

The proximal point algorithm of a resolvent for equilibrium
problems in geodesic spaces with negative curvature
曲率上限が負の測地距離空間における均衡問題の
リゾルベントと近接点法

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Abstract

In this paper, we consider properties of a resolvent of equilibrium problems. We prove a Δ -convergence theorem with the proximal point algorithm using a resolvent of equilibrium problems in a $CAT(-1)$ space having the convex hull finite property.

1 Introduction

Let K a nonempty set and $f: K \times K \rightarrow \mathbb{R}$. An equilibrium problem is defined as to find $z_0 \in K$ such that $f(z_0, y) \geq 0$ for $y \in K$. Equilibrium problems were first studied by Blum and Oettli [1]. Equilibrium problems include optimization problems, saddle point problems and fixed point problems, etc. In 2005, Combettes and Hirstoaga introduced the resolvent of equilibrium problems in Hilbert spaces [3].

Theorem 1.1 (Combettes and Hirstoaga [3]). *Let H be a Hilbert space and K a nonempty closed convex subset of H . Let $f: K \times K \rightarrow \mathbb{R}$ and S_f the set of solutions to the equilibrium problem for f . Suppose the following conditions:*

- $f(y, y) = 0$ for all $y \in K$;
- $f(y, z) + f(z, y) \leq 0$ for all $y, z \in K$;
- $f(y, \cdot): K \rightarrow \mathbb{R}$ is lower semicontinuous and convex for every $y \in H$;

- $f(\cdot, z): K \rightarrow \mathbb{R}$ is upper hemicontinuous for every $z \in K$.

Then the resolvent operator J_f defined by

$$J_f x = \left\{ z \in K \mid \inf_{y \in K} (f(z, y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

has the following properties:

- (i) $D(J_f) = X$;
- (ii) J_f is single-valued and firmly nonexpansive;
- (iii) $F(J_f) = S_f$;
- (iv) S_f is closed and convex.

In 2018, Kimura and Kishi [7] introduced a resolvent of equilibrium problems in a complete CAT(0) space having the convex hull finite property. In 2021, Kimura [6] introduced a resolvent of equilibrium problems in an admissible complete CAT(1) space having the convex hull finite property.

In this paper, we propose fundamental properties of a resolvent of equilibrium problems and prove a Δ -convergence theorem with the proximal point algorithm in a complete CAT(-1) space having the convex hull finite property.

2 Preliminaries

Let (X, d) a metric space, and T a mapping of X into itself. The set of all fixed points of T is denoted by $\mathcal{F}(T)$. Let $\{x_n\}$ be a bounded sequence of X . The set of $AC(\{x_n\})$ of all *asymptotic centers* of $\{x_n\} \subset X$ is defined by

$$AC(\{x_n\}) = \left\{ x_0 \in X \mid \limsup_{n \rightarrow \infty} d(x_n, x_0) = \inf_{x \in X} \limsup_{n \rightarrow \infty} d(x_n, x) \right\}.$$

A sequence $\{x_n\} \subset X$ is said to be Δ -convergent to $x_0 \in X$ if $AC(\{x_{n_i}\}) = \{x_0\}$ for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$. It is denoted by $x_n \xrightarrow{\Delta} x_0$. Let f be a function of X into $\mathbb{R}$. Then, $\text{Argmin}_{x \in X} f(x)$ is the set of all minimizers of f . Let T be a mapping of X into itself. Then, a mapping T is *hyperbolically nonspreading* if for $x, y \in X$, the inequality

$$2 \cosh d(Tx, Ty) \leq \cosh d(Tx, y) + \cosh d(x, Ty)$$

holds. A mapping T is *quasinonexpansive* if $\mathcal{F}(T)$ is nonempty and the inequality $d(Tx, z) \leq d(x, z)$ holds for $x \in X$ and $z \in \mathcal{F}(T)$. We know that if T is hyperbolically nonspreading and $\mathcal{F}(T)$ is nonempty, T is quasinonexpansive. In fact, for $x \in X$ and $z \in \mathcal{F}(T)$, by hyperbolical nonspreadingness of T , we get

$$2 \cosh d(Tx, z) \leq \cosh d(Tx, z) + \cosh d(x, z)$$

and hence T is quasinonexpansive.

Let $x, y \in X$ and γ_{xy} a mapping of $[0, d(x, y)]$ into X . A mapping γ_{xy} is called a *geodesic with endpoints x and y* if $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$ and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. X is called a *geodesic space* if for all $x, y \in X$, there exists geodesic with endpoints x and y . In what follows, we assume that for $x, y \in X$, X has a unique geodesic with endpoints x and y . The image of geodesic with endpoints x and y is denoted by $\text{Im } \gamma_{xy}$. For $x, y \in X$ and $t \in [0, 1]$, there exists $z \in \text{Im } \gamma_{xy}$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$, which is denoted by $z = tx \oplus (1-t)y$.

Let X be a geodesic space, and $\mathbb{H}^2$ the 2-dimensional hyperbolic space. A *geodesic triangle* $\triangle(x, y, z)$ with vertices $x, y, z \in X$ is defined by $\text{Im } \gamma_{xy} \cup \text{Im } \gamma_{yz} \cup \text{Im } \gamma_{zx}$. Further, a *comparison triangle* $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ to $\triangle(x, y, z)$ with vertices $\bar{x}, \bar{y}, \bar{z} \in \mathbb{H}^2$ is defined by $\text{Im } \gamma_{\bar{x}\bar{y}} \cup \text{Im } \gamma_{\bar{y}\bar{z}} \cup \text{Im } \gamma_{\bar{z}\bar{x}}$ with $d(x, y) = d_{\mathbb{H}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{H}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{H}^2}(\bar{z}, \bar{x})$, where $d_{\mathbb{H}^2}(\cdot, \cdot)$ is the hyperbolic metric on $\mathbb{H}^2$. A point $\bar{p} \in \text{Im } \gamma_{\bar{x}\bar{y}}$ is called a *comparison point* for $p \in \text{Im } \gamma_{xy}$ if $d(x, p) = d_{\mathbb{H}^2}(\bar{x}, \bar{p})$. X is a $\text{CAT}(-1)$ space if for $p, q \in \triangle(x, y, z) \subset X$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{H}^2$, the inequality $d(p, q) \leq d_{\mathbb{H}^2}(\bar{p}, \bar{q})$ holds for all geodesic triangles in X . In general, a $\text{CAT}(-1)$ space is a $\text{CAT}(0)$ space [2]. In $\text{CAT}(-1)$ spaces, the inequality

$$\begin{aligned} & \cosh d(tx \oplus (1-t)y, z) \sinh d(x, y) \\ & \leq \cosh d(x, z) \sinh td(x, y) + \cosh d(y, z) \sinh(1-t)d(x, y) \end{aligned}$$

always holds for $x, y, z \in X$ and $t \in [0, 1]$.

The following lemmas are important properties of a $\text{CAT}(0)$ space.

Lemma 2.1 (Kirk and Panyanak [9]). *Let X be a complete $\text{CAT}(0)$ space. Then every bounded sequence has a subsequence which is Δ -convergent to $x_0 \in X$.*

Lemma 2.2 (Dhompongsa, Kirk and Sims [4]). *Let X be a complete $\text{CAT}(0)$ space and $\{x_n\}$ a bounded sequence of X . Then the asymptotic center of $\{x_n\}$ consists of one point.*

Let X be a geodesic space and f a function of X into $\mathbb{R}$. A function f is said to be *lower semicontinuous* if the inequality

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$$

holds, wherever $\{x_n\} \subset X$ converges to $x \in X$. If f is continuous, then it is lower semicontinuous. A function f is said to be *convex* if

$$f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

holds for all $x, y \in X$ and $\alpha \in]0, 1[$. A function f is said to be *upper hemicontinuous* if the inequality

$$f(x) \geq \limsup_{t \rightarrow 0^+} f((1-t)x \oplus ty)$$

holds for all $x, y \in X$.

We will consider the following conditions for a function used by an equilibrium problems.

Condition 2.1. Let X be a geodesic space and K a nonempty closed convex subset of X . We suppose that a bifunction $f: K \times K \rightarrow \mathbb{R}$ satisfies the following conditions:

- $f(x, x) = 0$ for all $x \in K$;
- $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
- for every $x \in K$, $f(x, \cdot): K \rightarrow \mathbb{R}$ is lower semicontinuous and convex;
- for every $y \in K$, $f(\cdot, y): K \rightarrow \mathbb{R}$ is upper hemicontinuous.

The following theorem is important to show a Δ -convergence theorem with the proximal point algorithm.

Theorem 2.1 (Kajimura and Kimura [5]). *Let X be a complete $\text{CAT}(-1)$ space, $\{z_n\}$ a bounded sequence in X , $\{\beta_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \beta_n = \infty$ and*

$$g(y) = \limsup_{n \rightarrow \infty} \frac{1}{\sum_{l=1}^n \beta_l} \sum_{k=1}^n \beta_k \cosh d(y, z_k)$$

for $y \in X$. Then, $\text{Argmin}_X g$ consists of one point.

The set of solutions to the equilibrium problem for f is denoted by $\text{Equil } f$, that is,

$$\text{Equil } f = \left\{ z \in K \mid \inf_{y \in K} f(z, y) \geq 0 \right\}.$$

Let X be a $\text{CAT}(-1)$ space and E a nonempty subset of X . Then a *convex hull* of E is defined by

$$\text{co } E = \bigcup_{n=0}^{\infty} X_n,$$

where $X_0 = E$ and $X_n = \{tu_{n-1} \oplus (1-t)v_{n-1} \mid u_{n-1}, v_{n-1} \in X_{n-1}, t \in [0, 1]\}$. X has the *convex hull finite property* if every continuous mapping T of $\text{clco } E$ into itself has a fixed point for all finite subsets E of X , where $\text{clco } E$ is the closure of $\text{co } E$; see [11].

In the following theorem shows the properties of a resolvent of equilibrium problems in a $\text{CAT}(-1)$ space having the convex hull finite property.

Theorem 2.2 (Kimura and Ogihara [8]). *Let X be a complete $\text{CAT}(-1)$ space having the convex hull finite property and K a nonempty closed convex subset of X . Suppose that $f: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2.1. Define a set-valued mapping $L_f: X \rightarrow 2^K$ by*

$$L_f x = \left\{ z \in K \mid \inf_{y \in K} (f(z, y) + \cosh d(x, y) - \cosh d(x, z)) \geq 0 \right\}$$

for all $x \in X$. Put $C_z = \cosh d(z, L_f z)$ for $z \in X$. Then the following hold:

- (i) $D(L_f) = X$;

(ii) L_f is single-valued and the inequality

$$(C_x + C_y) \cosh d(L_fx, L_fy) \leq \cosh d(L_fx, y) + \cosh d(x, L_fy)$$

holds for $x, y \in X$, and thus L_f is hyperbolically nonspreading;

(iii) $\text{Equil } f = \mathcal{F}(L_f)$, and thus it is closed and convex.

3 Fundamental properties of resolvents

In this section, we prove the lemmas which is necessary to prove a Δ -convergence theorem in a $\text{CAT}(-1)$ space having the convex hull finite property.

Lemma 3.1. *Let X be a complete $\text{CAT}(-1)$ space having the convex hull finite property and K a nonempty closed convex subset of X . Suppose that $f: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2.1. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the inequality*

$$0 \leq f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)} (\cosh d(x, w) - \cosh d(x, L_{\lambda f}x) \cosh d(w, L_{\lambda f}x))$$

holds for $x \in X$ and $w \in K$ with $w \neq L_{\lambda f}x$.

Proof. Let $x \in X$ and $w \in K$ with $w \neq L_{\lambda f}x$. Put $\tau_t = tw \oplus (1-t)L_{\lambda f}x \in K$ for $t \in]0, 1[$. Then, we get

$$\begin{aligned} 0 &\leq \lambda f(L_{\lambda f}x, \tau_t) + \cosh d(x, \tau_t) - \cosh d(x, L_{\lambda f}x) \\ &\leq \lambda t f(L_{\lambda f}x, w) + \cosh d(x, \tau_t) - \cosh d(x, L_{\lambda f}x) \\ &\leq \lambda t f(L_{\lambda f}x, w) + \frac{L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)}{\sinh d(L_{\lambda f}x, w)} \end{aligned}$$

where

$$L(t) = \cosh d(x, w) \sinh td(L_{\lambda f}x, w) + \cosh d(x, L_{\lambda f}x) \sinh(1-t)d(L_{\lambda f}x, w).$$

Dividing by λt and letting $t \searrow 0$, we obtain

$$\begin{aligned} 0 &\leq f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t \searrow 0} \frac{L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)}{t} \\ &= f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t \searrow 0} \frac{d}{dt} (L(t) - \cosh d(x, L_{\lambda f}x) \sinh d(L_{\lambda f}x, w)) \\ &= f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)} (\cosh d(x, w) - \cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, w)) \end{aligned}$$

and hence we get the desired result. $\square$

Corollary 3.1. *Let X be a complete CAT(-1) space having the convex hull finite property and K a nonempty closed convex subset of X . Suppose that $f: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2.1. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequalities hold:*

$$(\mu C_{\lambda, x} + \lambda C_{\mu, y}) \cosh d(L_{\lambda f} x, L_{\mu f} y) \leq \mu \cosh d(x, L_{\mu f} y) + \lambda \cosh d(L_{\lambda f} x, y)$$

and

$$(\lambda + \mu) \cosh d(L_{\lambda f} x, L_{\mu f} y) \leq \mu \cosh d(x, L_{\mu f} y) + \lambda \cosh d(L_{\lambda f} x, y)$$

for all $x, y \in X$ and $\lambda, \mu > 0$, where $C_{\eta, z} = \cosh d(z, L_{\eta f} z)$ for $z \in X$ and $\eta > 0$.

Proof. Let $x, y \in X$ and $\lambda, \mu > 0$ with $D = d(L_{\lambda f} x, L_{\mu f} y) > 0$ and put $C_{\eta, z} = \cosh d(L_{\eta f} z, z)$ for $z \in X$ and $\eta > 0$. By Lemma 3.1, we get

$$0 \leq f(L_{\lambda f} x, L_{\mu f} y) + \frac{D}{\lambda \sinh D} (\cosh d(x, L_{\mu f} y) - C_{\lambda, x} \cosh D).$$

Similarly, it holds that

$$0 \leq f(L_{\mu f} y, L_{\lambda f} x) + \frac{D}{\mu \sinh D} (\cosh d(L_{\lambda f} x, y) - C_{\mu, y} \cosh D).$$

From Condition 2.1, adding these inequalities, we get

$$\begin{aligned} 0 &\leq f(L_{\lambda f} x, L_{\mu f} y) + f(L_{\mu f} y, L_{\lambda f} x) + \frac{D}{\lambda \sinh D} (\cosh d(x, L_{\mu f} y) - C_{\lambda, x} \cosh D) \\ &\quad + \frac{D}{\mu \sinh D} (\cosh d(L_{\lambda f} x, y) - C_{\mu, y} \cosh D) \\ &\leq \frac{D}{\sinh D} \left(\frac{\cosh d(x, L_{\mu f} y) - C_{\lambda, x} \cosh D}{\lambda} + \frac{\cosh d(L_{\lambda f} x, y) - C_{\mu, y} \cosh D}{\mu} \right) \end{aligned}$$

Since $t/(\sinh t) > 0$ for $t > 0$, we get

$$(\mu C_{\lambda, x} + \lambda C_{\mu, y}) \cosh D \leq \mu \cosh d(x, L_{\mu f} y) + \lambda \cosh d(L_{\lambda f} x, y).$$

Since $\cosh t \geq 1$ for $t \geq 0$, we get

$$(\lambda + \mu) \cosh D \leq \mu \cosh d(x, L_{\mu f} y) + \lambda \cosh d(L_{\lambda f} x, y).$$

If $D = 0$, the inequalities obviously hold. It completes the proof. $\square$

Corollary 3.2. *Let X be a complete CAT(-1) space having the convex hull finite property and K a nonempty closed convex subset of X . Suppose that $f: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2.1 and that $\text{Equil } f$ is nonempty. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequality holds:*

$$\cosh d(x, L_{\lambda f} x) \cosh d(L_{\lambda f} x, z) \leq \cosh d(x, z)$$

for all $x \in X$ and $z \in \text{Equil } f$.

Proof. Let $x \in X$, $z \in \text{Equil } f$ and $\lambda > 0$. By Corollary 3.1, we get

$$(\cosh d(x, L_{\lambda f} x) + \lambda \cosh d(z, L_f z)) \cosh d(L_{\lambda f} x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f} x, z)$$

and hence

$$(\cosh d(x, L_{\lambda f} x) + \lambda) \cosh d(L_{\lambda f} x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f} x, z).$$

Therefore, we have

$$\cosh d(x, L_{\lambda f} x) \cosh d(L_{\lambda f} x, z) \leq \cosh d(x, z)$$

and get the desired result. $\square$

Lemma 3.2. *Let X be a complete $\text{CAT}(-1)$ space having the convex hull finite property and K a nonempty closed convex subset of X . Suppose that $f: K \times K \rightarrow \mathbb{R}$ satisfies Condition 2.1. Let $\{\lambda_n\} \subset]0, \infty[$ such that $\limsup_{n \rightarrow \infty} \lambda_n > 0$, $L_{\lambda_n f}$ a resolvent of $\lambda_n f$, and $\{x_n\}$ a bounded sequence of X such that $x_n \xrightarrow{\Delta} x_0 \in X$ and $\lim_{n \rightarrow \infty} d(x_n, L_{\lambda_n f} x_n) = 0$. Then $x_0 \in \text{Equil } f$.*

Proof. Put $\lambda_0 = \limsup_{n \rightarrow \infty} \lambda_n$. By Corollary 3.1, we get

$$\cosh d(L_{\lambda_n f} x_n, L_f x_0) \leq \frac{\lambda_n}{1 + \lambda_n} \cosh d(L_{\lambda_n f} x_n, x_0) + \frac{1}{1 + \lambda_n} \cosh d(x_n, L_f x_0).$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ arbitrarily. For $y \in K$, then, we get

$$d(L_{\lambda_{n_j} f} x_{n_j}, y) \leq d(L_{\lambda_{n_j} f} x_{n_j}, x_{n_j}) + d(x_{n_j}, y) \leq 2d(L_{\lambda_{n_j} f} x_{n_j}, x_{n_j}) + d(L_{\lambda_{n_j} f} x_{n_j}, y).$$

Since $d(x_{n_j}, L_{\lambda_{n_j} f} x_{n_j}) \rightarrow 0$, letting $j \rightarrow \infty$, we get

$$\limsup_{j \rightarrow \infty} d(L_{\lambda_{n_j} f} x_{n_j}, y) = \limsup_{j \rightarrow \infty} d(x_{n_j}, y).$$

Suppose $\lambda_0 = \infty$. Then we take a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lim_{i \rightarrow \infty} \lambda_{n_i} = \infty$. It implies that

$$\begin{aligned} \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, L_f x_0)) &= \limsup_{i \rightarrow \infty} (\cosh d(L_{\lambda_{n_i} f} x_{n_i}, L_f x_0)) \\ &\leq \limsup_{i \rightarrow \infty} (\cosh d(L_{\lambda_{n_i} f} x_{n_i}, x_0)) \\ &= \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, x_0)). \end{aligned}$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \text{Equil } f$. We next suppose $\lambda_0 < \infty$. Then we get

$$\begin{aligned}
& \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, L_f x_0)) \\
&= \limsup_{i \rightarrow \infty} (\cosh d(L_{\lambda_{n_i} f} x_{n_i}, L_f x_0)) \\
&\leq \frac{\lambda_0}{1 + \lambda_0} \limsup_{i \rightarrow \infty} (\cosh d(L_{\lambda_{n_i} f} x_{n_i}, x_0)) + \frac{1}{1 + \lambda_0} \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, L_f x_0)) \\
&= \frac{\lambda_0}{1 + \lambda_0} \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, x_0)) + \frac{1}{1 + \lambda_0} \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, L_f x_0))
\end{aligned}$$

and hence

$$\limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, L_f x_0)) \leq \limsup_{i \rightarrow \infty} (\cosh d(x_{n_i}, x_0)).$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \text{Equil } f$. Consequently, we complete the proof. $\square$

4 A Δ -convergence theorem

In this section, we prove a Δ -convergence theorem with the proximal point algorithm in a $\text{CAT}(-1)$ space having the convex hull finite property.

Theorem 4.1. *Let X be a complete $\text{CAT}(-1)$ space having the convex hull finite property, K a nonempty closed convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ satisfying Condition 2.1 and $\{\lambda_n\} \subset]0, \infty[$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. For given $x_1 \in X$, define $\{x_n\}$ by*

$$x_{n+1} = L_{\lambda_n f} x_n = \left\{ z \in K \mid \inf_{y \in K} (f(z, y) + \cosh d(x_n, y) - \cosh d(x_n, z)) \geq 0 \right\}$$

for all $n \in \mathbb{N}$. Then, the following hold:

- (i) $\text{Equil } f$ is nonempty if and only if $\{x_n\}$ is bounded;
- (ii) if $\text{Equil } f$ is nonempty and $\liminf_{n \rightarrow \infty} \lambda_n > 0$, $\{x_n\}$ is Δ -convergent to an element of $\text{Equil } f$.

Proof. (i) We first suppose that $\text{Equil } f$ is nonempty and show that $\{x_n\}$ is bounded. Let $u \in \text{Equil } f$. Since $L_{\lambda_n f}$ is quasinonexpansive, we get

$$d(x_{n+1}, u) = d(L_{\lambda_n f} x_n, u) \leq d(x_n, u)$$

and hence $\{d(x_n, u)\}$ is nonincreasing and $\{x_n\}$ is bounded for $n \in \mathbb{N}$. We next suppose $\{x_n\}$ is bounded and show that $\text{Equil } f$ is nonempty. For $k \in \mathbb{N}$ with $k \leq n$, by Corollary 3.1 we get

$$(1 + \lambda_k) \cosh d(L_{\lambda_k f} x_k, L_f y) \leq \lambda_k \cosh d(L_{\lambda_k f} x_k, y) + \cosh d(x_k, L_f y)$$

and hence

$$\lambda_k \cosh d(x_{k+1}, L_f y) \leq \lambda_k \cosh d(x_{k+1}, y) + (\cosh d(x_k, L_f y) - \cosh d(x_{k+1}, L_f y))$$

for all $y \in X$. Adding both sides of the inequality above from $k = 1$ to $k = n$ and dividing both sides by $\sum_{l=1}^n \lambda_l$, we get

$$\begin{aligned} & \frac{1}{\sum_{l=1}^n \lambda_l} \sum_{k=1}^n \lambda_k \cosh d(x_{k+1}, L_f y) \\ & \leq \frac{1}{\sum_{l=1}^n \lambda_l} \sum_{k=1}^n \lambda_k \cosh d(x_{k+1}, y) + \frac{1}{\sum_{l=1}^n \lambda_l} (\cosh d(x_1, L_f y) - \cosh d(x_{n+1}, L_f y)) \\ & \leq \frac{1}{\sum_{l=1}^n \lambda_l} \sum_{k=1}^n \lambda_k \cosh d(x_{k+1}, p) + \frac{1}{\sum_{l=1}^n \lambda_l} \cosh d(x_1, p). \end{aligned}$$

By Theorem 2.1, we know that $\text{Argmin}_X g$ consists of one point, where

$$g(z) = \limsup_{n \rightarrow \infty} \frac{1}{\sum_{l=1}^n \lambda_l} \sum_{k=1}^n \lambda_k \cosh d(x_{k+1}, z)$$

for all $z \in X$. Let $p \in \text{Argmin}_X g$. Since $\sum_{l=1}^{\infty} \lambda_l = \infty$, letting $n \rightarrow \infty$, we get

$$g(L_f p) \leq g(p) \leq g(L_f p)$$

and hence $p = L_f p$. This implies that $p \in \text{Equil } f$.

(ii) Suppose $\text{Equil } f$ is nonempty and $\liminf_{n \rightarrow \infty} \lambda_n > 0$. Let $p \in \text{Equil } f$. Since $L_{\lambda_n f}$ and $\text{Equil } f$ is nonempty, $L_{\lambda_n f}$ is quasicontractive. Then, we get

$$0 \leq d(x_{n+1}, p) = d(L_{\lambda_n f} x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n \rightarrow \infty} d(x_n, p)$. By Corollary 3.1, we have

$$1 \leq \cosh d(x_n, L_{\lambda_n f} x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)}.$$

Letting $n \rightarrow \infty$, we obtain $d(x_n, L_{\lambda_n f} x_n) \rightarrow 0$. Put $\text{AC}(\{x_n\}) = \{x_0\}$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Since $\{x_{n_i}\}$ is bounded and by Lemma 2.2, $\text{AC}(\{x_{n_i}\}) = \{y_0\}$. Further, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, $\text{AC}(\{x_{n_{i_j}}\}) = \{z_0\}$. By Lemma 3.2, we get $z_0 \in \text{Equil } f$. Then, we get

$$\limsup_{n \rightarrow \infty} d(x_n, z_0) = \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, z_0)$$

$$\begin{aligned}
&\leq \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, y_0) \\
&\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, y_0) \\
&\leq \limsup_{i \rightarrow \infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n \rightarrow \infty} d(x_n, x_0) \leq \limsup_{n \rightarrow \infty} d(x_n, z_0)
\end{aligned}$$

and hence $y_0 = x_0 = z_0 \in \text{Equil } f$. Therefore $x_n \xrightarrow{\Delta} x_0 \in \text{Equil } f$. Consequently, we get the desired result. $\square$

Corollary 4.1. *Let X be a complete CAT(−1) space having the convex hull finite property, K a nonempty closed convex subset of X , $f: K \times K \rightarrow \mathbb{R}$ satisfying Condition 2.1 and L_f a resolvent of f . Then the following hold:*

- (i) *Equil f is nonempty if and only if $\{L_f^n x\}$ is bounded for each $x \in X$;*
- (ii) *If Equil f is nonempty, $\{L_f^n x\}$ is Δ -convergent to an element of Equil f for each $x \in X$.*

Proof. Let $\{\lambda_n\} \subset]0, \infty[$ such that $\lambda_n = 1$ for $n \in \mathbb{N}$. Using Theorem 4.1, we get (i) and (ii). Consequently, we complete the proof. $\square$

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