

Dynamical Systems for the Frobenius-Perron Operator

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1 Introduction

The Frobenius-Perron and Koopman operators are useful for various mathematical fields. We consider the following transformation.

$$S(x, y) = (ax + by + \alpha, cx + dy + \beta) \pmod{1}, \quad (1.1)$$

where $0 \leq x, y < 1$, $a, b, c, d \in \mathbf{R}$ and $0 \leq \alpha, \beta < 1$. This transformation may display three levels of irregular behavior (ergodicity, mixing and exactness) depending on the coefficients a, b, c, d, α and β . We investigate the relation between the coefficients and the behavior using these operators. We first give a necessary and sufficient condition for S to be measure preserving [Theorem 4], because measure preserving is supposed in the definition of mixing and exactness. In the case of $a, b, c, d \in \mathbf{Z}$ and $\alpha = \beta = 0$ in (1.1), we show a necessary and sufficient condition for S to be mixing [Theorem 9]. In Theorem 10, we show S displays the following behaviors depending on $a, b, c, d \in \mathbf{Z}$ and $0 \leq \alpha, \beta < 1$ in (1.1):

- (i) S is mixing;
- (ii) S is ergodic, but not mixing;
- (iii) S is not ergodic.

2 The Frobenius-Perron and Koopman Operators

Definition (Markov operator). Let $(X, \mathcal{A}, \mu)$ be a measure space. Any linear operator $P : L^1 \rightarrow L^1$ satisfying

- (a) $Pf \geq 0$ for $f \geq 0, f \in L^1$;
- (b) $\|Pf\| = \|f\|$, for $f \geq 0, f \in L^1$

is called a **Markov operator**.

Definition (nonsingular). A measurable transformation $S : X \rightarrow X$ on a measure space $(X, \mathcal{A}, \mu)$ is **nonsingular** if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ such that $\mu(A) = 0$.

Definition . Let $(X, \mathcal{A}, \mu)$ be a measure space. If $S : X \rightarrow X$ is a nonsingular transformation, the unique operator $P : L^1 \rightarrow L^1$ defined by

$$\int_A Pf(x)\mu(dx) = \int_{S^{-1}(A)} f(x)\mu(dx) \quad \text{for } A \in \mathcal{A} \quad (2.1)$$

is called the **Frobenius-Perron operator** corresponding to S .

Definition . Let $(X, \mathcal{A}, \mu)$ be a measure space, $S : X \rightarrow X$ a nonsingular transformation, and $f \in L^\infty$. The operator $U : L^\infty \rightarrow L^\infty$ defined by

$$Uf(x) = f(S(x))$$

is called the **Koopman operator** with respect to S .

Definition (measure-preserving). Let $(X, \mathcal{A}, \mu)$ be a measure space and $S : X \rightarrow X$ a measurable transformation. Then S is said to be **measure preserving** if

$$\mu(S^{-1}(A)) = \mu(A) \quad \text{for all } A \in \mathcal{A}.$$

Definition (ergodic). Let $(X, \mathcal{A}, \mu)$ be a measure space and let a nonsingular transformation $S : X \rightarrow X$ be given. The S is called **ergodic** if every invariant set $A \in \mathcal{A}$ is such that either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Definition (mixing). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, and $S : X \rightarrow X$ a measure-preserving transformation. S is called **mixing** if

$$\lim_{n \rightarrow \infty} \mu(A \cap S^{-n}(B)) = \mu(A)\mu(B) \quad \text{for all } A, B \in \mathcal{A}.$$

Definition (exact). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space and $S : X \rightarrow X$ a measure-preserving transformation such that $S(A) \in \mathcal{A}$ for each $A \in \mathcal{A}$. If

$$\lim_{n \rightarrow \infty} \mu(S^n(A)) = 1 \quad \text{for every } A \in \mathcal{A}, \mu(A) > 0,$$

then S is called **exact**.

Remark 1. If S is exact, then S is mixing. If S is mixing, then S is ergodic.

The proof of ergodicity, mixing, or exactness using these definitions is difficult. So we will use the following theorem and proposition.

Theorem 1 ([1]). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, $S : X \rightarrow X$ a measure-preserving transformation, and P the Frobenius-Perron operator corresponding to S . Then

(a) S is ergodic if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle P^k f, g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \quad \text{for } f \in L^1, g \in L^\infty;$$

(b) S is mixing if and only if

$$\lim_{n \rightarrow \infty} \langle P^n f, g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \quad \text{for } f \in L^1, g \in L^\infty;$$

(c) S is exact if and only if

$$\lim_{n \rightarrow \infty} \|P^n f - \langle f, 1 \rangle\| = 0 \quad \text{for } f \in L^1.$$

Proposition 2 ([1]). Let $(X, \mathcal{A}, \mu)$ be a normalized measure space, $S : X \rightarrow X$ a measure-preserving transformation, and U the Koopman operator corresponding to S . Then

(a) S is ergodic if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle f, U^k g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \quad \text{for } f \in L^1, g \in L^\infty;$$

(b) S is mixing if and only if

$$\lim_{n \rightarrow \infty} \langle f, U^n g \rangle = \langle f, 1 \rangle \langle 1, g \rangle \quad \text{for } f \in L^1, g \in L^\infty.$$

3 The dynamics of $S^n(x, y)$

Consider first $\alpha = \beta = 0$ in (1.1), i.e.

$$S(x, y) = (ax + by, cx + dy) \pmod{1},$$

where $a, b, c, d \in \mathbf{R}$. Let $X = [0, 1) \times [0, 1)$ and $X^\circ = (0, 1) \times (0, 1)$ and O, P, Q and R be the points $(0, 0), (a, c), (a + b, c + d)$ and $(b, d) \in \mathbf{R}^2$, respectively.

Proposition 3. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1},$$

where $a, b, c, d \in \mathbf{R}$ and the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\det A = ad - bc = 1$$

and $|a + d| < 2$. If there exist $(x_0, y_0) \in X^\circ$ such that $S(x_0, y_0) = (x_0, y_0)$, then S is not ergodic.

Proof. We will show that there exists a nontrivial invariant set.

Let eigenvalues of A be $\mu \pm i\nu$. There exist $\theta \in [0, 2\pi]$ and $r, t \in \mathbf{R}$ satisfying

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix} \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

$S(x_0, y_0) = (x_0, y_0)$ means that there exists $m, n \in \mathbf{Z}$ such that $A(x_0, y_0) + (m, n) = (x_0, y_0)$. By putting $T(x_0, y_0) = A(x_0, y_0) + (m, n)$, we see that the set $\Gamma(x, y) = \{T^n(x, y) \mid n = 0, 1, \dots\}$ is on the ellipse with center (x_0, y_0) , since $ad - bc = 1$ and $|a + d| < 2$. If (x, y) is very near to (x_0, y_0) , $\Gamma(x_0, y_0) \subset X^\circ$ and $T^n(x, y) = S^n(x, y)$. So, if we take a sufficiently small set B such that $\mu(B) > 0$ and $(x_0, y_0) \in B$, then $\Gamma(B)$ is an invariant set under S , which implies S is not ergodic. $\square$

Example 1. Suppose $A = \begin{pmatrix} -\frac{13}{10} & -\frac{7}{10} \\ \frac{113}{70} & \frac{1}{10} \end{pmatrix}$ ($\det A = 1$).

Put $(x_0, y_0) = (\frac{9}{32}, \frac{113}{224})$ or $(x_0, y_0) = (\frac{25}{32}, \frac{65}{224})$. Then $S(x_0, y_0) = (x_0, y_0)$. Thus, S is not ergodic by Proposition 3.

Suppose $A = \begin{pmatrix} 1 & -\frac{1}{1000} \\ 1 & \frac{999}{1000} \end{pmatrix}$ ($\det A = 1$). There doesn't exist $(x_0, y_0) \in X^\circ$ such that

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad (m, n \in \mathbf{Z}).$$

Now let's be back to the definitions of mixing and exact. Since measure preserving is supposed in the definition of mixing and exactness (i.e. $\mu(S^{-1}(A)) = \mu(A)$ for $\forall A \in \mathcal{A}$), we first give a necessary and sufficient condition for S to be measure preserving.

Let $A(X) = \bigcup_{l=1}^M B_l$, where $B_l \subset [m_l, m_l + 1) \times [n_l, n_l + 1)$, $m_l, n_l \in \mathbf{Z}$. We define ϕ_l as

$$\phi_l(B_l) = \{(x - m_l, y - n_l) | (x, y) \in B_l\} \subset X.$$

Lemma 1. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1}$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$. The following statements are equivalent:

- (1) S is measure preserving;
- (2) The following statements hold:
 - (i) $|\det A| = n \in \mathbf{N}$;
 - (ii) There exist the sets K_l ($l = 1, 2, \dots, M$) and a partition of X $\{D_j\}_{j=1}^k$ such that $B_l = \bigcup_{j_l \in K_l} \phi_l^{-1}(D_{j_l})$;
 - (iii) The number of elements of the set $\{l \mid D_j^\circ \cap \phi_l(B_l) \neq \emptyset\}$ is equal to n .
- (3) $|\det A| = n \in \mathbf{N}$ and either (a) or (b) holds:
 - (a) $a, c \in \mathbf{Z}$ and there exist $(x_0, y_0) \in \mathbf{Z}^2$ on the line $\overline{RQ}$;
 - (b) $b, d \in \mathbf{Z}$ and there exist $(x_0, y_0) \in \mathbf{Z}^2$ on the line $\overline{PQ}$.

Proof. We show (1) implies (2). There exists a partition of X $\{D_j\}_{j=1}^k$ such that $D_j = \bigcap_{j_l=1}^{t_j} \phi_{j_l}(B_{j_l})$ ($1 \leq \forall j \leq k, \exists t_j \geq 1$), $\phi_l(B_l) = \bigcup_{l_i=1}^{h_l} D_{l_i}$ ($1 \leq \forall j \leq k$) and $\mu(D_j) > 0$ ($1 \leq j \leq k$), where μ is Lebesgue measure.

Then for any $j \in \{1, 2, \dots, k\}$ there exists $l \in \mathbf{N}$ such that $\mu(A^{-1}\phi_l^{-1}D_j) > 0$. Put $K_j = \{l \mid D_j^\circ \cap \phi_l(B_l) \neq \emptyset\}$ and k_j be the number of elements of K_j . Since $S^{-1}(D_j) = \bigcup_{l \in K_j} A^{-1}\phi_l^{-1}(D_j)$,

$$\begin{aligned} \mu(S^{-1}(D_j)) &= k_j \mu(A^{-1}\phi_l^{-1}D_j) \\ &= k_j |\det A|^{-1} \mu(D_j). \end{aligned}$$

We have $k_j = |\det A|$ for $1 \leq \forall j \leq k$ by $\mu(S^{-1}(D_j)) = \mu(D_j)$. Since

$$\begin{aligned} \sum_{j=1}^k |\det A| \mu(D_j) &= \sum_{l=1}^M \mu(\phi_l^{-1}(B_l)) \\ &= \mu(A(X)) = |\det A|, \end{aligned}$$

we have $\sum_{j=1}^k \mu(D_j) = 1$.

We show (2) implies (1). Let $G \in \mathcal{A}$. There exist $k_0 \in \mathbb{N}$, $\{G_i\}_{i=1}^{k_0}$ and $\{j_i\}_{i=1}^{k_0}$ ($j_i \in \{1, 2, \dots, k\}$) such that $G \cap D_{j_i} = G_i$ and $G = \bigcup_{i=1}^{k_0} G_i$ ($G_i^\circ \cap G_j^\circ = \emptyset$ $i \neq j$). There exist $\{i_m\}_{m=1}^n$ such that $G_i \subset \phi_{i_m}(B_{i_m})$. We have

$$\begin{aligned} \mu(S^{-1}(G)) &= \mu(S^{-1}(\bigcup_{i=1}^{k_0} G_i)) = \sum_{m=1}^n \sum_{i=1}^{k_0} \mu(A^{-1} \phi_{i_m}^{-1}(G_i)) \\ &= n \sum_{i=1}^{k_0} \mu(A^{-1} \phi_{i_1}^{-1}(G_i)) = \sum_{i=1}^{k_0} \mu(G_i) = \mu(G). \end{aligned}$$

We show (3) implies (2). Put $B'_l = B_l \bmod 1$. Since there exist $(t_i, s_i) \in \mathbb{Z}^2$ ($i = 1, 2$) such that the line $\{(x - t_i, y - s_i) | (x, y) \in A(X)\} \cap A(X)$ is parallel to either $y = \frac{c}{a}x$ or $y = \frac{d}{b}x$, there exists $l' \in \{1, \dots, M\}$ for any $l \in \{1, \dots, M\}$ such that the line $B'_l \cap B'_{l'}$ is parallel to either $y = \frac{c}{a}x$ or $y = \frac{d}{b}x$. Then there exists a partition $\{D_j\}$ which satisfies the condition (2).

We show (2) implies (3). Consider the case of $a, b, c, d > 0$, $\det A > 0$ and $d > c$. There exists $j \in \{1, \dots, M\}$ such that $(0, 0) \in B_j$. Then there exist l_1, l_2 and $l_3 \in \{1, \dots, k\}$ such that $(0, 0) \in D_{l_1} \cap D_{l_2} \cap D_{l_3}$, $D_{l_1}^\circ \cap D_{l_2}^\circ \cap D_{l_3}^\circ = \emptyset$, the line $D_{l_1} \cap D_{l_2}$ is parallel to $y = \frac{c}{a}x$ and the line $D_{l_1} \cap D_{l_3}$ is parallel to $y = \frac{d}{b}x$. The following statements hold:

- (I) There exists $j \in \{1, \dots, M\}$ and $(m_1, n_1) \in \mathbb{Z}^2$ such that $(m_1, n_1) \in \phi_j(D_{l_2})$;
- (II) There exists $i \in \{1, \dots, M\}$ and $(m_2, n_2) \in \mathbb{Z}^2$ such that $(m_2, n_2) \in \phi_i(D_{l_3})$.

Suppose $(m_1, n_1) \neq (b, d)$ and $(m_2, n_2) \neq (a, c)$. There exists $l_4 \in \{1, \dots, k\}$ such that $(1, 1) \in D_{l_4}$, $\partial D_{l_4} \cap X^\circ$ is parallel to $y = \frac{c}{a}x$ and there is $\phi_{j_0}^{-1}(\partial D_{l_4} \cap X^\circ)$ on $y = \frac{c}{a}x$ for $\exists j_0 \in \{1, 2, \dots, M\}$. There exists $(m_3, n_3) \in \mathbb{Z}^2$ such that $(m_3, n_3) \in \phi_{j_0}^{-1}(D_{l_4})$ and $(m_3, n_3) \neq (a, c)$. The parallelogram which has the vertices $(0, 0), (m_3, n_3), (b + m_3, d + n_3)$ and (b, d) satisfies the condition of (3). Put the parallelogram be B' . Suppose $B'' = \{(x - m_3, y - n_3) | (x, y) \in A(X) \setminus B'\}$. If we repeat a similar procedure for B'' , there are no lattice point on the line $\overline{OP}$, which contradicts the assumption. Either $(a, c) \in \mathbb{Z}^2$ or $(b, d) \in \mathbb{Z}^2$ holds. So (3) follows from (I) and (II). In the other cases, we may prove in a similar way. $\square$

By Lemma 1, we shall show the following theorem.

Theorem 4. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1}$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$. The following (1) and (2) are equivalent.

(1) S is measure preserving;

(2) $|\det A| = n \in \mathbf{N}$ and $(a, c) | n$ ($a, c \in \mathbf{Z}$)

or

$|\det A| = n \in \mathbf{N}$ and $(b, d) | n$ ($b, d \in \mathbf{Z}$),

where (a, c) indicates a greatest common divisor of a and c .

Proof. We shall show that Lemma 1 (3) and Theorem 4 (2) are equivalent.

(Lemma 1 (3) $\implies$ Theorem 4 (2))

Let $a, c \in \mathbf{Z}$, $ad - bc = n$ and let l be the line $\overline{RQ}$. Then $l : y = \frac{c}{a}(x - b) + d$. Let $(x_0, y_0) \in \mathbf{Z}^2$,

$$\begin{aligned} y_0 &= \frac{c}{a}(x_0 - b) + d \\ &= \frac{c}{a}x_0 + \frac{n}{a} \end{aligned}$$

Suppose $(a, c) = p$, and $(n, p) = m < p$. Put $p = mp'$, $a = pa'$, $c = pc'$, $n = mn'$ then $(n', p') = 1$

So $y_0 = \frac{c'}{a'}x_0 + \frac{n'}{a'p'}$ and $a'y_0 - c'x_0 = \frac{n'}{p'}$ holds. $a'y_0 - c'x_0 \in \mathbf{Z}$ contradicts $\frac{n'}{p'} \notin \mathbf{Z}$. So $(n, p) = p$ holds.

(Theorem 4 (2) $\implies$ Lemma 1 (3))

Let $|\det A| = n$, $a, c \in \mathbf{Z}$, $l : y = \frac{c}{a}(x - b) + d = \frac{c}{a}x + \frac{n}{a}$. Put $(a, c) = p \in \mathbf{Z}$ then $a = pa'$, $c = pc'$, $(a', c') = 1$ (i.e. $\exists s, t \in \mathbf{Z}$ s.t. $a's + c't = 1$) holds. By $p | n$, put $n = pn'$ ($n' \in \mathbf{Z}$). $a'n's + c'n't = n'$ holds. If $x_1 = -n't \in \mathbf{Z}$ then

$$y_1 = \frac{c'}{a'}(-n't) + \frac{n'}{a'} = \frac{-c'tn' + n'}{a'} = \frac{a'n's}{a'} = n's \in \mathbf{Z}.$$

Hence $(x_1, y_1) \in \mathbf{Z}^2$. □

By the above theorem, we can consider the case of $\det A \in \mathbf{Z}$ hereafter.

Lemma 2. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1}$$

and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$. Put $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ and $\det A = \pm m$ ($m \geq 1$).

Then

$$\begin{cases} a_{n+1} = az_n \mp mz_{n-1} \\ b_{n+1} = bz_n \\ c_{n+1} = cz_n \\ d_{n+1} = dz_n \mp mz_{n-1}, \end{cases}$$

where

$$\begin{cases} z_{-1} = 0 \\ z_0 = 1 \\ z_{n+1} = (a + d)z_n \mp mz_{n-1}. \end{cases}$$

Put $D = \{f(x, y) = \exp[2\pi i(px + qy)] | p, q \in \mathbf{Z}\}$. Since the linear span of D is dense in $L^1(X)$, we have the following.

Theorem 5. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1},$$

where $a, b, c, d \in \mathbf{Z}$. Put $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then the following statements are equivalent:

- (1) S is not mixing;
- (2) There exist $\{n_j\}_{j=1}^\infty$ and $(p, q, k, l) \neq (0, 0, 0, 0)$ ($p, q, k, l \in \mathbf{Z}$) such that

$$ka_{n_j} + lc_{n_j} - p = kb_{n_j} + ld_{n_j} - q = 0;$$

- (3) There exists $\{z_{n_j}\}_{j=1}^\infty$ which satisfies either (i) or (ii).

(i) $z_{n_j} = z_{n_1}$ and $z_{n_j-1} = z_{n_1-1}$ for any j .

(ii) There exists an eigenvalue λ of matrix A such that

$$\lambda \in \mathbf{Q} \text{ and } \frac{z_{n_j} - z_{n_1}}{z_{n_j-1} - z_{n_1-1}} = \frac{\det A}{\lambda} \text{ for any } j, l (j \neq l).$$

Proof. ((1) $\implies$ (2))

If S is not mixing, then $\lim_{n \rightarrow \infty} \langle f, U^n g \rangle \neq \langle f, 1 \rangle \langle 1, g \rangle = \begin{cases} 1 & k = l = p = q = 0 \\ 0 & \text{otherwise} \end{cases}$, i.e.

for any n_0 , there exists $n_1 \geq n_0$ such that $\langle f, U^{n_1} g \rangle = 1$ with $(k, l, p, q) \neq (0, \dots, 0)$. Repeating the relation, we can show that there exists $n_2 \geq n_1$ such that $\langle f, U^{n_2} g \rangle = 1$ with $(k, l, p, q) \neq (0, \dots, 0)$. Taking this sequence $\{n_j\}_{j=1}^\infty$, the next holds: $\langle f, U^{n_j} g \rangle = 1$ with $(k, l, p, q) \neq (0, \dots, 0)$, i.e. $ka_{n_j} + lc_{n_j} - p = 0$ and $kb_{n_j} + ld_{n_j} - q = 0$. This means (2) holds.

((2) $\implies$ (1))

We shall show that S is not mixing by Proposition 2(b). (S is mixing $\Leftrightarrow \lim \langle f, U^n g \rangle = \langle f, 1 \rangle \langle 1, g \rangle$ with g in a linearly dense set in $L^\infty(X)$). We define the Koopman operator as $U^n g(x, y) = g(S^n(x, y))$. If we take $g(x, y) = \exp[2\pi i(kx + ly)]$ and $f(x, y) = \exp[-2\pi i(px + qy)]$ with $k, l, p, q \in \mathbf{Z}$ then we have $U^n g(x, y) = g(a_n x + b_n y, c_n x + d_n y)$ and

$$\begin{aligned} \langle f, U^n g \rangle &= \int_0^1 \int_0^1 \exp[2\pi i \{(ka_n + lc_n - p)x + (kb_n + ld_n - q)y\}] dx dy \\ &= \begin{cases} 1 & \text{if } ka_n + lc_n - p = kb_n + ld_n - q = 0 \\ 0 & \text{otherwise} \end{cases} \quad \dots (A) \end{aligned}$$

On the other hand,

$$\langle f, 1 \rangle \langle 1, g \rangle = \begin{cases} 1 & k = l = p = q = 0 \\ 0 & \text{otherwise} \end{cases}$$

By (2), for any $n_0 \in N$, there exists $t \geq n_0$ ($t \in \{n_j\}$) and $p, q, k, l \in Z$ such that $(p, q, k, l) \neq (0, 0, 0, 0)$ and $ka_t + lc_t - p = kb_t + ld_t - q = 0 \dots (B)$. By (A) and (B), $\langle f, U^n g \rangle$ does not converge to $\langle f, 1 \rangle \langle 1, g \rangle$. So S is not mixing.

(2) $\Leftrightarrow$ (3)

Put $|\det A| = N$.

(2) $\Leftrightarrow \exists \{n_j\}$ and $\exists (p, q, k, l) \neq (0, 0, 0, 0)$ s.t. $ka_{n_j} + lc_{n_j} - p = kb_{n_j} + ld_{n_j} - q = 0$

$$\Leftrightarrow \begin{cases} ka_{n_j} + lc_{n_j} - p = (ka + lc)zn_j - 1 \mp kNz_{n_j-2} - p = 0 \\ ka_{n_i} + lc_{n_i} - p = (ka + lc)zn_i - 1 \mp kNz_{n_i-2} - p = 0 \end{cases}$$

$$\Leftrightarrow k(a_{n_j} - a_{n_i}) + l(c_{n_j} - c_{n_i}) = (ka + lc)(z_{n_j-1} - z_{n_i-1}) \mp kN(z_{n_j-2} - z_{n_i-2}) = 0$$

$$\Leftrightarrow \begin{cases} k = c = p = 0, l \neq 0, d \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm \frac{N}{d} \\ \text{or} \\ k = d = 0, l \neq 0, z_{n_j-1} = z_{n_i-1}, z_{n_j-2} = z_{n_i-2} \\ \text{or} \\ l = b = q = 0, k \neq 0, a \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm \frac{N}{a} \\ \text{or} \\ l = a = 0, k \neq 0, z_{n_j-1} = z_{n_i-1}, z_{n_j-2} = z_{n_i-2} \\ \text{or} \\ ka + lc \neq 0, kb + ld \neq 0, l \neq 0, \frac{z_{n_j-1} - z_{n_i-1}}{z_{n_j-2} - z_{n_i-2}} = \pm \frac{kN}{ka+lc} = \pm \frac{lN}{kb+ld} \end{cases}$$

$\Leftrightarrow$ (3). □

Theorem 6. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1},$$

where $a, b, c, d \in \mathbf{Z}$. Put $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. If there exist $\{n_j\}_{j=1}^\infty$ and $(p, q, k, l) \neq (0, 0, 0, 0)$ ($p, q, k, l \in \mathbf{Z}$) such that $n_{j+1} - n_j = n_2 - n_1$ for any j and

$$ka_{n_j} + lc_{n_j} - p = kb_{n_j} + ld_{n_j} - q = 0,$$

then S is not ergodic.

In order to obtain a criterion for demonstrating either mixing, exactness or ergodicity, we first show the following propositions using Theorem 5.

Proposition 7. Suppose $\det A > 0$. Then the following statements holds:

- (1) If $a + d = \det A + 1$, then $\{z_n\}$ satisfies the condition of Theorem 5(3)(ii);
- (2) If $a + d = -(\det A + 1)$, then $\{z_{2n}\}$ satisfies the condition of Theorem 5(3)(ii);
- (3) If $|a + d| \neq \det A + 1$ ($\det A \neq 1$), then there doesn't exist $\{z_{n_j}\}$ which satisfies the condition of Theorem 5(3);
- (4) Let $\det A = 1$.

- (i) If $|a + d| = 0$, then $\{z_{4n}\}$ satisfies the condition of Theorem 5(3)(i).
- (ii) If $|a + d| = 1$, then $\{z_{6n}\}$ satisfies the condition of Theorem 5(3)(i).
- (iii) If $|a + d| = -1$, then $\{z_{3n}\}$ satisfies the condition of Theorem 5(3)(i).

Proposition 8. Suppose $\det A < 0$. Then the following statements holds:

- (1) If $a + d = \det A + 1 \neq 0$, then $\{z_n\}$ satisfies the condition of Theorem 5(3)(ii);
- (2) If $a + d = -(\det A + 1) \neq 0$, then $\{z_{2n}\}$ satisfies the condition of Theorem 5(3)(ii);
- (3) If $a + d = \det A + 1 = 0$, $\{z_{2n}\}$ satisfies the condition of Theorem 5(3)(i);
- (4) If $|a + d| \neq |\det A| - 1$, there doesn't exist $\{z_{n_j}\}$ which satisfies the condition of Theorem 5(3).

Using the next theorem, we can know the behavior of S calculating $\det A$ and $|a + d|$.

Theorem 9. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by, cx + dy) \pmod{1},$$

where $a, b, c, d \in \mathbf{Z}$. The following statements are equivalent:

- (i) S is mixing;
- (ii) S is ergodic;
- (iii) Either (a), (b) or (c) holds:
 - (a) $\det A \geq 2$ and $|a + d| \neq \det A + 1$;
 - (b) $\det A = 1$ and $|a + d| \geq 3$;
 - (c) $\det A < 0$ and $|a + d| \neq |\det A| - 1$.

We consider the following transformation:

$$S(x, y) = (ax + by + \alpha, cx + dy + \beta) \pmod{1},$$

where $a, b, c, d \in \mathbf{Z}$ and $0 \leq \alpha, \beta < 1$.

Theorem 10. Let $(X, \mathcal{A}, \mu)$ be a normalized measure space. Suppose $S : X \rightarrow X$ is defined by

$$S(x, y) = (ax + by + \alpha, cx + dy + \beta) \pmod{1},$$

where $a, b, c, d \in \mathbf{Z}$ and $0 \leq \alpha, \beta < 1$. Let $S_0(x, y) = (ax + by, cx + dy) \pmod{1}$.

The following statements hold:

- (1) If either $\det A = 1$ and $|a + d| \geq 3$ or $|a + d| \neq \operatorname{sgn}(\det A)(\det A + 1)$, then S is mixing, where $\operatorname{sgn}(\det A)$ indicates the sign of $\det A$;
- (2) If either (i) or (ii) holds, then S is ergodic, but not mixing;

- (i) $|a + d| = \text{sgn}(\det A)(\det A + 1)$, $A = \pm I$ (I is an 2×2 identity matrix) and $\alpha, \beta \notin \mathbf{Q}$.
 - (ii) $a + d = \det A + 1$, $A \neq I$ and either $\alpha c - (a - 1)\beta \notin \mathbf{Q}$ or $\alpha(d - 1) - \beta b \notin \mathbf{Q}$.
- (3) If either (i), (ii), (iii) or (iv) holds, S is not ergodic.
- (i) $\det A = 1$ and $|a + d| \leq 1$.
 - (ii) $|a + d| = \text{sgn}(\det A)(\det A + 1)$, $A \neq \pm I$ and either $\alpha \in \mathbf{Q}$ or $\beta \in \mathbf{Q}$.
 - (iii) $|a + d| = \det A + 1$, $A \neq I$ and either $\alpha c - (a - 1)\beta \in \mathbf{Q}$ or $\alpha(d - 1) - \beta b \in \mathbf{Q}$.
 - (iv) $|a + d| = -\det A - 1$ and $A \neq -I$.

References

- [1] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise*, Springer Verlag (1995)