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idiosyncratic risk in an OLG model”

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Taxing capital is a good idea: the role of idiosyncratic risk in an OLG model*

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Abstract

We investigate an overlapping generations model (OLG) model in which agents who live for two periods receive idiosyncratic productivity shocks when they are old. We show that a combination of lump-sum and linear capital taxes can always Pareto-improve the allocation, that is, it can raise the equilibrium welfare of one generation without affecting that of the others. As Dávila et al. (*Econometrica* (2012)) show, a capital reduction in one period raises the welfare levels of agents who are old in that period, but lowers that of the young agents, because it reduces their wages. We show that the government can compensate for these wage losses by additionally taxing the old agents, such that their welfare gains remain positive.

Keywords: idiosyncratic risk; capital tax, incomplete markets, overlapping generations

JEL classification: E5;

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1 Introduction

It is widely known that in most of the infinite horizon, representative agent models with complete markets, the optimal capital tax rate is zero (see Chamley (1986) and Judd (1985, 1987)). On the basis of these findings, Atkeson et al. (1999) and Chari and Kehoe (1999) suggest that taxing capital income should be avoided. A capital tax distorts decisions on current consumption and future consumption more severely than a labor income tax. Fiscal policies relying on capital income taxation leads to higher welfare costs than policies without such a tax.

Recently, several authors investigated the validity of the zero capital tax principle in an economy with incomplete markets and agent heterogeneity. By introducing uninsurable labor income shocks and credit constraints into a model with infinitely-lived agents, Aiyagari (1995) shows that the steady state optimal capital income tax rate is positive. Dávila et al. (2012) investigate a two-period production economy in which agents are identical ex-ante, but are heterogenous ex-post, that is, their second-period labor endowment is subject to idiosyncratic shocks. They find that the marginal effect of an increase in capital on each individual's expected utility is proportional to the covariance of the marginal utility of second period consumption and the labor endowment shock, the sign of which is negative. This means that the welfare effect of a reduction in capital is strictly positive. Dávila et al. (2012) then prove that a linear capital tax reduces capital and improves social welfare. They also investigate an infinite horizon version of the model with borrowing constraints and find numerically that a positive capital tax rate is still welfare-improving. A recent paper by Gottardi et al. (2012) extends the framework developed by Dávila et al. (2012) to allow for idiosyncratic shocks to investment and an endogenous labor supply. They show that when shocks to labor productivity are larger than are those to investment, the optimal capital tax rate is positive.

Moreover, in line with Dávila et al. (2012) and Gottardi et al. (2012), Carvajal and Polemarchakis (2011) incorporate uninsurable shocks into an overlapping generations (OLG) economy, as developed by Diamond (1965). In their model, agents who live for two periods are born in each period, and are subject to idiosyncratic productivity shocks

when they are old. Carvajal and Polemarchakis (2011) restrict their attention to the steady state equilibrium and study the marginal effect of a capital increase on steady state welfare. As a result, they find that the welfare effect consists of two terms; the first representing an intergenerational effect, proportional to the difference between the real interest rate and the population growth rate, and the second representing welfare losses owing to the idiosyncratic shocks. As is well known, the intergenerational effect is observed in homogenous agent OLG models such as that in the standard Diamond (1965) model. The second term, which is always negative, is because of idiosyncratic shocks and is almost the same as the effect found by Dávila et al. (2012). Without any shocks, we only observe the first term. In this case, the famous *golden rule applies*, which states that when the real interest rate is lower than the population growth rate, there is over-accumulation of the capital stock in the steady state. On the basis of their findings, Carvajal and Polemarchakis (2011) show that when an economy is subject to idiosyncratic labor endowment shocks, the golden rule argument should be modified. In other words, even when the real interest rate exceeds the population growth rate, a reduction in capital can raise *steady state* welfare. However, whether such a capital reduction in an OLG model will be a Pareto improvement in the economy is ambiguous.¹ This is because a capital reduction in one period may benefit old agents in that period, but would be harmful to young agents, as it lowers their wage income.

In this paper, we investigate an overlapping generation economy in which agents who live for two periods are born in each period.² Agents are ex-ante identical but are subject to idiosyncratic labor endowment shocks when they are old. This set-up is closely related to that of Carvajal and Polemarchakis (2011). However, we do *not* limit our attention to steady states and we study whether capital income taxation can raise the utility of one generation without affecting that of other generations. This paper shows that, for any

¹Carvajal and Polemarchakis (2011) also state that a capital reduction can Pareto-improve an economy. However, because they restrict their analyses to stationary equilibria and do not consider intergenerational heterogeneity, our analysis differs from theirs significantly.

²Fiorini (2008) investigates the effect of idiosyncratic shocks on the optimal policy in a monetary OLG economy and finds that in some situations inflation can raise steady state welfare.

competitive equilibrium allocation in which the government can only use a lump-sum tax, we can find a combination of lump-sum and linear capital taxes that Pareto-improves the market equilibrium. Here, we assume that the lump-sum tax can be age specific. We also show that the government can improve the welfare of every generation, including the initial old generation, through using the two tax policy instruments. Here, the initial old generation is also subject to idiosyncratic labor endowment shocks, but he does not make any savings decisions and simply consumes the after-tax wage income. Just as Dávila et al. (2012) show, a capital reduction in any period, for example period 1, raises the utility of old agents, who are born in period 0. Meanwhile, the capital reduction lowers the wage income of young agents in period 1 and reduces their welfare. However, the benefit from the capital reduction for the old agents is so large that their welfare gain is still positive, even if they are taxed additionally to compensate for the young agents' wage losses. Since this policy change in period 1 only affects the old and young agents in that period, it Pareto-improves the economy. We further show that we can find a government policy which improves the welfare of *every* generation, including the initial olds. In this paper, unlike Carvajal and Polemarchakis (2011), we do not restrict our attention to steady state welfare but treat the welfare of each generation separately.

It should be noted the welfare effect of capital taxation differs significantly from that of government bonds. As Diamond (1965), Ihori (1978) and Tirole (1985) show, government bonds reduce the equilibrium capital level. In this sense, their effect can be said to be similar to that of capital taxation. However, the welfare improvement seen in our economy requires an intertemporal wedge between current consumption and future consumption, which cannot be created by government bonds. Hence, government bonds cannot be a substitute for the distortionary capital tax of our model with agent heterogeneity.

Our paper also relates to that of Conesa et al. (2009), which investigates the optimal capital and income taxes in an overlapping generation economy, in which agents have an elastic labor supply and receive idiosyncratic productivity shocks in each period. Conesa et al. (2009) find that when agents are heterogeneous in their productivity, the optimal capital income tax rate is positive. Although this result seems to be very similar to our

owns, there are two significant differences between our study and that of Conesa et al. (2009). First, Conesa et al. (2009) allow labor income tax to be non-linear, whereas, in our paper, we only consider a lump-sum tax. Second, they focus on steady state welfare, as do Carvajal and Polemarchakis (2011), whereas we treat all generations differently and consider welfare improvements in the Pareto sense.³

This paper is organized as follows. Section 2 describes the basic structure of the model and characterizes the competitive equilibrium allocation. Section 3 shows that a linear capital tax can improve social welfare. Section 4 describes the way in which the welfare of every generation can be improved. Section 5 concludes the paper. The Appendix presents proofs of our propositions.

2 The Model

2.1 Production

Time is discrete and ranges from 0 to $+\infty$. A continuum of ex-ante identical individuals, with measure $N_t = N_0(1+n)^t$ ($t \geq 0$), is born in period t , where n is the population growth rate and $N_0 > 0$ is the number of agents born in period 0. Each agent lives for two periods. In period 0, there are initial old agents, with measure $N_{-1} = N_0/(1+n)$; they also receive the same idiosyncratic labor endowment shock at the beginning of period 0, but they make no savings decisions. In what follows, the group of individuals born in period t ($t \geq -1$) is called *generation t* , and the generation's population is N_t .

Agents supply their labor and receive wage incomes when they are young and old. Thus in period t , both generations $t-1$ and t work. The labor supply of the young, $\bar{l}$, is constant, but that of the old, e , is stochastic and takes a value $e^z > 0$ with probability

³Dávila (2012) studies the role of capital tax in a homogenous-agent, OLG economy with no productivity shocks. He assumes that only old agents are subject to the lump-sum tax and shows that a capital tax can improve steady state welfare. However, his conclusion does not hold when young agents are also taxed. Here, we show that when each agent receives idiosyncratic shocks, a linear capital tax is welfare improving, even if the government uses age-specific, lump-sum taxes on both the young and the old.

$\pi_z > 0$ ($z = 1, 2, \dots, Z$). The number of possible states is denoted by $Z (\geq 2)$. Here, we assume $e^z < e^{z+1}$ ($z = 1, 2, \dots, Z-1$) and $\sum_{z=1}^Z \pi_z = 1$. If the labor endowment of an old agent is e^z , we say that the agent's state is z . By the law of large numbers, the total amount of labor being provided at time t is $L_t = E[e]N_{t-1} + \bar{l}N_t = \{(1+n)\bar{l} + E[e]\}N_{t-1}$, where $E[e] = \sum_{z=1}^Z \pi_z e^z$. We normalize the unit of labor so that the following holds:

$$(1+n)\bar{l} + E[e] = 1. \quad (1)$$

Under this normalization, the level of total labor in a period is equal to the population in the previous period:

$$L_t = N_{t-1} \text{ for all } t \geq 0. \quad (2)$$

Later we prove that under condition (2) the equilibrium level of individual savings coincides with the capital-labor ratio.

There are many identical firms. The production function has constant returns to scale and is given by $F(K_t, L_t) \equiv f(k_t)L_t$, where K_t is aggregate capital in period t , $k_t = K_t/L_t$ denotes the capital-labor ratio in period t , and the per-capital production function, f , satisfies $f(0) = 0$, $f' > 0$, $f'' < 0$, $\lim_{k \rightarrow \infty} f(k) = +\infty$, and $\lim_{k \rightarrow 0} f'(k) = +\infty$. Factor markets are perfectly competitive; the equilibrium wage rate in period t , w_t , and the capital rental rate in period t , R_t , are respectively given by

$$\begin{aligned} w_t &= w(k_t) \equiv f(k_t) - k_t f'(k_t), \\ R_t &= R(k_t) \equiv f'(k_t) + 1 - \delta, \end{aligned}$$

where δ is the rate of capital depreciation. The initial old agents (generation -1) are endowed with the initial aggregate capital, K_0 , which is exogenous. Thus, the level of individual capital holdings is $K_0/N_{-1} = k_0$.

2.2 The consumer's problem

For all $t \geq 0$, generation t chooses their level of savings and maximizes their expected utility

$$U = u(c_t) + \beta E[u(d_{t+1,z})]$$

where u is the instantaneous utility function, β is the discount factor, c_t is the agent's consumption when he is young, and $d_{t+1,z}$ denotes his consumption when he is old and the state is z . The utility function u satisfies $u' > 0$, $u'' < 0$, and $\lim_{c \rightarrow 0} u'(c) = +\infty$. The budget constraints when an agent is young and old are, respectively,

$$\begin{aligned} c_t &= w_t \bar{l} - s_t + T_t^y, \\ d_{t+1,z} &= w_{t+1} e^z + (1 - \tau_{t+1}) R_{t+1} s_t + T_{t+1}^o. \end{aligned}$$

Here, s_t is the level of individual savings in period t , T_t^y is the lump-sum transfer to the young in period t , τ_{t+1} is the linear capital income tax rate in period $t + 1$, and T_{t+1}^o is the lump-sum transfer to the old in period $t + 1$. Here, the lump-sum tax is age-specific.⁴ The Euler equation of generation t is given by

$$u'(c_t) = (1 - \tau_{t+1}) R_{t+1} \beta E[u'(d_{t+1,z})]. \quad (3)$$

The initial old agents do not make any intertemporal choices and their utility with labor endowment e^z is $u(w_0 e^z + (1 - \tau_0^k) R_0 k_0 + T_0^o)$, which is exogenous.

2.3 Competitive equilibrium

In this section, we characterize the competitive equilibrium allocation. We first define an allocation.

Definition 1 *An allocation is a sequence $\{k_t, c_t, (d_{t,z})_{z=1}^Z\}_{t=0}^\infty$, where k_t is the capital-labor ratio in period t , c_t is the consumption of generation t when they are young, and $d_{t,z}$ is the consumption of generation $t - 1$ when they are old, and the state of the labor endowment shock is z .*

By the law of large numbers, the total consumption of generation $t - 1$ when he is old is equal to $E[d_{t,z}]$. Thus the allocation is feasible if and only if $N_t c_t + N_{t-1} E[d_{t,z}] + K_{t+1} =$

⁴Age-specific tax instruments are also considered by Erosa and Gavaix (2002). They show that if the government is not able to condition labor taxes on age then a positive capital income tax rate can be optimal. In contrast, in this paper, we show that if the government conditions lump-sum transfers on age, then a marginal introduction of capital income tax can always be Pareto-improving.

$F(K_t, L_t)$ for all $t \geq 0$. Dividing both sides of the feasibility condition by $L_t = N_{t-1}$, we obtain

$$(1+n)c_t + E[d_{t,z}] + (1+n)k_{t+1} = f(k_t). \quad (4)$$

The government policy is a sequence $\{\tau_t, T_t^y, T_t^o\}_{t=0}^\infty$, where τ_t is the rate of capital income tax in period t , T_t^y is the lump-sum transfer to young agents in period t , and T_t^o is the lump-sum transfer to old agents in period t . The budget constraint of the government is $\tau_t R_t K_t = N_t T_t^y + N_{t-1} T_t^o$. Dividing both sides of the equation by N_{t-1} , we get

$$\tau_t R_t k_t = (1+n)T_t^y + T_t^o, \quad (5)$$

for $t \geq 0$.

In a competitive equilibrium, the aggregate level of savings is equal to aggregate capital holdings, that is, $s_t N_t = K_{t+1}$. Using Eq. (2), we obtain

$$s_t = \frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{N_t} = k_{t+1}. \quad (6)$$

Therefore, the value of the level of individual savings in one period is equal to the capital-labor ratio in the next period.

We now formally define the competitive equilibrium.

Definition 2 A competitive equilibrium consists of an allocation, $\{k_t, c_t, (d_{t,z})_{z=1}^Z\}_{t=0}^\infty$, and a government policy, $\{\tau_t, T_t^y, T_t^o\}_{t=0}^\infty$, that satisfy the Euler equation, (3), the government budget constraint, (5), and the following budget constraints of agents for all $t \geq 0$:

$$c_t = w_t \bar{l} - k_{t+1} + T_t^y, \quad (7)$$

$$d_{t,z} = w_t e^z + (1 - \tau_t) R_t k_t + T_t^o. \quad (8)$$

Here, $R_t = R(k_t)$ and $w_t = w(k_t)$ are the factor prices and the initial capital K_0 is exogenously given.

It is straightforward to verify that Eqs. (5), (7), and (8) together imply the feasibility constraint (4).

3 The role of capital taxation

In this section, we show that for any competitive equilibrium allocation without capital taxation there exists another competitive equilibrium allocation with a linear capital tax that Pareto dominates the original allocation. We also characterize such Pareto-improving government policies.

3.1 Capital reduction

Let us pick any competitive equilibrium allocation $A = \{k_t, c_t, (d_{t,z})_{z=1}^Z\}_{t=0}^\infty$ under the zero capital tax policy $\{0, T_t^y, T_t^o\}_{t=0}^\infty$ which satisfies the government budget constraint $0 = (1+n)T_t^y + T_t^o$. This allocation is characterized by the following three equations:

$$c_t = w_t \bar{l} - k_{t+1} + T_t^y, \quad (9)$$

$$d_{t,z} = w_t e^z + R_t k_t + T_t^o, \quad (10)$$

$$u'(c_t) = \beta R_{t+1} E[u'(d_{t+1,z})], \quad (11)$$

where $w_t = w(k_t)$ and $R_t = R(k_t)$.

We next consider the following reallocations in period 0 and 1:

Reallocation in period 0: Reduce the savings level of each young individual (generation 0) from k_1 to x , with $k_1 > x$.

Reallocation in period 1: Transfer $m_1(x) = (1+n)(w_1 - w(x))\bar{l}$ units of consumption good from each old individual (generation 0) to the young (generation 1).

Following the first reallocation, the capital-labor ratio is changed to x . This raises the consumption of all agents in generation 0 by $k_1 - x > 0$ when they are young. Since the population of generation 1 is $(1+n)$ times larger than that of generation 0, the second reallocation raises the income of each individual in generation 1 by $m_1(x)/(1+n)$. Note that generations other than 0 and 1 are not affected by this allocation change. The sequence of the capital-labor ratio after the reallocation is given by $\{k_0, x, k_2, k_3, \dots\}$.

The income transfer from generation 0 to generation 1 is balanced and thus the allocation following the reallocation is feasible. However, at this point, it is not clear whether

we can implement the new allocation as a competitive equilibrium. Later we show that this is possible. The consumption bundle of generation 0 depends on x and is written as

$$\begin{aligned} c_0(x) &= c_0 + k_1 - x, \\ d_{1,z}(x) &= w(x)e^z + R(x)x - m_1(x) + T_1^o. \end{aligned}$$

It is obvious that $c_0(k_1) = c_0$ and $d_{1,z}(k_1) = d_{1,z}$. The reallocations do not alter the level of savings of generation 1 and it is equal to k_2 . Thus his consumption when he is young is $w(x)\bar{l} - k_2 + T_1^y + (1+n)^{-1}m_1(x) = w_1\bar{l} - k_2 + T_1^y$, which is the same the original level of consumption c_1 . This means that the loss in his wage income, owing to the capital reduction, is perfectly compensated by the subsidy $m_1(x)/(1+n)$. Since the allocation following period 2 is unchanged, the consumption bundle of generation t is $(\hat{c}_0(x), \hat{d}_{1,z}(x))$ when $t = 0$, and $(c_t, d_{t+1,z})$ when $t \geq 1$. It should be noted we do not yet know whether generation t can optimally choose the new consumption bundle under certain tax instruments.

3.2 Welfare improvement

We now show that the new allocation with lower capital raises the utility of generation 0. The utility depends on x , and, when we write it in the form of $v_0(x)$, is given by $v_0(x) = u(c_0 + k_1 - x) + \beta E[u(d_{1,z}(x))]$. From the assumption normalizing the labor unit, (1), the term $d_{1,z}(x) = w(x)(e^z + (1+n)\bar{l}) + R(x)x - w_1\bar{l}(1+n) + T_1^o$ can be re-written as

$$d_{1,z}(x) = w(x)(e^z - E[e^z]) + f(x) + (1-\delta)x - w_1\bar{l}(1+n) + T_1^o.$$

Differentiating the welfare function v_0 by x yields

$$v'_0(x) = -u'(c_0 + k_1 - x) + \beta E[\{w'(x)(e^z - E[e^z]) + R(x)\}u'(\hat{d}_{1,z}(x))]. \quad (12)$$

The first order condition under the original equilibrium allocation A is $u'(c_0) = \beta E[R_1 u'(d_{1,z})]$.

Thus when we evaluate Eq. (12) at $x = k_1$, we obtain

$$v'_0(k_1) = \beta w'(k_1) \text{cov}(e^z, u'(d_{1,z})) < 0. \quad (13)$$

Here, the strict inequality holds, because the labor endowment shock and the old period consumption is positively correlated, and the marginal utility, $u'(\cdot)$, is a decreasing function of consumption. Thus, as long as the new capital level, x , is less than k_1 , but is sufficiently close to k_1 , we have $v(x) > v(k_1)$. This implies that the new allocation improves the welfare of generation 0. As we have already shown above, the welfare of other generations are unaffected by the allocation change. Therefore, the new allocation is welfare improving in the Pareto sense.

Our conclusion regarding the desirability of capital reduction is closely related to the result obtained by Dávila et al. (2012), in their investigation of a two-period model with idiosyncratic shocks. The main difference between our OLG model and theirs is that, in Dávila et al. (2012), new agents do not enter the economy in the second period, and thus, they do not consider the negative effect of a capital reduction caused by one generation on the next. In our set-up, when we decrease the capital holdings of one generation, their utility increases, however, the utility of the next generation is reduced. Therefore, a Pareto-improvement cannot be obtained through capital reductions alone. In our study, we find that the birth of new agents in the second period increases the positive welfare effect of capital reduction. Therefore, even if we perfectly compensate for the wage loss experienced by generation 1, we can still raise the welfare of generation 0. Of course, if $n = -1$, that is, the new generation is not born in the first period, our model coincides with Dávila et al. (2012) and the income transfer from generation 0 to generation 1, $m_1(x) = (1 + n)(w_1 - w(x))\bar{l}$ becomes zero.

Our result is also related to that of Carvajal and Polemarchakis (2011), who investigate the welfare of agents in an OLG model. The main difference between the work of Carvajal and Polemarchakis (2011) and our owns is that unlike us, they focus on steady state welfare. When the analysis is restricted to steady states, a capital reduction affects the wage level both when an agent is young and when he is old. Carvajal and Polemarchakis (2011; p.20) argue "*...the golden rule may fail, and in an economy where the interest rate is higher than the growth of population, it may be that a Pareto improvement requires for every generation to save less.*" However, they use the words *Pareto improvement*

differently from us. They do not take into account the negative effect of a savings reduction on future generations. In our paper, we show that regardless of whether the interest rate is higher than the population growth rate, a combination of a linear capital tax and lump-sum tax can always raise the welfare of one generation without affecting that of other generations.

Finally, we fix the new capital labor ratio in period 1, x_1 , such that $v(x_1) > v(k_1)$ and let $\hat{A} = \{\hat{k}_t, \hat{c}_t, (\hat{d}_{t,z})_{z=1}^Z\}_{t=0}^\infty$ denote the new allocation. This is given by

$$\{\hat{k}_t\}_{t=0}^\infty = \{k_0, x_1, k_2, k_3, \dots\}, \quad (14)$$

$$\{\hat{c}_t\}_{t=0}^\infty = \{c_0(x_1), c_1, c_2, c_3, \dots\}, \quad (15)$$

$$\{\hat{d}_{t,z}\}_{t=0}^\infty = \{d_{0,z}, d_{1,z}(x_1), d_{2,z}, d_{3,z}, \dots\}. \quad (16)$$

3.3 Implementation with capital taxation

In this section, we show that the new allocation, $\hat{A}$, can be implemented as a competitive equilibrium using a linear capital tax. Let us consider the government policy, $\hat{G} = \{\hat{\tau}_t, \hat{T}_t^y, \hat{T}_t^o\}_{t=0}^\infty$, such that

$$(\hat{\tau}_t, \hat{T}_t^y, \hat{T}_t^o) = \begin{cases} \left(1 - \frac{u'(\hat{c}_0)}{\hat{R}_1 \beta E[u'(\hat{d}_{1,z})]}, T_1^y + \frac{m_1}{1+n}, T_1^o - m_1 + \hat{\tau}_1 \hat{R}_1 x_1\right) & \text{if } t = 1, \\ (0, T_t^y, T_t^o) & \text{if } t \neq 1, \end{cases} \quad (17)$$

where $\hat{R}_1 = R(x_1)$ and $m_1 = (1+n)(w_1 - w(x_1))\bar{l}$. The policy, $\hat{G}$, only differs from G when $t = 1$. It satisfies the government budget constraint because $(1+n)\hat{T}_1^y + \hat{T}_1^o - \hat{\tau}_1 R(x_1)x_1 = (1+n)T_1^y + T_1^o = 0$. To show that policy $\hat{G}$ implements $\hat{A}$ as a competitive equilibrium, we must first confirm that, the consumption bundle of agents in generation t , $(\hat{c}_t, \hat{d}_{t+1,z})$, is on their budget constraint and satisfies the Euler equation, if the sequence of capital labor ratio is $\{\hat{k}_t\}_{t=0}^\infty$. We must also verify that the level of savings of individuals in generation t coincides with $\hat{k}_{t+1}$ for all t .

As the policy change only affects generations 0 and 1, in the following analysis, we focus on their utility maximization problems. We first investigate the problem of generation 0 when the capital labor ratio in period 1 is x_1 . If an agent in generation 0 has a level of savings equal to s , his consumption bundle is $(c_0 + k_1 - s, w(x_1)e^z + (1 - \hat{\tau}_1)R(x_1)s + \hat{T}_1^o)$.

If $s = x_1$, one can easily check that the bundle coincides with $(\hat{c}_0, \hat{d}_{1,z})$ and then it is budget-feasible. Moreover, Eq. (17) implies that it satisfies the consumption Euler equation. Therefore, generation 0 optimally chooses the consumption bundle $(\hat{c}_0, \hat{d}_{1,z})$ and his optimal level of savings is x_1 . Second, we study generation 1. His consumption bundle when his level of savings is s is written as $(w(x)\bar{l} - s + \hat{T}_1^y, w_2 e^z + R_2 s + T_2^o)$. The bundle coincides with $(\hat{c}_1, \hat{d}_{2,z}) = (c_1, d_{2,z})$ when $s = k_2$. Moreover, as there is no capital tax in period 2, the first order condition on consumption coincides with Eq. (11). Therefore, generation 1 optimally chooses $(\hat{c}_1, \hat{d}_{2,z})$ and the optimal level of savings is equal to k_2 .

From these arguments, we obtain the following proposition.

Proposition 1 *For any equilibrium allocation A without linear capital income taxation, there exists an equilibrium allocation with a linear capital tax that Pareto dominates A .*

Eq. (17) shows that in the new competitive equilibrium allocation, the capital tax rate in period 1 is non-zero in general. However, the sign of the tax rate is uncertain. We now assume with regard to the per-capita production function, f .

Assumption 1. The per-capita production function, $f(k)$, satisfies $f'(k) + f''(k)k > 0$.

Example 1. The Cobb-Douglas function, $f(k) = Ak^\alpha$, with $A > 0$ and $\alpha < 1$ always satisfies Assumption 1, as $f'(k) + f''(k)k = \alpha^2 Ak^{\alpha-1} > 0$.

We have the following proposition regarding the sign of the linear capital tax rate.

Proposition 2 *Under Assumption 1, a strictly positive capital income tax Pareto improves the competitive equilibrium allocation.*

Proof. See the Appendix. ■

3.4 Government bonds

As many authors including Diamond (1965), Ithori (1978), and Tirole (1985), have shown, government bonds can also reduce the level of capital. In this sense, government bonds may have an effect similar to that of the linear capital tax. However, government bonds cannot be a substitute for a linear capital tax. To show this, let us consider a zero capital tax policy, $\{0, T_t^y, T_t^o\}_{t=0}^\infty$, together with a debt policy, $\{b_t\}_{t=0}^\infty$, where b_t denotes the per-capita bond holding of generation $t - 1$. Given the two policies, the sequence $\{k_t, c_t, (d_{t,z})_{z=1}^Z\}_{t=0}^\infty$ is a competitive equilibrium allocation if and only if it satisfies (i) the budget constraint of generation t ,

$$\begin{aligned} c_t &= w_t \bar{l} - k_{t+1} - b_{t+1} + T_t^y, \\ d_{t+1,z} &= w_{t+1} e^z + R_{t+1}(k_{t+1} + b_{t+1}) + T_{t+1}^o, \end{aligned}$$

(ii) the Euler equation,

$$u'(c_t) = R_{t+1} \beta E[u'(d_{t+1,z})], \quad (18)$$

and (iii) the government's budget constraint,

$$b_{t+1} = R_t b_t + T_t^y + T_t^o,$$

where $w_t = w(k_t)$ and $R_t = R(k_t)$.

For such an allocation, we can easily see that a zero capital tax policy without government bonds, $\{0, T_t^{by}, T_t^{bo}\}_{t=0}^\infty = \{0, T_t^y - b_{t+1}, T_t^o + R_t b_t\}_{t=0}^\infty$, implements the same allocation as a competitive equilibrium. This implies that the public debt can play a similar role to lump-sum taxes⁵, but it *cannot* be a substitute for a linear capital tax. As we see in Eq. (17), a welfare improvement requires a distortion to the Euler equation; however, public debt cannot generate an intertemporal wedge, as Eq. (18) shows. Therefore, for any competitive equilibrium allocation that involves government bonds but not capital taxation, we can Pareto-improve the allocation by introducing a linear capital tax.

⁵Ithori (1978) shows this point in the standard OLG model.

4 Welfare improvement of all generations

The previous section showed that a linear capital tax could improve the welfare of one generation without reducing the welfare of other generations. In this section, we characterize the government policy that, with a linear capital tax, improves the utility of every generation, including the initial old.

As shown in the previous section, for any competitive equilibrium allocation, $A = \{k_t, c_t, (d_{t,z})_{z=1}^Z\}_{t=0}^\infty$, under a zero capital tax policy, $\{0, T_t^y, T_t^o\}_{t=0}^\infty$, there exists another equilibrium allocation in which the welfare of generation 0 is improved and the welfare levels of other generations are unchanged. Here, we find another equilibrium allocation, namely $A^* = \{k_t^*, c_t^*, (d_{t,z}^*)_{z=1}^Z\}_{t=0}^\infty$, in which the welfare of all individuals is improved. First, given $\hat{A} = \{\hat{k}_t, \hat{c}_t, (\hat{d}_{t,z})_{z=1}^Z\}_{t=0}^\infty$ which satisfies Eqs. (14), (15) and (16), we consider the following reallocation, which raises the welfare of the initial old individuals:

Step 0: In period 0, transfer ϵ units of consumption good from each individual in generation 0 (the young) to generation -1 (the old) by reducing the young's consumption, where $\epsilon > 0$.

If ϵ is sufficiently small, the consumption of generation 0 when he is young is strictly positive and his welfare is still higher than the one under the original allocation, A . Fixing ϵ as such, and $A^1 = \{k_t^1, c_t^1, (d_{t,z}^1)_{z=1}^Z\}_{t=0}^\infty$ denotes the new allocation after Step 0. The new allocation is of the form

$$(k_t^1, c_t^1, d_{t,z}^1) = \begin{cases} (k_0, c_0 + k_1 - x_1 - \epsilon, d_{0,z} + \frac{\epsilon}{1+n}) & \text{if } t = 0, \\ (x_1, c_1, d_{1,z}(x_1)) & \text{if } t = 1, \\ (k_t, c_t, d_{t,z}) & \text{if } t > 1. \end{cases} \quad (19)$$

Note that we have already defined x_1 at the end of Section 3.2. For both the initial old (generation -1) and generation 0, the new allocation A^1 is better than the original allocation, A ; yet, other agents are indifferent between these two allocations. It should be noted that A^1 can no longer be implemented as a competitive equilibrium allocation when the government policy is $\hat{G}$.

We now consider the following reallocation on A^1 that improves the welfare of generation 1 without changing the welfare of generation 2.

Step 1: Find x_2 such that the following reallocations are feasible and raise generation 1's welfare:

- i) In period 1, reduce the level of savings of generation 1 (the young) from k_2 to x_2 .
- ii) In period 2, transfer $m_2 = (1+n)(w_2 - w(x_2))\bar{l} > 0$ units of consumption good from generation 1 (the old) to generation 2 (the young).

We showed in the previous section that such x_2 always exists. The new allocation, say $A^2 = \{k_t^2, c_t^2, (d_{t,z}^2)_{z=1}^Z\}_{t=0}^\infty$, is of the form

$$(k_t^2, c_t^2, d_{t,z}^2) = \begin{cases} (k_0, c_0 + k_1 - x_1 - \epsilon, d_{0,z} + \frac{\epsilon}{1+n}) & \text{if } t = 0, \\ (x_1, c_1 + k_2 - x_2, d_{1,z}(x_1)) & \text{if } t = 1, \\ (x_2, c_2, d_{2,z}(x_2)) & \text{if } t = 2, \\ (k_t, c_t, d_{t,z}) & \text{if } t \geq 3. \end{cases} \quad (20)$$

Note that generations -1 , 0 and 1 are welfare-improved at A^2 and the welfare of the other generations are unchanged.

We repeat similar reallocations to raise the welfare of every generation. Suppose that a sequence of new capital labor ratio $\{x_1, x_2, \dots, x_i\}$ is fixed for some integer $i \geq 2$. (It is actually fixed when $i = 2$). Let the allocation $A^i = \{k_t^i, c_t^i, (d_{t,z}^i)_{z=1}^Z\}_{t=0}^\infty$ be of the form

$$(k_t^i, c_t^i, d_{t,z}^i) = \begin{cases} (k_0, c_0 + k_1 - x_1 - \epsilon, d_{0,z} + \frac{\epsilon}{1+n}) & \text{if } t = 0, \\ (x_t, c_t + k_{t+1} - x_{t+1}, d_{t,z}(x_t)) & \text{if } 1 \leq t \leq i-1, \\ (x_i, c_i, d_{i,z}(x_i)) & \text{if } t = i, \\ (k_t, c_t, d_{t,z}) & \text{if } t \geq i+1, \end{cases} \quad (21)$$

where $d_{t,z}(x) = w(x)e^z + R(x)x + T_t^o - (1+n)(w_t - w(x))\bar{l}$. Then consider the following reallocation on A^i that improves the welfare of generation i without changing the welfare of generation $i+1$:

Step i : Find x_{i+1} such that the following reallocations raise generation i 's welfare:

- i) In period i , reduce the level of savings of generation i (the young) from k_{i+1} to x_{i+1} .
- ii) In period $i+1$, transfer $m_{i+1} = (1+n)(w_{i+1} - w(x_{i+1}))\bar{l} > 0$ units of consumption good from generation i (the old) to generation $i+1$ (the young).

After Step i , the sequence $\{x_1, x_2, \dots, x_i, x_{i+1}\}$ is fixed and we can determine A^{i+1} . Continuing Step i for all integers i , we get the following allocation $A^* = \{k_t^*, c_t^*, (d_{t,z}^*)_{z=1}^Z\}_{t=0}^\infty$:

$$(k_t^*, c_t^*, d_{t,z}^*) = \begin{cases} (k_0, c_0 + k_1 - x_1 - \epsilon, d_{0,z} + \frac{\epsilon}{1+n}) & \text{if } t = 0, \\ (x_t, c_t + k_{t+1} - x_{t+1}, d_{t,z}(x_t)) & \text{if } t \geq 1. \end{cases} \quad (22)$$

As step i improves the welfare of generation i while that of other generations remains unchanged, the welfare of every generation is higher under allocation A^* than A .

In the following proposition, we show that the new allocation can be implemented as a competitive equilibrium.

Proposition 3 *Let $(R_t^*, w_t^*) = (R(k_t^*), w(k_t^*))$ and $m_t = (1+n)(w_t - w_t^*)\bar{l}$. The government policy $\{\tau_t^*, T_t^{*y}, T_t^{*o}\}_{t=0}^\infty$ such that*

$$(\tau_t^*, T_t^{*y}, T_t^{*o}) = \begin{cases} (0, T_0^y - \epsilon, \frac{\epsilon}{1+n} + T_0^o) & \text{if } t = 0, \\ \left(1 - \frac{u'(c_t^*)}{R_{t+1}^* \beta E[u'(d_{t+1,z}^*)]}, \frac{m_t}{1+n} + T_t^y, \tau_t^* R_t^* k_t^* + T_t^o - m_t\right) & \text{if } t \geq 1, \end{cases} \quad (23)$$

implements $A^ = \{k_t^*, c_t^*, (d_{t,z}^*)_{z=1}^Z\}_{t=0}^\infty$ as a competitive equilibrium allocation. Thus it raises the equilibrium welfare of every generation.*

Proof. See the Appendix. ■

5 Conclusion

In this paper, we investigated a two-period overlapping generations model with capital accumulation in which agents receive idiosyncratic productivity shocks when they are old. We showed that a combination of lump-sum and linear capital taxes always Pareto improves the competitive equilibrium allocation. In other words, governments could raise the welfare of members of some generation without changing the welfare levels of other generations. As shown by previous studies, capital reduction in a period raises the welfare of agents who are old in that period. However, it lowers the welfare of young agents because it reduces their wage income. We showed that the government could compensate for the wage loss of the young agents by additionally taxing the old generation while keeping the welfare gains of the old agents positive.

Appendix

A Proof of Proposition 2

As $x_1 < k_1$, we have $u'(c_0) > u'(\hat{c}_0)$ and $R_1 < R(x_1)$. Moreover, under Assumption 1, $d\{R(k)k\}/dk > 0$. Thus given z , $d_{1,z}(x) = w(x)(e^z + (1+n)\bar{l}) + R(x)x - w_1\bar{l}(1+n) + T_1^o$ is an increasing function of x and we have $d_{1,z} > d_{1,z}(x_1)$. Therefore, the welfare-improving capital tax satisfies

$$1 = \frac{u'(c_0)}{R_1\beta E[u'(d_{1,z})]} > \frac{u'(\hat{c}_0)}{R(x)\beta E[u'(\hat{d}_{1,z}(x))]} = 1 - \tau_1,$$

and this implies $\tau_1 > 0$. ■

B Proof of Proposition 3

We first show that when the capital-labor ratio in period t is k_t^* , generation t optimally chooses the consumption bundle $(c_t^*, d_{t+1,z}^*)$ and his level of savings coincides with k_{t+1}^* . When the level of savings of generation t is s , his consumption bundle, say $(\tilde{c}_t, \tilde{d}_{t+1,z})$, is given by

$$\begin{aligned}\tilde{c}_t &= w_t^*\bar{l} - s + \frac{m_t}{1+n} + T_t^y, \\ \tilde{d}_{t+1,z} &= w_{t+1}^*e^z + (1 - \tau_{t+1}^*)R_{t+1}^*s + \tau_{t+1}^*R_{t+1}^*k_{t+1}^* + T_{t+1}^o - m_{t+1}.\end{aligned}$$

Since $k_t^* = x_t$ for all t , we have $\tilde{c}_t = c_t + k_{t+1} - k_{t+1}^* = c_t^*$ and $\tilde{d}_{t+1,z} = w_{t+1}^*e^z + R_{t+1}^*k_{t+1}^* + T_{t+1}^o - m_{t+1} = d_{t+1,z}(k_{t+1}^*) = d_{t+1,z}^*$ when $s = k_{t+1}^*$. Thus, the consumption bundle $(c_t^*, d_{t+1,z}^*)$ is feasible. Moreover, capital tax rate τ_{t+1}^* is determined such that the consumption bundle satisfies the first order condition $u'(c_t^*) = (1 - \tau_{t+1}^*)R_{t+1}^*\beta E[u'(d_{t+1,z}^*)]$. Hence, generation t optimally chooses the level of savings k_{t+1}^* . Next, the consumption of the initial old is $w_0^*e^z + \frac{\epsilon}{1+n} + T_0^o = d_{0,z} + \frac{\epsilon}{1+n}$, which is equal to $d_{0,z}^*$. Finally, we can easily check that the policy satisfies the government budget constraint if the capital labor ratio in period t is k_t^* . Thus the tax policy implements A^* as a competitive equilibrium. ■

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