

A modulus of uniform continuity with some order in $L^s_{loc}(\Omega; R^N)$ ($2 \leq s < \infty$)
and a sharp estimate of *Lebesgue points* of the first-derivatives
of minimizers of a Quasi-convex functional in the calculus of variations .

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Abstract. This paper establishes that minimizers of *strictly quasi-convex* variational functionals , satisfy a modulus of uniform continuity with some order in the norm of $L^s_{loc}(\Omega; R^N)$ with $2 \leq s \leq \infty$. This modulus of uniform continuity combined with a result in the present author's paper and one of Evans's results implies a local Hölder continuity and a sharp estimate for the *Hausdorff dimension* of *Lebesgue points* of the first derivatives of minimizers .

1. INTRODUCTION

In this paper we establish that minimizers for certain functionals in the calculus of variations satisfy a modulus of uniform continuity of some order in the norm of $L^s_{loc}(\Omega; R^N)$ with $2 \leq s < +\infty$. This functional is given as follows : Let n, N be positive integers . We denote by $M^{n \times N}$ the space of all real $n \times N$ matrices and suppose that $\Omega \subset R^n$ is a bounded with smooth boundary . Then for $v : \Omega \mapsto R^N$, we consider the functional

$$(1.1) \quad I[v] \equiv \int_{\Omega} F(\nabla v) dx ,$$

where $v = (v^i)$, $\nabla v = (\partial v^i / \partial x_{\alpha})$ ($\alpha = 1, \dots, n, i = 1, \dots, N$) is the gradient matrix of v and $F : M^{n \times N} \mapsto R$ is any given mapping , which is strictly defined later. Here we introduce another notation which will be used in this paper : $L^s(\Omega; R^N)$ is sth- power integrable function space. We also denote by $L^s_{loc}(\Omega; R^N)$ locally sth- power integrable function space . $H^{1,s}(\Omega; R^N)$ and $\dot{H}^{1,s}(\Omega; R^N)$ are the usual Sobolev spaces. Also $|A|$ and $H^{\gamma}(A)$ means the Lebesgue measure and the γ - dimensional Hausdorff measure of measurable set A in R^n , respectively , (see *Giaquinta* [Gm1] and *Giusti* [Gi] for detailed definition).

We introduce a forward translation operator and also a forward difference operator of a map in $f \in L^s(\Omega; R^N)$: Let h be any small number and e be a unit vector in R^n . We define a forward translate operator $\cdot^+$ by

$$(1.2) \quad f^+(x) \equiv f(x + he)$$

and define a forward difference operator τ_h by

$$(1.3) \quad \tau_h f = f^+ - f .$$

We adopt the summation convention : For $\forall A, P, Q \in M^{n \times N}$, we define

$$\begin{aligned} DF(A) &= \left(\frac{\partial F}{\partial p_\alpha^i}(A) \right), \\ D^2 F(A) &= \left(\frac{\partial^2 F}{\partial p_\alpha^i \partial p_\beta^j}(A) \right) \\ (\alpha, \beta &= 1, \dots, n, i, j = 1, \dots, N), \end{aligned}$$

$$DF(A) \cdot P = \sum_{\alpha=1}^n \sum_{i=1}^N \frac{\partial F}{\partial p_\alpha^i}(A) P_\alpha^i,$$

and

$$D^2 F(A) \langle P, Q \rangle = \sum_{\alpha, \beta=1}^n \sum_{i, j=1}^N \frac{\partial^2 F}{\partial p_\alpha^i \partial p_\beta^j}(A) P_\alpha^i Q_\beta^j.$$

Let $F(x, z, p) : \Omega \times R^N \times M^{n \times N} \mapsto R$ be a function satisfying

- (H1) $F(x, z, p) \leq K[1 + p^s]$
- (H2) $F(x, z, p) \geq m$
- (H3) $|F(x, z, p_1) - F(x, z, p_2)| \leq K[1 + |p_1|^{s-1} + |p_2|^{s-2}]|p_1 - p_2|$
- (H4) $|F(x_1, z_2, p) - F(x_1, z_1, p)| \leq K[1 + |p|^s][|x_1 - x_2| + |z_1 - z_2|]$

for $\exists m, K > 0$ and $s(1 \leq s < \infty)$. The first question in the calculus of variations can be considered as the existence problem of minimizers in some function space. Under the above condition, *Morrey* [Mo] has isolated that a necessary and sufficient condition of certain functional $F(x, z, p)$ for the lower semicontinuity of $I[\cdot]$ on some *Sobolev space* is *quasi-convex* :

$$\int_O F(x_0, z_0, p_0) dy \leq \int_O F(x_0, z_0, p_0 + \nabla \phi) dy \quad \text{for } \forall (x_0, z_0, p_0) \in \Omega \times R^N \times M^{n \times N},$$

for an arbitrary smooth, bounded, open set $O \subset R^n$, $\forall A \in M^{n \times N}$ and $\forall \phi \in C_0^1(O; R^N)$.

Recently *Acerbi* and *Fusco* [AF] has refined *Morrey's theorem*, who have obtained the following for $F(p)$:

THEOREM 0 ([AF]). Assume that $F : M^{n \times N} \mapsto R$ is continuous and for some positive numbers C and s the following

$$0 \leq F(p) \leq C(1 + |p|^s)$$

holds for $\forall p \in M^{n \times N}$. Then $I[\cdot]$ is weakly sequentially lower semicontinuous on the Sobolev space $H^{1,m}(\Omega; R^N)$ if and only if F is *quasi-convex*.

Also the second question can be considered as the regularity problem of such minimizers. However one often encounters that a minimizer is not necessarily regular everywhere in Ω , even when F is uniform convex (see [Gm1], [Gm2], [Gm3], [GG2] and [GI]). For the study of partial regularity, *Evans* [Ev] (see also [EG] and [GM]) has showed that minimizers has Hölder continuous first derivatives on some open subset $\Omega_0 \subset \Omega$ satisfying $|\Omega/\Omega_0| = 0$, when $F \in C^2(M^{n \times N}; R^N)$ and $D^2 F(p)$ is uniform continuous in $M^{n \times N}$ and strictly *quasi-convex* : For $\exists \gamma > 0$ and $\exists s(2 \leq s < \infty)$ F satisfies

$$(1.4) \quad \gamma \int_\Omega (1 + |\nabla \phi|^{s-2}) |\nabla \phi|^2 dy \leq \int_\Omega [F(A + \nabla \phi) - F(A)] dy$$

for $\forall A \in M^{n \times N}$ and $\forall \phi \in \overset{\circ}{C}^1(\Omega; R^N)$.

and suppose that

$$(H5) \quad |D^2 F(p)| \leq C_0(1 + |p|^{s-2})$$

for some constant C_0 and $\forall p \in M^{n \times N}$.

We remark that assumption (H5) implies that there exist positive constants C_1 and C_2 such that

$$(H6) \quad |F(p)| \leq C_1(1 + |p|^s)$$

$$(H7) \quad |DF(p)| \leq C_2(1 + |p|^{s-1})$$

for all $p \in M^{n \times N}$. Under the above condition, *Evans* has proved

THEOREM 1 ([Ev]). Assume that $2 \leq s < +\infty$, the function F satisfies (1.5) and (H5). Let $u \in H^{1,s}(\Omega; R^N)$ be a minimizer of $I[\cdot]$. Then there exists an open subset Ω_0 of Ω such that

$$(1.5) \quad |(\Omega/\Omega_0)| = 0$$

and the first derivatives of a minimizer u are locally Hölder continuous on Ω_0 :

$$\nabla u \in C^\alpha(\Omega_0; M^{n \times N})$$

for each $0 < \alpha < 1$.

This proof is performed by combining a *blow-up argument* with the following *Caccioppoli inequality* :

THEOREM 2 ([Ev]). There exists a constant C_3 independent of r such that a minimizer u satisfies

$$(1.6) \quad \int_{B_{r/2}(x)} (1 + |\nabla u|^{s-2}) |\nabla u|^2 dx \leq C_3 [(1/r)^2 \int_{B_r(x)} |u - a|^2 dx + (1/r)^s \int_{B_r(x)} |u - a|^s dx]$$

for $\forall B_r(x) \subset\subset \Omega$ and $\forall a \in R^N$.

From Theorem 2 and a *Gehring inequality* [Gm], it follows that

THEOREM 3. When ∇u satisfies the inequality (1.7) of Theorem 2, there exist positive numbers t ($t > s$) depending only on C_3, s, Ω and C_4 depending only on C_3, s, Ω and $\tilde{\Omega}$ such that $\nabla u \in L_{loc}^t(\Omega; R^N)$ and moreover the following holds :

$$(1.7) \quad \left[\frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} (1 + |\nabla u|)^t dx \right]^{1/t} \leq C_4 \left[\frac{1}{|\Omega|} \int_{\Omega} (1 + |\nabla u|)^s dx \right]^{1/s}$$

for $\forall \tilde{\Omega} \subset\subset \Omega$.

2. MAIN RESULT

Now we can state the main theorem

THEOREM 4 (MAIN THEOREM). Assume that $2 \leq s < +\infty$, the function F satisfies (1.4) and (H5). Let u be minimizer of $I[\cdot]$ in $H^{1,s}(\Omega; R^N)$. Then for an arbitrary open set $\tilde{\Omega}$ compactly contained in Ω , the following holds :

$$(2.1) \quad \int_{\tilde{\Omega}} |\tau_h \nabla u|^2 dx \leq C_5 \cdot h \quad \text{for } 0 < h < \frac{1}{8} \text{dist}(\tilde{\Omega}, \partial\Omega),$$

where C_5 is a constant depending only on $n, N, \gamma, C_0, \|\nabla u\|_{L^s}, \tilde{\Omega}$ and Ω .

Here we notice that as in the same way of author's previous result, one finds

THEOREM 5([Ho]). Let f be a function belonging to $L^p_{loc}(\Omega; R^N)$ ($1 \leq p < \infty$) with the following condition: Let $\tilde{\Omega}$ be an arbitrary open set compactly contained in Ω and suppose that there exist positive numbers C_6 and α ($0 < \alpha < n/p$) independent of h such that f satisfies

$$(2.2) \quad \int_{\tilde{\Omega}} |\tau_h f|^p dx \leq C_6 \cdot h^{p\alpha}$$

for any number h with $0 < h < \frac{1}{4} \text{dist}(\tilde{\Omega}, \partial\Omega)$. Then for the singular set S_f of the map f defined by

$$(2.3) \quad S_f = \{x \in \Omega : \lim_{\rho \rightarrow +0} f_{x,\rho}\} \cup \{x \in \Omega : \lim_{\rho \rightarrow +0} |f_{x,\rho}| = +\infty\} \cup \{x \in \Omega : \lim_{\rho \rightarrow +0} \int_{B_\rho(x)} |f - f_{x,\rho}|^p dy > 0\}$$

where $f_{x,\rho} = 1/|B_\rho| \int_{B_\rho(x)} f(y) dy$, the following holds:

$$(2.4) \quad H^{(\beta)}(S) = 0$$

for any positive number β with $n - p\alpha < \beta$.

From Theorem 4 and Theorem 5, we obtain

THEOREM 6. A singular set $S_{\nabla u}$ of the first derivatives of such minimizers, have at most

$$(2.5) \quad H^{n-1+\epsilon}(S) = 0$$

$$\text{for } \forall \epsilon > 0.$$

In addition, noting [Ev] and [EG], one finds that (2.5) shows the first derivatives of minimizers satisfy local Hölder continuity on Ω/S :

$$\nabla u \in C^\alpha(\Omega/S; M^{n \times N}) \quad \text{for } 0 < \alpha < 1.$$

3. PROOF OF THEOREM 4

Since u is a minimizer of $I[\cdot]$ in $H^{1,s}(\Omega; R^N)$, u satisfies the following first-variational formula :

$$(3.1) \quad \int_{\Omega} DF(\nabla u) \cdot \nabla \phi dx = 0 \quad \text{for } \forall \phi \in \dot{H}^{1,s}(\Omega; R^N).$$

Transferring x to $x + he$ along the direction of a unit vector e , we have

$$(3.2) \quad \int_{\Omega} DF(\nabla u^+) \cdot \nabla \phi dx = 0 \quad \text{for} \quad \forall \phi \in \mathring{H}^{1,s}(\Omega_1; R^N).$$

where $\Omega_0 = \tilde{\Omega}$, $\Omega_k = \{x \in \Omega : \text{dist}(x, \tilde{\Omega}) < \frac{k}{4} \text{dist}(\tilde{\Omega}, \partial\Omega)\}$ ($k = 0, 1, \dots, 4$). (3.1) subtracted after (3.2) gives

$$(3.3) \quad \int_{\Omega} [DF(\nabla u^+) - DF(\nabla u)] \cdot \nabla \phi dx = 0 \quad \text{for} \quad \forall \phi \in \mathring{H}^{1,s}(\Omega_1; R^N).$$

Thus we have

$$(3.4) \quad \int_{\Omega} \int_0^1 D^2 F(\nabla u + t \nabla(\tau_h u)) < \nabla(\tau_h u), \nabla \phi > dt dx = 0 \quad \text{for} \quad \forall \phi \in \mathring{H}^{1,m}(\Omega_1; R^N).$$

Substituting $\tau_h u \eta^2$ for ϕ , where a cut-off function $\eta \in C_0^\infty(\Omega)$ satisfies

$$\eta = \begin{cases} 1 & \text{in } \Omega_0, \\ 0 & \text{outside } \Omega_1 \end{cases} \quad \text{with} \quad \begin{cases} |\nabla \eta| \leq \frac{2}{\text{dist}(\Omega_0, \Omega_1)}, \\ 0 \leq |\eta| \leq 1. \end{cases}$$

We can proceed the calculation of (3.3) as follows :

$$(3.5) \quad \begin{aligned} & \int_{\Omega} < \tau_h [DF(\nabla u)], \nabla(\tau_h u) \eta^2 > dx \\ &= \int_{\Omega} \int_0^1 D^2 F(\nabla u + t \nabla(\tau_h u)) \\ & \quad [< \nabla(\tau_h u), \nabla(\tau_h u) \eta^2 > + 2 < \nabla(\tau_h u), \tau_h u \eta \nabla \eta >] dt dx \end{aligned}$$

Consequently, the following

$$(3.6) \quad \begin{aligned} & \int_{\Omega} D^2 F(A) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dx \\ &= \int_{\Omega} [D^2 F(A) - \int_0^1 D^2 F(\nabla u + t \nabla(\tau_h u))] \\ & \quad < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dt dx \\ & \quad - 2 \int_{\Omega} \int_0^1 D^2 F(\nabla u + t \nabla(\tau_h u)) dt < \nabla(\tau_h u) \eta, \tau_h u \nabla \eta > dt dx, \end{aligned}$$

holds for $\forall A \in M^{n \times N}$. Now let Ω_1 be approximated by a union of hypercubes $D_{k,i}$ with each edge length $1/k$ sufficiently large $k > 0$:

$$(3.7) \quad \begin{aligned} \Omega_1 &\subset \bigcup_{i=1}^I D_{k,i} \quad \text{with} \quad \Omega_1 \subset H_k \subset \Omega_2, \\ \mathring{D}_{k,i} \cap \mathring{D}_{k,j} &= \emptyset \quad \text{in} \quad i \neq j, \\ |H_k - \Omega_2| &\rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty, \\ |D_{k,i}| &= (1/k)^n. \end{aligned}$$

Moreover we remark that there exists subsequence of I which we call $I(k)$ such that $H_k = \bigcup_{i=1}^{I(k)} D_{k,i}$ satisfies $\Omega_1 \subset H_k \subset \Omega_2$ and $|\Omega_2 - H_k| \rightarrow 0$ as $k \rightarrow +\infty$. For $x \in H_k$, we define

$$\overline{\nabla u}(x) \equiv \frac{1}{|D_{k,i}|} \int_{D_{k,i}} \nabla u(y) dy \quad \text{for } x \in D_{k,i} \quad \text{and } i = 1, \dots, I.$$

When we adopt $\overline{\nabla u}(x) + s \overline{\nabla(\tau_h u)}(x)$ ($0 \leq s \leq 1$), $\overline{\nabla \tau_h u}(x) \equiv \overline{\nabla u^+}(x) - \overline{\nabla u}(x)$ as A , then it follows from (3.5), (3.6) and (3.7) that

$$\begin{aligned} & \int_{\Omega} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dx \\ &= \int_{\Omega} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) dx - \int_0^1 \int_{\Omega} D^2 F(\nabla u + t \nabla(\tau_h u)) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dt dx \\ (3.8) \quad & - 2 \int_{\Omega} \int_0^1 D^2 F(\nabla u + t \nabla(\tau_h u)) < \nabla(\tau_h u) \eta, \tau_h u \nabla \eta > dt dx. \end{aligned}$$

By integrating (3.8) over $[0, 1]$ for s , we obtain

$$\begin{aligned} & \int_{\Omega} \int_0^1 D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ &= \int_{\Omega} \int_0^1 [D^2 F(\overline{\nabla u} + t \overline{\nabla(\tau_h u)}) - D^2 F(\nabla u + t \nabla(\tau_h u))] < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dt dx \\ (3.9) \quad & - 2 \int_{\Omega} \int_0^1 (D^2 F)(\nabla u + t \nabla(\tau_h u)) dt < \nabla(\tau_h u) \eta, (\tau_h u) \nabla \eta > dx. \end{aligned}$$

The above (3.9) is a starting point to our proof. The original technique used here is seen in [Da] and [Mo]. At first, we estimate the left-hand side in (3.9) from below:

$$\begin{aligned} & \int_{\Omega} \int_0^1 D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ & \geq \int_0^1 \int_{H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ & + \int_0^1 \int_{\Omega_2/H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ & = \sum_{i=1}^{I(k)} \int_0^1 \int_{D_{k,i}} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ (3.10) \quad & + \int_0^1 \int_{\Omega_2/H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx. \end{aligned}$$

If we use the mean value theorem for s , then there exist positive numbers $s_{0,i}$ ($i = 1, \dots, I(k)$) such that

$$\begin{aligned} & = \sum_{i=1}^{I(k)} \int_{D_{k,i}} D^2 F(\overline{\nabla u} + s_{0,i} \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \\ & + \int_0^1 \int_{\Omega_2/H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx \end{aligned}$$

Here we remark that from *Morrey* ([Mo] , Th 4.4.3) and *Federer* ([Fe] , Th 5.1.10) assumption (1.4) implies the strong *Legendre - Hadamard* condition :

$$(3.11) \quad \sum_{\alpha, \beta} \sum_{i, j} \frac{\partial^2 F}{\partial p_\alpha^i \partial p_\beta^j}(A) \xi_\alpha \xi_\beta \eta^i \eta^j \geq \gamma |\xi|^2 |\eta|^2$$

$$\text{for } \forall A \in M^{n \times N}, \forall \xi \in R^n \text{ and } \forall \eta \in R^N.$$

Thus by noting that $\overline{\nabla u}$ is a constant on each hypercube $D_{k,i}$ ($i = 1, \dots, I$) , we have

$$(3.10) \geq \gamma \sum_{i=1}^{I(k)} \int_{D_{i,k}} |\nabla(\tau_h u)|^2 dx + \int_0^1 \int_{\Omega_2/H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx$$

$$(3.12) \quad = \gamma \int_{H_k} |\nabla(\tau_h u)|^2 dx + \int_0^1 \int_{\Omega_2/H_k} D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 ds dx.$$

Next we estimate the first term on the right - hand side in (3.9) : From uniform continuity assumption of $D^2 F(p)$, there exists a non-negative function $w(t)$ increasing in t , and $w(0) = 0$ concave , continuous and bounded and a constant C_7 , such that we obtain

$$(3.13) \quad \begin{aligned} & \int_{\Omega} \int_0^1 [D^2 F(\overline{\nabla u} + s \overline{\nabla(\tau_h u)}) - D^2 F(\nabla u + s \nabla(\tau_h u))] < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dt dx \\ & \leq C_7 \int_{\Omega_1} \int_0^1 [1 + |\overline{\nabla u} + s \overline{\nabla(\tau_h u)}|^{s-2} + |\nabla u + s \nabla(\tau_h u)|^{s-2}] \\ & w(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2) |\nabla(\tau_h u)|^2 dx \\ & \leq 2C_7 2^{s-1} \int_{\Omega_1} [1 + |\overline{\nabla u}|^{s-2} + |\overline{\nabla u^+}|^{s-2} + |\nabla u|^{s-2} + |\nabla u^+|^{s-2}] \\ & [|\nabla u|^2 + |\nabla u^+|^2] \cdot w(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2) dx. \end{aligned}$$

Since $\nabla u \in L_{loc}^t(\Omega; R^N)$ ($t > s$) from (1.8) of Theorem 3 , we can apply Hölder inequality to (3.13) as follows : For $s_1 = t/(s-2)$, $s_2 = t/2$ and $s_3 = t/(t-s)$, we estimate the right-hand in (3.13)

$$\begin{aligned} & \leq 2^s C_7 5 \cdot 2 \left\{ \int_{\Omega_1} [1 + |\overline{\nabla u}|^t + |\overline{\nabla u^+}|^t + |\nabla u|^t + |\nabla u^+|^t] dx \right\}^{(s-2)/t} \\ & \left\{ \int_{\Omega_1} [|\nabla u|^t + |\nabla u^+|^t] dx \right\}^{2/t} \left\{ \int_{\Omega_1} w^{t/(t-s)} (|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2) dx \right\}^{(t-s)/t}. \end{aligned}$$

Successively by using bounded and concave properties of $w(t)$, we have

$$\begin{aligned}
&\leq 2^s 10 C_7 \left\{ \int_{\Omega_2} [1 + |\overline{\nabla} u|^t + |\nabla u|^t] dx \right\}^{(s-2)/t} \\
&\quad \left\{ \int_{\Omega_2} |\nabla u|^t dx \right\}^{2/t} \left\{ \int_{\Omega_1} w(|\overline{\nabla} u - \nabla u|^2 + |\overline{\nabla} u^+ - \nabla u^+|^2) dx \right\}^{(t-s)/t} \\
&\leq 2^s 10 C_7 |\Omega_1|^{(t-s)/t} \left\{ \int_{\Omega_2} [1 + |\overline{\nabla} u|^t + |\nabla u|^t] dx \right\}^{s/t} \\
&\quad \left\{ \frac{1}{|\Omega_1|} \int_{\Omega_1} w(|\overline{\nabla} u - \nabla u| + |\overline{\nabla} u^+ - \nabla u^+|) dx \right\}^{(t-s)/t} \\
&\leq 2^s C_7 10 |\Omega|^{1-s/t} \left\{ \int_{\Omega_2} [1 + |\overline{\nabla} u|^t + |\nabla u|^t] dx \right\}^{s/t} \\
(3.14) \quad &\cdot w\left(\frac{1}{|\Omega_1|} \int_{\Omega_1} [|\overline{\nabla} u - \nabla u| + |\overline{\nabla} u^+ - \nabla u^+|] dx\right)^{(t-s)/t}.
\end{aligned}$$

From L_1 - norm continuity of integrable function, for $\forall \epsilon > 0$, there exists $k = k(\epsilon)$ such that

$$(3.14) \leq 2^s 10 C_7 |\Omega|^{1-s/t} \cdot \epsilon \cdot \left\{ \int_{\Omega_2} [1 + |\overline{\nabla} u|^t + |\nabla u|^t] dx \right\}^{s/t}.$$

Finally we shall estimate the second term on the right-hand side in (3.9) : From assumption (H5) and using *Newton - Leibnitz* formula we obtain

$$\begin{aligned}
&-2 \int_{\Omega_1} \int_0^1 (D^2 F)(\nabla u + t \nabla(\tau_h u)) dt < \nabla(\tau_h u) \eta, \tau_h u \nabla \eta > dt dx \\
&\leq 2 C_0 \int_{\Omega_1} \int_0^1 (1 + |\nabla u + t \nabla(\tau_h u)|^{s-2}) |\nabla(\tau_h u)| \cdot |\tau_h u| \cdot |\nabla \eta| dx \\
&\leq 2^s C_0 \int_{\Omega_1} (1 + |\nabla u^+|^{s-2} + |\nabla u|^{s-2}) |\nabla(\tau_h u)| \cdot |\tau_h u| \cdot |\nabla \eta| dx \\
&\leq 2^s C_0 \frac{2}{\text{dist}(\Omega_0, \Omega_1)} \left\{ \int_{\Omega_1} [1 + |\nabla u^+|^{s-2} + |\nabla u|^{s-2}]^{s/(s-2)} dx \right\}^{(s-2)/s} \\
&\quad \left\{ \int_{\Omega_1} [|\nabla u^+| + |\nabla u|]^s dx \right\}^{1/s} \left\{ \int_{\Omega_1} |\tau_h u|^s dx \right\}^{1/s} \\
(3.16) \quad &\leq 2^s C_0 3 \cdot 2 \frac{2}{\text{dist}(\Omega_0, \Omega_1)} \left\{ \int_{\Omega_2} [1 + |\nabla u|^s] dx \right\}^{1-1/s} \left\{ \int_{\Omega_1} |\tau_h u|^s dx \right\}^{1/s} \\
&\leq 2^s 12 C_0 \frac{h}{\text{dist}(\Omega_0, \Omega_1)} \left\{ \int_{\Omega_2} [1 + |\nabla u|^s] dx \right\}^{1-1/s} \left\{ \int_{\Omega_2} |\nabla u|^s dx \right\}^{1/s}.
\end{aligned}$$

Consequently it follows from (3.12), (3.15) and (3.16) that

$$\begin{aligned}
&\gamma \int_{H_k} |\nabla(\tau_h u)|^2 dx \\
&\leq 2^s 10 C_7 |\Omega_1|^{1-s/t} \epsilon \left\{ \frac{1}{|\Omega_1|} \int_{\Omega_2} (1 + |\nabla u|^t + |\overline{\nabla} u|^t) dx \right\}^{s/t} \\
(3.17) \quad &+ \frac{122^s C_0 h}{\text{dist}(\Omega_0, \Omega_1)} \left\{ \int_{\Omega_2} (1 + |\nabla u|^s) dx \right\}^{1-1/s} \left\{ \int_{\Omega_2} |\nabla u|^s dx \right\}^{1/s}.
\end{aligned}$$

Now letting pass to the limit $k \rightarrow \infty$, we deduce the desired estimates :

$$(3.18) \quad \begin{aligned} & \int_{\tilde{\Omega}} |\nabla(\tau_h u)|^2 dx \\ & \leq \gamma^{-1} \frac{2^s 80 C_0 h}{\text{dist}(\tilde{\Omega}, \partial\Omega)} \left\{ \int_{\Omega_2} (1 + |\nabla u|^s) dx \right\}. \end{aligned}$$

This completes our proof.

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