c^*$ ($T \rightarrow \omega + 0$) for a number $c_{min} > 0$ and that for each $c > c^*$, the solution denoted by $U(x, z; c, T)$, converges to the planer traveling wave of (1.1), say $U^{1a}(z; c)$, connecting $u = 1$ ($z \rightarrow -\infty$) with $u = a$ ($z \rightarrow \infty$).

On the other hand, there is $c_{min} > 0$ such that $c_{min}(T) \rightarrow c_{min}$ ($T \rightarrow \infty$) and for each $c > c_{min}$ the solution $U(x, z; c, T)$ converges to $U(\cdot, \cdot; c)$ of (1.6) with $v = v_h$ as $T \rightarrow \infty$ with an appropriate shift.

In sequel they succeed to prove the existence of the traveling wave together with the characterization of the minimum speed for the solution to (1.6).

Theorem 3.1. ([14]) Assume (1.2) and (1.3). For $n = 1$ there is a solution $U = U(x, z; c)$ to (1.6) satisfying $U(-x, z; c) = U(x, z; c)$, $U_z(\cdot, z; c) < 0$ if and only if $c \in [c_{min}, \infty)$. This solution is unique up to z translation.

As for the higher dimensional case we have the next result.

Theorem 3.2. ([16]) *Let Φ be the positive solution to (1.5) which is unique. Then there is a solution to (1.6) satisfying $U_z(x, z) \leq 0$ if $c \geq 2\sqrt{\kappa}$, where*

$$\kappa := \max \left\{ \max_{0 \leq u \leq 1} |f'(u)|, \lambda_1 \right\}$$

(λ_1 is the first eigenvalue of $\mathcal{L}$ defined in (2.7)).

The proof is done by the comparison method, that is, constructing appropriate subsolution and supersolution¹. Unfortunately, this result is not so strong as Theorem 3.1 since the existence of the minimum speed is not discussed. Namely, $c \geq 2\sqrt{\kappa}$ is a technical condition to ensure the existence of the traveling wave. The uniqueness of the solution to (1.6) up to translation is neither discussed.

To apply the similar argument in [14] for $n \geq 2$, it is natural to consider the equation in the cylindrical domain given by

$$(3.1) \quad \begin{cases} \Delta W + W_{zz} + cW_z + f(w) = 0 & ((x, z) \in \{|x| < R\} \times \mathbb{R}), \\ \lim_{z \rightarrow -\infty} W(x, z) = v_R^+(x), & \lim_{z \rightarrow \infty} W(x, z) = v_R^-(x), \end{cases}$$

where $v_R^+(r)$ and $v_R^-(r)$ are the stable solution and the unstable solution of (2.4) with (1.2)~(1.4) respectively. As mentioned in the previous section, those solutions are radially symmetric and ordered as $0 < v_R^-(r) < v_R^+(r)$. Moreover,

$$\lim_{R \rightarrow \infty} v_R^-(r) = \Phi(r), \quad \lim_{R \rightarrow \infty} v_R^+(r) = 1 \quad (\text{in any bounded interval } [0, K])$$

hold. By the results of the previous section and applying the results found in [4] or [22, 23], we obtain the next result.

Lemma 3.3. *Assume (1.2)~(1.4). Then there is a number $c_m^R > 0$ such that for each $c \in [c_m^R, \infty)$ the equation (3.1) allows a unique solution $W(r, z)$ ($r = |x|$) satisfying $W_z < 0$ up to translation in z .*

Thus, the solution $u_R(x, y, t) := W(|x|, y - ct)$ gives the traveling wave connecting $u_R = v_R^+$ ($y = -\infty$) and $u_R = v_R^-$ ($y \rightarrow \infty$) in the cylindrical domain.

In conclusion we need a further discussion to prove the convergence of the minimum speed c_m^R as $R \rightarrow \infty$, which is a future work.

¹The definition of κ in [16] should be corrected as in the above theorem.

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