

τ -tilting modules over Nakayama algebras

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1 Introduction

The notion of tilting modules plays a central role in representation theory of algebras. As an important result, there is a bijection between basic tilting modules and functorially finite faithful torsion classes for a given algebra. The bijection produces fruitful results for tilting modules. To give a class of modules corresponding bijectively functorially finite torsion classes, the author in [AIR] introduced the notion of (support) τ -tilting modules. Indeed, they showed that there is a bijection between basic support τ -tilting modules and functorially finite torsion classes. By the bijection, support τ -tilting modules hold various properties of tilting modules. Moreover, they correspond bijectively with many important objects in representation theory, *e.g.*, basic two-term silting complexes and basic cluster-tilting objects. Therefore it is important to give a classification of (support) τ -tilting modules.

In this report, we give a classification of τ -tilting modules over Nakayama algebras. The following theorem is our main result.

Theorem 1.1. *Let Λ be a Nakayama algebra with n simple modules. Assume that the length of every indecomposable projective Λ -module is at least n . Then there is a bijection between*

- (1) *the set $\tau\text{-tilt}\Lambda$ of isomorphism classes of basic τ -tilting Λ -modules,*
- (2) *the set $\mathcal{T}(n)$ of triangulations of an n -regular polygon with a puncture.*

Throughout this report, we use the following notation. By an algebra we mean basic, ring-indecomposable and finite dimensional algebra over an algebraically closed field K , and by a module we mean a finitely generated right module. For an algebra Λ , we denote by $\text{mod}\Lambda$ the category of finitely generated right Λ -modules. We denote by $[i, j]$ the interval $\{i, i+1, \dots, j-1, j\}$ of integers $i \leq j$. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of an algebra Λ . For each $i \in [1, n]$, we put $P_i = e_i\Lambda$ and $S_i = P_i/\text{rad}P_i$.

2 Preliminaries

Let Λ be a finite dimensional K -algebra. In this section, we collect some results which are necessary in this report. We start with basic fact for representation theory of finite dimensional algebras.

- (1) The category $\text{mod}\Lambda$ is *Krull-Schmidt*, that is, any module M in $\text{mod}\Lambda$ is isomorphic to a finite direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_m$, where $M_1, M_2, \dots, M_m$ are indecomposable Λ -modules. Moreover, the direct sum is uniquely determined up to isomorphism and permutation. Then we let $|M| := m$. We say that M is *basic* if $M_1, \dots, M_m$ are pairwise nonisomorphic.
- (2) We say that Λ is *basic* if it is basic as a Λ -module. A basic algebra Λ is isomorphic to the bounded quiver algebra KQ/I , where Q is a finite quiver and I is an admissible ideal of the path algebra KQ . Then, for the vertex set $\{1, 2, \dots, n\}$ in Q , we have a decomposition $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ as a Λ -module, where P_i is an indecomposable projective Λ -module. Moreover, each indecomposable projective (respectively, simple) Λ -module is isomorphic to P_i (respectively, $S_i := P_i/\text{rad}P_i$) for some $i \in \{1, 2, \dots, n\}$.
- (3) For a Λ -module M with a minimal projective presentation $P^{-1} \xrightarrow{p} P^0 \rightarrow M \rightarrow 0$, we define τM in $\text{mod}\Lambda$ by an exact sequence

$$0 \rightarrow \tau M \rightarrow \nu P^{-1} \xrightarrow{\nu p} \nu P^0,$$

where $\nu := \text{Hom}_K(\text{Hom}_\Lambda(-, \Lambda), K)$. We call τ the AR translation of Λ . We have $\tau P = 0$ if P is a projective Λ -module.

2.1 τ -tilting modules

In this subsection, we recall the definition of τ -tilting modules.

Definition 2.1. (1) We call M in $\text{mod}\Lambda$ *τ -rigid* if $\text{Hom}_\Lambda(M, \tau M) = 0$.

(2) We call M in $\text{mod}\Lambda$ *τ -tilting* if it is τ -rigid and $|M| = |\Lambda|$.

(3) We call M in $\text{mod}\Lambda$ *support τ -tilting* if there exists an idempotent $e \in \Lambda$ such that M is a τ -tilting $(\Lambda/\Lambda e\Lambda)$ -module.

We denote by $\tau\text{-rigid}\Lambda$ the set of isomorphism classes of indecomposable τ -rigid Λ -modules and by $\tau\text{-tilt}\Lambda$ the set of isomorphism classes of basic τ -tilting Λ -modules.

Example 2.2. Every projective Λ -module is a τ -rigid Λ -module by the definition of τ . In particular, Λ is a τ -tilting Λ -module.

By the following remark, τ -tilting modules are a generalization of tilting modules. A Λ -module T is said to be *tilting* if it satisfies three conditions: (a) the projective dimension is at most one, (b) rigid (i.e., $\text{Ext}_\Lambda^1(T, T) = 0$), and (c) $|T| = |\Lambda|$.

Remark 2.3. Every τ -rigid Λ -module is rigid and the converse holds if the projective dimension of a rigid Λ -module is at most one. Thus, if a Λ -module satisfies the condition (a) and (b) above, then it is τ -rigid. Hence tilting Λ -modules are τ -tilting Λ -modules. Moreover, if Λ is hereditary (i.e., the global dimension of Λ is at most one), then τ -tilting Λ -modules are exactly tilting.

We obtain a close connection between support τ -tilting modules and other important objects in representation theory.

Theorem 2.4. ([AIR, Theorem 0.5]) *Let Λ be a finite dimensional K -algebra. Then there are bijections between*

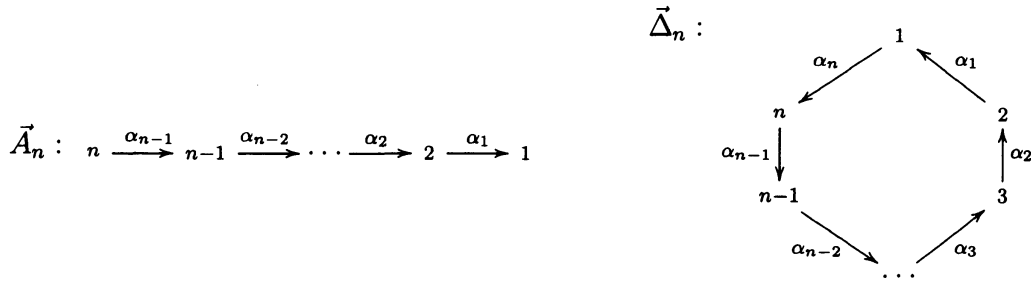
- (1) *the set of isomorphism classes of basic support τ -tilting Λ -modules,*
- (2) *the set of functorially finite torsion classes in $\text{mod } \Lambda$,*
- (3) *the set of isomorphism classes of basic two-term silting complexes for Λ ,*
- (4) *the set of isomorphism classes of basic cluster-tilting objects in a 2-Calabi-Yau triangulated category $\mathcal{C}$ if Λ is an associated 2-Calabi-Yau tilted algebras.*

Hence it is an important to give a classification of (support) τ -tilting modules.

2.2 Nakayama algebras

In this subsection, we recall properties of Nakayama algebras. A module M is said to be *uniserial* if it has a unique composition series. A finite dimensional algebra is said to be *Nakayama* if every indecomposable projective module and every indecomposable injective module are uniserial. Nakayama algebras is given by the following quivers.

Proposition 2.5. ([ASS, V.3.2]) *A basic ring-indecomposable algebra is Nakayama if and only if its quiver is either $\vec{A}_n$ or $\vec{\Delta}_n$.*

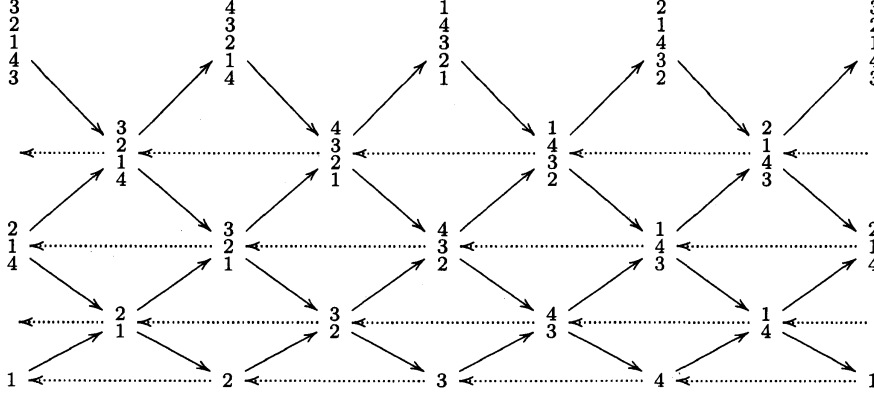


In the following, we assume that Λ is a basic ring-indecomposable Nakayama algebra with $n := |\Lambda|$. We give a concrete description of indecomposable Λ -modules. We denote by $\ell(M)$ the length of a Λ -module M .

Proposition 2.6. ([ASS, V.3.5, V.4.1 and V.4.2]) *For any indecomposable Λ -module M , there exist $i \in [1, n]$ and $l \in [1, \ell(P_i)]$ such that $M \simeq P_i / \text{rad}^l P_i$ and $l = \ell(M)$. Moreover, if M is not projective, then we have $\tau M \simeq \text{rad} P_i / \text{rad}^{l+1} P_i$ and $\ell(\tau M) = \ell(M)$.*

By Proposition 2.6, each indecomposable Λ -module M is uniquely determined, up to isomorphism, by its simple top S_j and the length $l := \ell(M)$. In this case, M has a unique composition series with the associated composition factors $S_j, S_{j-1}, \dots, S_{j-l+1}$.

We give an example of Nakayama algebras. We let $\Lambda_n^r := K\vec{\Delta}_n/J^r$, where J is the arrow ideal of $K\vec{\Delta}_n$. The Auslander-Reiten quiver of Λ_n^r can be drawn easily [ASS, V.4.1]. For example, the Auslander-Reiten quiver of Λ_4^5 is given by the following quiver, where the broken arrows mean the action of the AR translation τ :



2.3 Triangulations

In this subsection, we recall the definition and properties of triangulations. Let $\mathcal{G}_n$ be an n -regular polygon with a puncture. We label the points of $\mathcal{G}_n$ counterclockwise around the boundary by $1, 2, \dots, n$.

Definition 2.7. Let $i, j \in [1, n]$.

- (1) An *inner arc* $\langle i, j \rangle$ in $\mathcal{G}_n$ is a path from the point i to the point j homotopic to the boundary path $i, i+1, \dots, i+l = j \pmod{n}$, where l is the smallest positive integer satisfying $i+l = j \pmod{n}$ and $l \geq 2$. Then we call i an *initial* point, j a *terminal* point, and $\ell(\langle i, j \rangle) := l$ the *length* of the inner arc. By definition, $2 \leq \ell(\langle i, j \rangle) \leq n$ holds for any inner arc in $\mathcal{G}_n$.
- (2) A *projective arc* $\langle \bullet, j \rangle$ in $\mathcal{G}_n$ is a path from the puncture to the point j . Then we call j a *terminal* point.
- (3) An *admissible arc* is an inner arc or a projective arc. Namely,

$$\text{Arc}(n) := \{\text{admissible arcs in } \mathcal{G}_n\} = \{\langle i, j \rangle \mid i, j \in [1, n]\} \coprod \{\langle \bullet, j \rangle \mid j \in [1, n]\}.$$

Note that, if $i \neq j$, $\langle i, j \rangle$ and $\langle j, i \rangle$ are different arcs as the picture in Figure 1 shows.

Definition 2.8. (1) Two admissible arcs in $\mathcal{G}_n$ are called *compatible* if they do not intersect in $\mathcal{G}_n$ (except their initial and terminal points).

- (2) A *triangulation* of $\mathcal{G}_n$ is a maximal set of distinct pairwise compatible admissible arcs. We denote by $\mathcal{T}(n)$ the set of triangulations of $\mathcal{G}_n$.

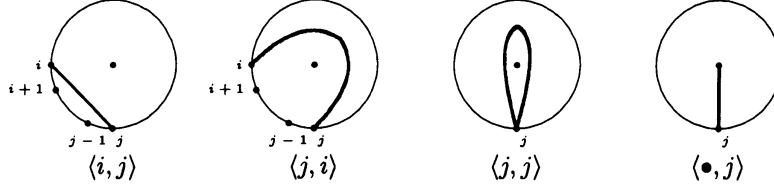
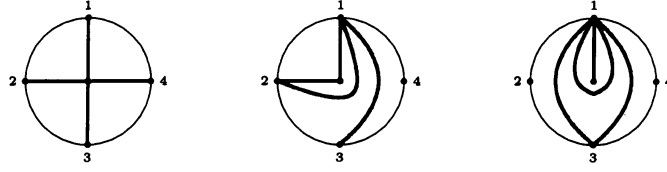


Figure 1: Admissible arcs in a polygon with a puncture

- (3) For integers $l_1, l_2, \dots, l_n \geq 1$, we denote by $T(n; l_1, l_2, \dots, l_n)$ the subset of $T(n)$ consisting of triangulations such that the length of every inner arc with the terminal point j is at most l_j for any $j \in [1, n]$.

For example, the set of all projective arcs gives a triangulation of $\mathcal{G}_n$.

Figure 2: Triangulations of $\mathcal{G}_4$

By easy observation, triangulations have the following properties.

Proposition 2.9. *Each triangulation of $\mathcal{G}_n$ consists of exactly n admissible arcs and contains at least one projective arc.*

3 Main result

In this section, we give a proof of Theorem 1.1. First, we give a criterion for indecomposable modules to be τ -rigid.

Proposition 3.1. *Let M be an indecomposable nonprojective Λ -module. Then M is τ -rigid if and only if $\ell(M) < n$ holds.*

Proof. By Proposition 2.6, we may assume that M (respectively, τM) has a (unique) composition series with associated composition factors $\text{top} M = S_j, S_{j-1}, \dots, S_{j-l+1} = \text{soc} M$ (respectively, $\text{top} \tau M = S_{j-1}, S_{j-2}, \dots, S_{j-l} = \text{soc} \tau M$), where $l := \ell(M) = \ell(\tau M)$. If $l < n$ holds, then M (respectively, τM) does not have S_{j-l} (respectively, S_j) as a composition factor. Thus we have $\text{Hom}_\Lambda(M, \tau M) = 0$, and hence M is τ -rigid. On the other hand, if $l \geq n$ holds, then M (respectively, τM) has S_{j-l} (respectively, S_j) as a composition factor. Thus there exists a non-zero morphism $M \rightarrow \tau M$ in $\text{mod } \Lambda$, and hence M is not τ -rigid. $\square$

Secondly, we give a correspondence between indecomposable τ -rigid modules and admissible arcs. By Proposition 3.1, every indecomposable nonprojective τ -rigid Λ -module M is uniquely determined by its simple top S_j and its simple socle S_k . Such an indecomposable τ -rigid module is denoted by $(k-2, j)$. Moreover, let $(\bullet, j) := P_j$.

Proposition 3.2. *Let Λ be a Nakayama algebra and $\ell_j := \ell(P_j)$. The following hold.*

(1) *There is a bijection*

$$f : \tau\text{-rigid}\Lambda \longrightarrow \{\langle \bullet, i \rangle \mid i \in [1, n]\} \coprod \{\langle i, j \rangle \mid i, j \in [1, n], \ell(\langle i, j \rangle) \leq \ell_j\}$$

given by $(i, j) \mapsto \langle i, j \rangle$ for $i \in [1, n] \coprod \{\bullet\}$ and $j \in [1, n]$.

(2) *For any $i, k \in [1, n] \coprod \{\bullet\}$ and $j, l \in [1, n]$, $(i, j) \oplus (k, l)$ is τ -rigid if and only if $\langle i, j \rangle$ and $\langle k, l \rangle$ are compatible.*

Proof. (1) By Proposition 3.1, every indecomposable Λ -module M is either a projective Λ -module or a Λ -module with $\ell(M) < \min\{\ell(P), n\}$, where P is a projective cover of M . Thus there are one-to-one correspondences

$$\begin{aligned} \{P_j \mid j \in [1, n]\} &\longleftrightarrow \{\langle \bullet, j \rangle \mid j \in [1, n]\} \\ \{(i, j) \mid i, j \in [1, n], \ell(\langle i, j \rangle) < \min\{\ell_j, n\}\} &\longleftrightarrow \{\langle i, j \rangle \mid i, j \in [1, n], \ell(\langle i, j \rangle) \leq \ell_j\}. \end{aligned}$$

(2) Assume that $(i, j) \oplus (k, l)$ is not τ -rigid. We may assume that $\text{Hom}_\Lambda((i, j), \tau(k, l)) \neq 0$ and $k \neq \{\bullet\}$. Then (i, j) (respectively, $\tau(k, l)$) has S_{k-1} (respectively, S_j) as a composition factor. Thus $\langle i, j \rangle$ and $\langle k, l \rangle$ are compatible. Conversely, we can easily check, if $\langle i, j \rangle$ and $\langle k, l \rangle$ are compatible, then $(i, j) \oplus (k, l)$ is τ -rigid. $\square$

As a conclusion, we obtain the following theorem. This is a generalization of Theorem 1.1.

Theorem 3.3. ([Ad]) *Let Λ be a Nakayama algebra with n simple modules and $\ell_j := \ell(P_j)$ for any $j \in [1, n]$. Then there is a bijection*

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n; \ell_1, \ell_2, \dots, \ell_n)$$

given by $M = M_1 \oplus M_2 \oplus \dots \oplus M_n \mapsto \{f(M_1), f(M_2), \dots, f(M_n)\}$.

Proof. It follows from Proposition 3.2. $\square$

As an application of Theorem 3.3, we give a proof of the following well-known result.

Corollary 3.4. *Let $\Lambda := K\vec{A}_n$ be a path algebra. Then there is a bijection between*

- (1) *the set $\text{tilt}\Lambda$ of isomorphism classes of basic tilting Λ -modules,*
- (2) *the set of triangulations of an $(n+2)$ -regular polygon (with no puncture).*

Proof. By Theorem 3.3 and $\ell(P_i) = i$ for any $i \in [1, n]$, we have a bijection

$$\tau\text{-tilt}\Lambda \longrightarrow \mathcal{T}(n; 1, 2, \dots, n).$$

Since Λ is hereditary, we have $\tau\text{-tilt}\Lambda = \text{tilt}\Lambda$ by Remark 2.3. On the other hand, we show that

$$\mathcal{T}(n; 1, 2, \dots, n) = \{X \in \mathcal{T}(n) \mid \langle \bullet, n \rangle \in X\}.$$

Indeed, assume that $X \in \mathcal{T}(n)$ with $\langle \bullet, n \rangle \in X$. Then we have $\ell(\langle i, j \rangle) \leq j$ for each inner arc $\langle i, j \rangle \in X$. Thus, we have $X \in \mathcal{T}(n; 1, 2, \dots, n)$. Conversely, assume that $X \in \mathcal{T}(n; 1, 2, \dots, n)$. Clearly, the projective arc $\langle \bullet, n \rangle$ is compatible with all admissible arc in X . Thus, we have $\langle \bullet, n \rangle \in X$. Note that $\mathcal{T}(n; 1, 2, \dots, n)$ can identify the set of triangulations of an $(n+2)$ -regular polygon (with no puncture). $\square$

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