

# On the principal series representation of $SU(2, 2)$

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## 1 Introduction

Let  $G$  denote the special unitary group  $SU(2, 2)$ . In the paper, we will deal with the principal series representations of  $G$  which are parabolically induced by the minimal parabolic subgroup  $P_{min}$  with Langlands decomposition  $P_{min} = MAN$ ;

$$\pi_{\sigma, \nu} = \text{Ind}_{P_{min}}^G (\sigma \otimes e^{\nu+\rho} \otimes 1_N),$$

where  $\rho$  is the half sum associated to the root system of the pair  $(G, A)$ ,  $\nu$  is a complex valued real linear form on  $\mathfrak{a} = \text{Lie}(A)$ ,  $\sigma$  is a unitary character of  $M$ .

Let  $\eta$  be a continuous unitary character of  $N$ . We then have the Jacquet functional  $J_{\sigma, \nu}$  on the space of differentiable functions of  $L^2_{\sigma}(K)$ , the representation space of  $\pi_{\sigma, \nu}$ , such that  $J_{\sigma, \nu}(\pi_{\sigma, \nu}(n)f) = \eta(n)J_{\sigma, \nu}(f)$  for any  $n \in N$ . The functional defines an intertwiner  $J$  from  $\pi_{\sigma, \nu}|_K$  to  $\mathcal{A}_{\eta}(N \backslash G)$  by sending any  $v \in \pi_{\sigma, \nu}|_K$  to the function  $J_v(g) := J_{\sigma, \nu}(\pi_{\sigma, \nu}(g)v)$ , ( $g \in G$ ). Here the subspace of all  $K$ -finite vectors of  $\pi_{\sigma, \nu}$  is denoted by  $\pi_{\sigma, \nu}|_K$  and  $\mathcal{A}_{\eta}(N \backslash G)$  is the subspace of  $C^{\infty}(G)$  consisting of all moderate growth functions  $f(g)$  such that  $f/ng = \eta(n)f(g)$  for  $n \in N$  and  $g \in G$ . In fact,  $J$  is an intertwiner of  $K$  and  $\mathfrak{g}$ -equivariant, and hence the study of the image of  $J$  (the Whittaker model) leads us to the problem of the investigations of the  $(\mathfrak{g}, K)$ -module structure and the functions  $J_v(g)$  for certain  $K$ -types of  $\pi_{\sigma, \nu}$ .

The main goal of this paper is to describe the above mentioned objects in terms of parameters of the principal series representation  $\pi_{\sigma, \nu}$  explicitly. Note that our results are quite similar to that of Ishii [4] and Oda [5], for both  $Sp(2, \mathbb{R})$  and  $SU(2, 2)$  have the same restricted root system.

We also consider a matrix representations of the Knapp-Stein intertwining operator which have been motivated by a result of Goodman-Wallach [2].

## 2 Preliminaries

Let  $K$  be the compact group  $S(U(2) \times U(2))$ . Then  $K$  is the maximal compact subgroup of  $G$  fixed by the Cartan involution  $\theta$  for  $G$  given by

$$\theta(g) = {}^t \bar{g}^{-1}, \quad g \in G.$$

We fix the following basis for the 7 dimensional Lie algebra  $\mathfrak{k}_{\mathbb{C}}$ , the complexification of  $\mathfrak{k} = \text{Lie}(K)$ :

$$\begin{aligned} h^1 &= \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, & h^2 &= \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, & I_{2,2} &= \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \\ e_{\pm}^1 &= \begin{pmatrix} e_{\pm} & 0 \\ 0 & 0 \end{pmatrix}, & e_{\pm}^2 &= \begin{pmatrix} 0 & 0 \\ 0 & e_{\pm} \end{pmatrix}, \end{aligned}$$

where  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ .

For every  $K$ -module  $V$ , it is clear that  $I_{2,2} \in \mathfrak{k}_{\mathbb{C}}$  commutes with the action of  $K$  on  $V$ . If  $V$  is irreducible, then by Schur's lemma, the operator is a scalar of the identity map.

**Lemma 2.1** *Let  $m_1, m_2$  be positive integers and  $l$  be an integer. If  $m_1 + m_2 + l$  is an even integer, then there is an irreducible  $K$ -module  $(\tau_{[m_1, m_2; l]}, V_{m_1, m_2})$  with a basis  $\{f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$  of  $V_{m_1, m_2}$  such that  $I_{2,2}f_{pq} = lf_{pq}$  and*

$$\begin{aligned} h^1(f_{pq}) &= (2p - m_1)f_{pq}, & e_+^1(f_{pq}) &= (m_1 - p)f_{p+1,q}, & e_-^1(f_{pq}) &= pf_{p-1,q}, \\ h^2(f_{pq}) &= (2q - m_2)f_{pq}, & e_+^2(f_{pq}) &= (m_2 - q)f_{p,q+1}, & e_-^2(f_{pq}) &= qf_{p,q-1}. \end{aligned}$$

It follows from the fact that  $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$  is a twofold covering of  $K$  with the projection given by

$$pr(g_1, g_2; u) = \text{diag}(ug_1, u^{-1}g_2), \quad g_1, g_2 \in SU(2), \quad u \in \mathbb{C}^{(1)}.$$

### 3 $K$ -finite vectors in the principal series

In this section, for each simple  $K$ -module  $\tau \in \hat{K}$ , we associate a matrix function  $\mathbf{S}_{\sigma, \nu}^{(\tau)}(k)$ ,  $k \in K$ , whose entries give a basis for the  $\tau$ -isotypic component of  $\pi_{\sigma, \nu}$ . The main feature of this basis is that the both  $\mathfrak{g}$  and  $K$ -actions on  $\pi_{\sigma, \nu}|_K$  have simple expressions in terms of parameters of given representation. For more details about this theme, we refer to [5] which is our main reference.

**Proposition 3.1** *Let  $H(\tau)$  be the  $\tau$ -isotypic component of  $L^2(K)$ , and put  $\dim(\tau) = n$ . There exists a unique square matrix function  $\mathbf{S}^{(\tau)}(k)$ ,  $k \in K$ , of size  $n$  with entries in  $H(\tau)$ ,*

$$\mathbf{S}^{(\tau)}(k) = \begin{bmatrix} f_{11}(k) & \cdots & f_{n1}(k) \\ \vdots & \ddots & \vdots \\ f_{1n}(k) & \cdots & f_{nn}(k) \end{bmatrix} = \{f_{ij}(k)\}_{1 \leq i, j \leq n},$$

satisfying the following two conditions:

1.  $\mathbf{S}^{(\tau)}(1_K) = \text{diag}(1, \dots, 1) \in M_n(\mathbb{C})$ ,
2. For each  $\alpha$  ( $1 \leq \alpha \leq n$ ), the set  $\{f_{\alpha 1}(k), \dots, f_{\alpha n}(k)\}$  is a basis for  $\tau$  as in Lemma 2.1. Moreover, we have

$$H(\tau) = \bigoplus_{\alpha} W_{\alpha},$$

where  $W_{\alpha}$  denotes the space spanned by  $f_{\alpha 1}(k), \dots, f_{\alpha n}(k)$ .

*Proof.* The existence of the matrix function is similar to that of [5]. We consider the uniqueness. Assume that there exist two matrices  $\mathbf{F}^{(\tau)}(k) = \{f_{ij}(k)\}$  and  $\mathbf{G}^{(\tau)}(k) = \{g_{ij}(k)\}$  as required. Denote by  $F_{\alpha}$  the isomorphism between  $\tau$  and the space spanned by  $\{f_{\alpha j}(k), \dots, f_{\alpha n}(k)\}$ . Similarly, we define  $G_{\alpha}$  for the  $\alpha$ -th column of  $\mathbf{G}^{(\tau)}(k)$ . As a result, we obtain two ordered bases  $\{F_{\alpha}\}_{\alpha}$  and  $\{G_{\alpha}\}_{\alpha}$  for the  $n$ -dimensional vector space  $\text{Hom}_K(\tau, H(\tau))$ . Then we have the  $n$  by  $n$  matrix  $A = \{a_{\alpha\beta}\}$ , the change of coordinate matrix, such that

$$F_{\alpha} = \sum_{\beta} a_{\alpha\beta} G_{\beta}.$$

For a basis  $\{f_{\gamma}\}$  of  $\tau$ , one obtains

$$f_{\alpha\gamma}(k) = F_{\alpha}(f_{\gamma}) = \sum_{\beta} a_{\alpha\beta} G_{\beta}(f_{\gamma}) = \sum_{\beta} a_{\alpha\beta} f_{\beta\gamma}(k).$$

Evaluation at the point  $1_K$  shows that

$$a_{\alpha\gamma} = \delta_{\alpha\gamma}.$$

If  $v \neq 0 \in W_\alpha \cap W_\beta$ , then  $Kv = W_\alpha = W_\beta$ . Schur's lemma and second condition imply that  $\alpha = \beta$ . Assume there is a matrix  $\mathbf{S}^{(\tau)}(k)$  as required, we then have the direct sum decomposition of  $H(\tau)$ .  $\square$

For each  $\tau_m = \tau_{[m_1, m_2; l]} \in \hat{K}$ , define a finite set  $I(\tau_m)$  to be the collection of indices  $\alpha$  such that  $W_\alpha$  occurs in  $\pi_{\sigma, \nu} |_K$  as a  $K$ -module. Thus, the cardinality of  $I(\tau_m)$  is the  $K$ -multiplicity of  $\tau_m$  in  $\pi_{\sigma, \nu}$ . Let  $s$  be the integer parameter corresponding to  $\sigma \in \hat{M}$ . By setting  $n = (m_1 + m_2 + s)/2$ , one can see that  $p + q = n$  if  $\alpha \in I(\tau_m)$  with  $\alpha = (m_2 + 1)p + q + 1, (q \leq m_2)$ . We identify the index  $\alpha$  with the pair  $(p, q)$  defined by  $\alpha$ .

We define a matrix function  $\mathbf{S}_{\sigma, \nu}^{(\tau_m)}(k)$  attached to the  $\tau$ -isotypic component of  $\pi_{\sigma, \nu}$  by eliminating all the  $\alpha$ -th columns of  $\mathbf{S}^{(\tau_m)}(k)$  when  $p + q \neq n$  and change the  $\alpha$ -th columns by 0 if  $\alpha \notin I(\tau_m)$  and  $p + q = n$ .

## 4 The $(\mathfrak{g}, K)$ -module structure on $\pi_{\sigma, \nu}$

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the Cartan decomposition of  $\mathfrak{g} = \text{Lie}(G)$  corresponding to  $\theta$ . In this section, we explicitly describe  $\mathfrak{p}_\mathbb{C}$ -action on the space

$$\pi_{\sigma, \nu} |_K \cong \bigoplus_{\tau_m \in \hat{K}} \bigoplus_{\alpha \in I(\tau_m)} W_\alpha.$$

Since the adjoint representation of  $K$  on  $\mathfrak{p}_\mathbb{C}$  splits into two irreducible components, the antiholomorphic part  $\mathfrak{p}_-$  and the holomorphic part  $\mathfrak{p}_+$ , it is enough to investigate the  $\mathfrak{p}_+$ -action for our purpose. Let  $E_{ij}$  be the matrix unit of  $M_4(\mathbb{R})$  with 1 in the  $(i, j)$ -entry and zero elsewhere. Then the set  $\{E_{ij} \mid i = 1, 2, j = 3, 4\}$  forms a basis for  $\mathfrak{p}_+$ . For a fixed pair  $(e_1, e_2)$ ,  $e_j \in \{\pm 1\}$  with  $j = 1, 2$ , we define  $\mathbf{c}_t^j$  by

$$\mathbf{c}_t^j = \frac{t}{m_j + 1} \quad (0 \leq t \leq m_j + e_j).$$

Let  $(\tau_m, V_m)$  be an irreducible representation of  $K$  with parametrization  $m = [m_1, m_2; l]$ . By the well known Clebsch-Gordan theorem, the irreducible components in the  $K$ -module  $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$  are precisely the  $K$ -representations

$$T = \{ \tau_{[m_1+e_1, m_2+e_2; l+2]} \mid e_1, e_2 \in \{\pm 1\} \},$$

and we will denote these by  $\tau_{[\pm, \pm; +]}$  or  $\tau_{[e_1, e_2; +]}$ .

For each  $K$ -isomorphism between  $\tau_m$  and  $W_\alpha$  in Proposition 3.1, we have the following surjective homomorphism  $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \rightarrow \mathfrak{p}_+ W_\alpha$  of  $K$ -modules. Therefore, we obtain an injection

$$\mathfrak{p}_+ H_{\sigma, \nu}(\tau_m) \hookrightarrow \bigoplus_{\tau_{m'} \in T} H_{\sigma, \nu}(\tau_{m'})$$

which implies the following theorem. Here  $H_{\sigma, \nu}(\tau_m)$  stands for the  $\tau_m$ -isotypic component of  $\pi_{\sigma, \nu}$ .

**Theorem 4.1** *Let  $\tau_{[e_1, e_2; +]}$  be a simple  $K$ -submodule of the  $K$ -module  $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$  for a given simple  $K$ -module  $\tau_m$  and the  $K$ -module  $(\text{Ad}, \mathfrak{p}_+)$ . Then we have that*

$$\mathcal{C}_{[e_1, e_2; +]} \mathbf{S}_{\sigma, \nu}^{(\tau_m)}(k) = \mathbf{S}_{\sigma, \nu}^{(\tau_{[e_1, e_2; +]})}(k) \Gamma_{[e_1, e_2; +]},$$

where the product of matrices of the left hand side is the differential operation. Here,  $r = (s + l)/2$  and

1.  $\Gamma_{[-,-,+]} = \{a_{ij}\}_{0 \leq i \leq n-1, 0 \leq j \leq n}$  is a matrix whose all non zero entries are given by

$$\begin{aligned} a_{t-1,t} &= \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t) & \text{if } (t, n-t) \in I(\tau_m), (t-1, n-t) \in I(\tau_{m'}), \\ a_{t,t} &= -\frac{1}{2}(\nu_1 - 1 - m_2 + r - 2t) & \text{if } (t, n-t) \in I(\tau_m), (t, n-t-1) \in I(\tau_{m'}). \end{aligned}$$

and  $C_{[-,-,+]} = \{C_{ij}\}$  is a matrix of size  $(m_1 m_2) \times (m_1 + 1)(m_2 + 1)$  with entries given by

$$\begin{aligned} C_{m_2 p + q + 1, (m_2 + 1)p + q + 1} &= -E_{14}, \\ C_{m_2 p + q + 1, (m_2 + 1)p + q + 2} &= -E_{13}, \\ C_{m_2 p + q + 1, (m_2 + 1)(p+1) + q + 1} &= E_{24}, \\ C_{m_2 p + q + 1, (m_2 + 1)(p+1) + q + 2} &= E_{23}, \end{aligned}$$

for each  $0 \leq p \leq m_1 - 1$  and  $0 \leq q \leq m_2 - 1$ , but all other entries are 0.

2.  $\Gamma_{[+,+,+]} = \{a_{ij}\}_{0 \leq i \leq n+1, 0 \leq j \leq n}$  is a matrix whose all non zero entries are given by

$$\begin{aligned} a_{t,t} &= \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)(1 - c_t^1) c_{\nu-t+1}^2 & \text{if } (t, n-t) \in I(\tau_m), (t, n-t+1) \in I(\tau_{m'}), \\ a_{t+1,t} &= \frac{1}{2}(\nu_1 + 3 + 2m_1 + m_2 + r - 2t) c_{t+1}^1 (c_{\nu-t}^2 - 1) & \text{if } (t, n-t) \in I(\tau_m), (t+1, n-t) \in I(\tau_{m'}). \end{aligned}$$

and  $C_{[+,+,+]} = \{C_{ij}\}$  is a matrix of size  $(m_1 + 2)(m_2 + 2) \times (m_1 + 1)(m_2 + 1)$  with entries given by

$$\begin{aligned} C_{(m_2+2)p+q+1, (m_2+1)p+q+1} &= -(1 - c_p^1)(1 - c_q^2) E_{23}, \\ C_{(m_2+2)p+q+1, (m_2+1)p+q} &= (1 - c_p^1) c_q^2 E_{24}, \\ C_{(m_2+2)p+q+1, (m_2+1)(p-1)+q+1} &= -c_p^1 (1 - c_q^2) E_{13}, \\ C_{(m_2+2)p+q+1, (m_2+1)(p-1)+q} &= c_p^1 c_q^2 E_{14}, \end{aligned}$$

for each  $0 \leq p \leq m_1 + 1$  and  $0 \leq q \leq m_2 + 1$ , but all other entries are 0.

3.  $\Gamma_{[-,+,+]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$  is a square matrix whose all non zero entries are given by

$$\begin{aligned} a_{t-1,t} &= \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t) c_{\nu-t+1}^2 & \text{if } (t, n-t) \in I(\tau_m), (t-1, n-t+1) \in I(\tau_{m'}), \\ a_{t,t} &= \frac{1}{2}(\nu_1 + 1 + m_2 + r - 2t)(1 - c_{\nu-t}^2) & \text{if } (t, n-t) \in I(\tau_m), (t, n-t) \in I(\tau_{m'}). \end{aligned}$$

and  $C_{[-,+,+]} = \{C_{ij}\}$  is a matrix of size  $m_1(m_2 + 2) \times (m_1 + 1)(m_2 + 1)$  with entries given by

$$\begin{aligned} C_{(m_2+2)p+q+1, (m_2+1)p+q+1} &= (1 - c_q^2) E_{13}, \\ C_{(m_2+2)p+q+1, (m_2+1)p+q} &= -c_q^2 E_{14}, \\ C_{(m_2+2)p+q+1, (m_2+1)(p+1)+q+1} &= -(1 - c_q^2) E_{23}, \\ C_{(m_2+2)p+q+1, (m_2+1)(p+1)+q} &= c_q^2 E_{24}, \end{aligned}$$

for  $0 \leq p \leq m_1 + 1$  and  $0 \leq q \leq m_2 - 1$ , but all other entries are 0.

4.  $\Gamma_{[+,-,+]} = \{a_{ij}\}_{0 \leq i \leq n, 0 \leq j \leq n}$  is a square matrix whose all non zero entries are given by

$$\begin{aligned} a_{t,t} &= \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)(1 - c_t^1) & \text{if } (t, n-t) \in I(\tau_m), (t, n-t) \in I(\tau_{m'}), \\ a_{t+1,t} &= \frac{1}{2}(\nu_1 + 1 + 2m_1 - m_2 + r - 2t) c_{t+1}^1 & \text{if } (t, n-t) \in I(\tau_m), (t+1, n-t-1) \in I(\tau_{m'}). \end{aligned}$$

and  $C_{[+, -, +]} = \{C_{ij}\}$  is a matrix of size  $(m_1 + 2)m_2 \times (m_1 + 1)(m_2 + 1)$  with entries given by

$$\begin{aligned} C_{m_2 p + q + 1, (m_2 + 1)p + q + 1} &= (1 - c_p^1)E_{24}, \\ C_{m_2 p + q + 1, (m_2 + 1)p + q + 2} &= (1 - c_p^1)E_{23}, \\ C_{m_2 p + q + 1, (m_2 + 1)(p - 1) + q + 1} &= c_p^1 E_{14}, \\ C_{m_2 p + q + 1, (m_2 + 1)(p - 1) + q + 2} &= c_p^1 E_{13}, \end{aligned}$$

for  $0 \leq p \leq m_1 + 1$  and  $0 \leq q \leq m_2 - 1$ , but all other entries are 0.

#### 4.0.1 The Knapp-Stein operator

In this subsection, we consider a matrix representation of the Knapp-Stein operator with respect to the basis for  $\pi_{\sigma, \nu} |_K$ . This is motivated by Theorem 6.7 in the paper of Goodman-Wallach [2].

Let us recall the Knapp-Stein intertwining operator  $A_{\sigma, \nu}^s$  from the space of all  $C^\infty$ -vectors of  $\pi_{\sigma, \nu}$  to that of  $\pi_{s(\sigma), s(\nu)}$  defined by

$$(A_{\sigma, \nu}^s f)(k) = \int_{\tilde{N}_s} a(n_s s^* k)^{\nu + \rho} f(k(n_s s^* k)) dn_s, \quad (f \in \pi_{\sigma, \nu}^\infty).$$

Here  $s^* \in K$  such that  $s := \text{Ad}(s^*) \in W(A)$ ,  $\tilde{N}_s = N \cap s^* N s^{*-1}$  and  $s(\sigma)$  is a character of  $M$  given by  $s(\sigma)(m) = \sigma(s^* m s^{*-1})$ ,  $m \in M$ . Since it is a linear map from  $\pi_{\sigma, \nu}$  to  $\pi_{s(\sigma), s(\nu)}$  satisfying

$$A_{\sigma, \nu}^s \pi_{\sigma, \nu}(x) f = \pi_{s(\sigma), s(\nu)}(x) A_{\sigma, \nu}^s f, \quad x \in G \text{ (or } U(\mathfrak{g})),$$

we have a linear map

$$A^s(\tau) : \text{Hom}_K(\tau, \pi_{\sigma, \nu} |_K) \rightarrow \text{Hom}_K(\tau, \pi_{s(\sigma), s(\nu)} |_K).$$

for any  $\tau \in \hat{K}$ .

Let  $[\alpha_i]$  be the  $K$ -isomorphism from  $\tau$  to  $W_{\alpha_i}$  for  $\alpha_i \in I(\tau)$ . We equip the space  $\text{Hom}_K(\tau, \pi_{\sigma, \nu} |_K)$  with the basis consisting of the  $K$ -homomorphisms  $[\alpha_i]$ . Similarly, we choose a basis for the space  $\text{Hom}_K(\tau, \pi_{s(\sigma), s(\nu)} |_K)$ . Then we want to compute all entries  $a_{ij}$  of the matrix  $A^s(\tau) = (a_{ij})$  such that

$$A^s(\tau)[\alpha_i] = \sum_{\alpha_j^s \in I} a_{ij} \cdot [\alpha_j^s]$$

where  $I = \{\alpha^s \mid W_{\alpha^s} \hookrightarrow \pi_{s(\sigma), s(\nu)} |_K\}$ . For each basis vector  $f_{pq}$  of  $\tau$  as in Lemma 2.1, we have that

$$(A^s(\tau)[\alpha_i])(f_{pq}) = \sum_{\alpha_j^s \in I} a_{ij} \cdot [\alpha_j^s](f_{pq}) = \sum_{\alpha_j^s \in I} a_{ij} \cdot f_{\alpha_j^s, pq}^{(\tau)}(k).$$

On the other hand, by definition of the map  $A^s(\tau)$ , one has

$$(A^s(\tau)[\alpha_i])(f_{pq}) = (A_{\sigma, \nu}^s f_{\alpha_i, pq}^{(\tau)})(k), \quad \alpha_i \in I(\pi_{\sigma, \nu}, \tau).$$

Thus we have the following formula for the coefficients  $a_{ij}$  of the matrix  $A^s(\tau)$  for each  $\tau \in \hat{K}$ .

**Lemma 4.2** *Let  $\alpha_i$  be in  $I(\pi_{\sigma, \nu}, \tau)$  and  $\alpha_j^s$  be in  $I(\pi_{s(\sigma), s(\nu)}, \tau)$ . Then the  $(i, j)$ -th coefficient of  $A^s(\tau)$*

$$a_{ij} = (A_{\sigma, \nu}^s f_{\alpha_i, \alpha_j^s}^{(m)})(1_4).$$

#### Example 4.3

Let  $s$  be a generator of  $W(A)$  whose image is the matrix  $\text{diag}(1, -1)$  under the representation of  $W(A)$  on  $\mathfrak{a}^*$ . Then we choose the corresponding  $s^* \in K$  as the matrix  $\text{diag}(1, -i, 1, i)$  and hence

$$\bar{N}_s = \exp(\mathfrak{g}_{-2\lambda_2}) = \left\{ n_s(t) = \kappa^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & t & & 1 \end{pmatrix} \kappa : t \in \mathbb{R} \right\}.$$

Since  $n_s \in \bar{N}_s$ , one has  ${}^t n_s I_{2,2} n_s = I_{2,2}$  and hence  $n_s s^* = I_{2,2} {}^t n_s^{-1} I_{2,2} s^*$ . Thus, we have the following.

Assume  $n_s = n_s(t) \in \bar{N}_s$ . Let  $n' \in N$ ,  $a(n_s s^*) \in A$  and  $k(n_s s^*) \in K$  be so that  $n_s s^* = n' a(n_s s^*) k(n_s s^*)$ . Then

$$\begin{aligned} a(n_s s^*)^{\nu+\rho} &= (1+t^2)^{-\frac{\nu_2+1}{2}}, \\ k(n_s s^*) &= \text{diag}(1, -iu, -1, -iu^{-1}) \end{aligned}$$

where  $u = ((1-it)/(1+it))^{\frac{1}{2}}$ .

For a fixed  $\tau_m \in \hat{K}$ , therefore

$$f_{\gamma_i, \beta_j}(k(n_s s^*)) = 0 \text{ when } \gamma_i \neq \beta_j$$

If  $\tau = \tau_{[m_1, m_2; l]}$  then we have

$$A^s(\tau) = 2\pi 2^{-\nu_2} \Gamma(\nu_2) \text{diag} \left[ \frac{(-1)^{(m_1+m_2)/2-p+1} i^{m_2+r}}{\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2} + d) \Gamma(\frac{1}{2}\nu_2 + \frac{1}{2} - d)} \right]_p$$

where  $d = \frac{1}{2}(m_1 + r - 2p)$  for  $(p, (m_1 + m_2)/2 - p) \in I(\pi_{\sigma, \nu}, \tau_m)$ .

## 5 Whittaker functions

The main focus of this section is on the integral expressions of Whittaker functions on  $G$  related to certain principal series. The results of the section 4.1 lead us to the study of Whittaker functions related to some  $K$ -types. For this purpose, we focus our investigation on the principal series representations which contain one dimensional  $K$ -types and apply the method used in [4] to evaluate such Whittaker functions. More precisely, in this setting, the character  $\sigma$  of  $M$  factors through a character  $\chi$  of  $\mu_2$ . Let  $(\pi_{\chi, \nu}, L_{\chi}^2(K))$  denote the principal representation series corresponding to such character  $\sigma$ .

For an integer  $u$ , define a function  $f_u(k)$  on  $K$  by  $f_u(k) := \det(k_2)^u$ ,  $k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K$ .

**Lemma 5.1** *Let  $f_u(k)$  be as above. Then  $\tau_{[0,0,2u]} \cong \mathbb{C} f_u(k)$  as  $K$ -modules. Moreover, if  $\chi(-1) = (-1)^u$  then  $f_u(k) \in L_{\chi}^2(K)$  and  $[\pi_{\chi, \nu} : \tau_{[0,0,2u]}] = 1$ .*

### 5.1 The Jacquet integral.

Let  $J_{\chi, \nu}$  be the Jacquet functional on the subspace of differentiable functions of  $L_{\chi}^2(K)$  given by

$$J_{\chi, \nu}(f) = \int_N \eta(n)^{-1} a(s^* n)^{\nu+\rho} f(k(s^* n)) dn$$

for a differentiable function  $f$  in  $L_{\chi}^2(K)$  and the longest element  $s \in W(A)$ . Here  $W(A)$  is the Weyl group defined as the quotient of  $M^* = N_K(\mathfrak{a})$ , the normalizer of  $\mathfrak{a}$  in  $K$ , by  $M$  and  $s^*$  is an element of  $M^*$  mapping to the longest element  $s \in W(A)$ .

Then one has  $J_{\chi,\nu}(\pi(n)f) = \eta(n)J_{\chi,\nu}(f)$  and hence

$$J \in \text{Hom}_{(\mathfrak{g},K)}(\pi_{\chi,\nu}|_K, \mathcal{A}_\eta(N \setminus G)), \quad (1)$$

where  $J$  associates  $v \in \pi_{\chi,\nu}|_K$  to the function  $J_v(g) := J_{\chi,\nu}(\pi_{\chi,\nu}(g)v)$ , ( $g \in G$ ). We want to have an explicit formula for the  $A$ -radial part:

$$J_{f_u}(a) = J_{\chi,\nu}(\pi_{\chi,\nu}(a)f_u) = a^{-\nu+\rho} \int_N \eta(ana)^{-1} a(s^*n)^{\nu+\rho} f_u(k(s^*n)) dn.$$

In our case, we can choose  $I_{2,2} \in K$  for  $s^* \in K$ .

### 5.1.1 The first modification

Let  $E_{ij}$  be the usual matrix with 1 in  $(i, j)$ -entry and zero elsewhere. Put

$$\begin{aligned} E_0 &= \kappa^{-1}(E_{12} - E_{43})\kappa, & E_1 &= i\kappa^{-1}(E_{12} + E_{43})\kappa, & E_2 &= \kappa^{-1}E_{24}\kappa, \\ F_0 &= \kappa^{-1}(E_{14} + E_{23})\kappa, & F_1 &= i\kappa^{-1}(E_{14} - E_{23})\kappa, & F_2 &= \kappa^{-1}E_{13}\kappa, \end{aligned}$$

by setting  $i = \sqrt{-1}$  and

$$\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}.$$

Then the corresponding root spaces of positive roots in  $\Phi(\mathfrak{g}, \mathfrak{a})$  are given by

$$\begin{aligned} \mathfrak{g}_{\lambda_1 - \lambda_2} &= E_0 \cdot \mathbb{R} \oplus E_1 \cdot \mathbb{R}, & \mathfrak{g}_{2\lambda_2} &= E_2 \cdot \mathbb{R}, \\ \mathfrak{g}_{\lambda_1 + \lambda_2} &= F_0 \cdot \mathbb{R} \oplus F_1 \cdot \mathbb{R}, & \mathfrak{g}_{2\lambda_1} &= F_2 \cdot \mathbb{R}. \end{aligned}$$

Let  $\mathfrak{n}$  be a subalgebra defined by  $\mathfrak{n} = \sum_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$ . We now describe elements of a maximal nilpotent subgroup  $N$  of  $G$  given by  $N = \exp(\mathfrak{n})$ .

The Killing form  $B(X, Y) = \text{tr}(\text{ad}X \cdot \text{ad}Y)$ , ( $X, Y \in \mathfrak{g}$ ) and Cartan involution  $\theta$  of  $\mathfrak{g}$  induce an inner product  $\langle, \rangle$  of  $\mathfrak{g}$  via

$$\langle X, Y \rangle = -B(X, Y^\theta), \quad (X, Y \in \mathfrak{g}).$$

Then one has that  $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$  if  $\alpha \neq \beta$ , because of the involution  $\theta$ . Moreover, one can see that the set  $\{E_i, F_i \mid i = 0, 1, 2\}$  is an  $\langle, \rangle$ -orthogonal basis for  $\mathfrak{n}$  such that a each element  $n = n(n_0, n_1, n_2, n_3)$  in the maximal unipotent group  $N = \exp(\mathfrak{n})$  is expressed in the form:

$$\kappa^{-1} \begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{n}_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \kappa$$

for  $n_1, n_3 \in \mathbb{R}$ ,  $n_0, n_2 \in \mathbb{C}$ .

**Lemma 5.2** *We have*

1. Set  $N_1 = \begin{pmatrix} n_1 & n_2 \\ \bar{n}_2 & n_3 \end{pmatrix}$  for  $n = n(n_0, n_1, n_2, n_3) \in N$ . Then

$$f_u(k(I_{2,2}n)) = \left( \det(1 - \sqrt{-1}N_1) / \det(1 + \sqrt{-1}N_1) \right)^{\frac{\pi}{2}}.$$

2. Let  $\eta$  be a character of  $N$  determined by a real number  $c_2$  and  $c = c_0 + \sqrt{-1}c_1 \in \mathbb{C}$ . Then

$$\eta(ana^{-1}) = \exp(2\sqrt{-1} \left( \frac{a_1}{a_2} \operatorname{Re}(\bar{c}n_0) + c_2 a_2^2 n_3 \right)),$$

where  $a_i = \exp(t_i)$ ,  $(i = 1, 2)$  for  $a = a(t_1, t_2) \in A$ .

3. For  $\nu = (\nu_1, \nu_2) \in \operatorname{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$ , we have that  $a(I_{2,2}n)^{\nu+\rho} = \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1} \Delta_2^{-\frac{\nu_2+1}{2}}$  where  
 $\Delta_1 = 1 + n_1^2 + \bar{n}_2 n_2 + (\bar{n}_0 n_2 + n_0 \bar{n}_2)(n_1 + n_3) + \bar{n}_0 n_0(1 + \bar{n}_2 n_2 + n_3^2)$ ,  
 $\Delta_2 = 1 + n_1^2 + 2n_2 \bar{n}_2 + n_3^2 + (n_1 n_3 - n_2 \bar{n}_2)^2$  for  $n = n(n_0, n_1, n_2, n_3) \in N$ .

For future convenience, we choose a new coordinate for  $A$  by

$$y = (y_1, y_2) = \left( \frac{a_1}{a_2}, a_2^2 \right).$$

Since  $f \rightarrow J_f(g)$  is the Whittaker realization of  $\pi_{\chi, \nu}$ ,  $J_{f_u}(a)$  is the radial part of a Whittaker function on  $G$  belonging to  $\pi_{\nu}$ . Thus, in the new coordinate system, we can summarize that the radial part of Whittaker function associated with the  $K$ -type  $\tau_u$  can be written in the form

$$y_1^{-\nu_1+3} y_2^{-\frac{\nu_1+\nu_2}{2}+2} \int_N \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1} \Delta_2^{-\frac{\nu_2+1}{2}} \times \exp(-2\sqrt{-1}(y_1 \operatorname{Re}(\bar{c}n_0) + c_2 y_2 n_3)) f_u(k(I_{2,2}n)) dn,$$

where  $dn$  is a multiplicative Haar measure on  $N$ . Now we shall give a normalization of Haar measure of  $N$ . Since the exponential map of  $\mathfrak{n}$  onto  $N$  is an analytic isomorphism, there exists a unique Haar measure  $dn$  on  $N$  that corresponds to Lebesgue measure on  $\mathfrak{n}$ .

**Lemma 5.3** *The radial part of the moderate growth Whittaker function  $W_{(\nu_1, \nu_2)}(y_1, y_2; u)$  (up to constant) associated with the  $K$ -type  $\tau_u$  can be written in the form*

$$y_1^{-\nu_1+3} y_2^{-\frac{\nu_1+\nu_2}{2}+2} \int_{\mathbb{R}^6} \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1} \Delta_2^{-\frac{\nu_2+1}{2}} \exp(-2\sqrt{-1}(c_0 z_0 y_1 + c_1 t_0 y_1 - n_3 y_2)) f_u(k(I_{2,2}n))$$

with respect to  $dz_0 dt_0 dn_1 dz_2 dt_2 dn_3$ . Here  $n_i = z_i + \sqrt{-1}t_i$  ( $i = 0, 2$ ).

In fact, it suffices to consider the cases  $u = 0$  and  $1$  for our purposes. Let  $K_{\mu}(z)$  be the Bessel function.

**Theorem 5.4** *Let  $\pi_{\chi, \nu}$  be irreducible and  $\eta$  be a nondegenerated unitary character  $N$ . Then we have the following assertions on the  $A$ -radial part of the primary Whittaker function  $W_{(\nu_1, \nu_2)}(y_1, y_2; u)$ .*

*If  $\chi$  is trivial then the Whittaker function  $W_{(\nu_1, \nu_2)}(y_1, y_2; 0)$  is identified with  $y_1^3 y_2^2$  times*

$$\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

*If  $\chi$  is non-trivial then the Whittaker function  $\bar{W}_{(\nu_1, \nu_2)}(y_1, y_2; 1)$  is identified with  $y_1^4 y_2^3/4$  times*

$$\int_0^\infty \int_0^\infty T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2) (\sqrt{t_1/t_2} - 1/\sqrt{t_1 t_2}) \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

where  $T_{\nu_1, \nu_2}(y_1, y_2, t_1, t_2)$  is the function

$$K_{\frac{\nu_1+\nu_2}{2}}(2\sqrt{t_2/t_1}) K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1 t_2}) \exp\left(-|c_2|y_2 t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2 + c_1^2)|c_2| \frac{y_1^2 y_2}{t_2}\right)$$

Note here that, we need the following formula to reduce the number of integral symbols corresponding to the root spaces  $\mathfrak{g}_{\lambda_1-\lambda_2}$  and  $\mathfrak{g}_{\lambda_1+\lambda_2}$ :

**Formula 5.5** *Let  $a, c \in \mathbb{R}_+^*$  and  $b, \alpha, \beta \in \mathbb{R}$  such that  $\alpha^2 + \beta^2 = 1$ . Then we have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \exp(-c(x^2 + y^2) - a(\alpha x + \beta y)^2 + 2\sqrt{-1}b(\alpha x + \beta y)) dx dy = \frac{\pi}{(c^2 + ac)^{\frac{1}{2}}} \exp\left(\frac{-b^2}{a+c}\right).$$

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