

Extremal transition and quantum cohomology

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2. INTRODUCTION

In this section, we will summarize the main results of this thesis, and then describe how this thesis is structured. When a singular projective variety X_{sing} admits a crepant resolution X_{res} and a smoothing X_{sm} , we say that X_{res} and X_{sm} are related by extremal transition. Since we want to study relationship between their quantum cohomology in algebraic setting, we will assume throughout this thesis X_{res} and X_{sm} are both projective manifolds. We attempt to understand the change of quantum cohomology under extremal transition and relate it with the following diagram:

$$(1) \quad X_{\text{res}} \xrightarrow{\pi} X_{\text{sing}} \xleftarrow{r} X_{\text{sm}}$$

where π is a resolution of singularities and r is a (continuous) retraction.

At the present stage, we don't have a unified understanding of the relationship between the quantum cohomology of X_{res} and X_{sm} related by general extremal transitions. One of the main difficulty is that under extremal transitions in higher dimensions (greater than three), the singularities of X_{sing} don't have clear common nature and vanishing cycles of X_{sm} vary from case to case. Consequently it is difficult to study the relationship between quantum cohomology of X_{res} and X_{sm} . However, when X_{sing} is a threefold with only ordinary double points, this is known as *conifold transition* and has been studied by many authors. For 3-fold conifold transition, we have a good understanding of how the quantum cohomology change. In fact, the relationship between quantum cohomology of X_{res} and X_{sm} can be derived from a result of Li and Ruan in [28]: under 3-fold conifold transition, the quantum cohomology of X_{sm} can be identified with a subquotient of $QH^*(X_{\text{res}})$ with quantum variables of exceptional curves specialized to one. In order to state the theorems in precise mathematical terms, let us recall that the (small) quantum product $\star$ of a smooth projective variety X defines a commutative ring structure

on

$$QH^*(X) = H^*(X) \otimes \mathbb{C}[[q_1, \dots, q_r]],$$

where q_i is the Novikov (quantum) variable associated to a curve class on X and $r = \dim H^2(X)$. This defines the quantum connection (or Dubrovin connection)

$$\nabla_{q_i \frac{\partial}{\partial q_i}} = q_i \frac{\partial}{\partial q_i} + \frac{1}{z}(\phi_i \star) \quad 1 \leq i \leq r$$

on the trivial bundle over the q -space with fiber the cohomology group $H^*(X)$. Here z is a parameter taking values in $\mathbb{C}^\times$. This bundle is flat for all values of z . Here $\phi_1, \dots, \phi_r$ is a basis of $H^2(X)$ dual to the variables $q_1, \dots, q_r$.

In [28], Li and Ruan studied the change of Gromov-Witten invariants and the functoriality of quantum cohomology. In terms of the quantum connection, their result can be restated as follows:

Theorem 2.1 (see Theorem 11.5 and Corollary 11.6). *Let $X_{\text{res}} \rightarrow X_{\text{sing}} \leftarrow X_{\text{sm}}$ be a 3-fold conifold transition. Let $E_1, \dots, E_k$ be exceptional curves of X_{res} . Then*

- (1) *The quantum connection of X_{res} is of the form*

$$\nabla^{\text{res}} = \nabla' + \sum_{i=1}^k N_i \frac{dq^{E_i}}{1 - q^{E_i}}$$

where ∇' is a connection which is regular along $\Delta_{\text{exc}} = \{q^{E_1} = q^{E_2} = \dots = q^{E_k} = 1\}$ and $N_i \in \text{End}(H^(X_{\text{res}}))$ is a nilpotent endomorphism.*

- (2) *The residue endomorphisms N_i along $q^{E_i} = 1$ define a filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ as follows:*

$$V := \bigcap_{i=1}^k \text{Ker}(N_i), \quad W := V \cap V^\perp = \bigcap_{i=1}^k \text{Ker}(N_i) \cap \sum_{i=1}^k \text{Im}(N_i).$$

This filtration arises from the diagram (1) as $V = \text{Im } \pi^$ and $W = \pi^*(\text{Ker } r^*)$.*

- (3) *The connection $\nabla'|_{\Delta_{\text{exc}}}$ induces a flat connection on the vector bundle $(V/W) \times \Delta_{\text{exc}} \rightarrow \Delta_{\text{exc}}$ which is isomorphic to the small quantum connection of X_{sm} , under the isomorphism $r^* \circ (\pi^*)^{-1}: V/W \cong H^*(X_{\text{sm}})$.*

In particular, the small quantum cohomology $QH^(X_{\text{sm}})$ of X_{sm} is isomorphic to the subquotient $(V/W, \star|_{q_{\text{exc}}=1})$ of the quantum cohomology of X_{res} along the locus where all the exceptional quantum variables $q_{\text{exc}} = (q^{E_1}, \dots, q^{E_k})$ equal one.*

Remark 2.2. A recent paper of Lee-Lin-Wang [27] studied the behaviour of $A + B$ -theory under conifold transition of Calabi-Yau threefolds.

In section 8, we study $\text{Fl}(1, 2, 3)$ and the resolution of its toric degeneration. This example illustrates the above theorem. The detailed results are as follows:

The complete flag $\text{Fl}(1, 2, 3)$ degenerates to a singular toric variety X_{sing} . The toric variety X_{sing} admits a small crepant resolution, which we denote by X_{res} . The contracted curve β_3 on X_{res} corresponds to quantum parameter q_3 whose degree

is zero. We denote the divisor dual to β_3 by $\mathfrak{p}_3$. The residue N_3 of quantum product matrix $\mathfrak{p}_3\star$ defines a weight filtration

$$0 \subset \text{Im } N_3 \subset \text{Ker } N_3 \subset H^*(X_{\text{res}}).$$

We have the following theorem:

Theorem 2.3. *The weight filtration defined by the nilpotent operator $N = \text{Res}_{q_3=1}(\mathfrak{p}_3\star)$ coincides with the filtration*

$$0 \subset \pi^*(\text{Ker } r^*) \subset \text{Im } \pi^* \subset H^*(X_{\text{res}}).$$

The quantum multiplication by $\mathfrak{p}_1, \mathfrak{p}_2$ on $H^(X_{\text{res}})$ are regular at $q_3 = 1$ and the operators induced by $\lim_{q_3 \rightarrow 1} \mathfrak{p}_1\star, \lim_{q_3 \rightarrow 1} \mathfrak{p}_2\star$ on*

$$\text{Ker } N / \text{Im } N \cong H^*(\text{Fl}(1, 2, 3))$$

coincide with the quantum multiplication by p_1, p_2 on $H^(\text{Fl}(1, 2, 3))$. Here note that $\mathfrak{p}_i \in \text{Im } \pi^*$ and $p_i = r^*(\pi^*)^{-1}\mathfrak{p}_i$ for $i = 1, 2$.*

In this thesis, we also calculate two higher dimensional extremal transition examples, and find that similar phenomena happen. As studied in [19, 2], a partial flag variety admits a flat degeneration to a singular Gorenstein toric variety X_{sing} , which admits a toric crepant resolution X_{res} . We study extremal transitions of $\text{Gr}(2, 4)$ and $\text{Gr}(2, 5)$ by explicit computations. We have the following theorems for $\text{Gr}(2, 4)$ and $\text{Gr}(2, 5)$. The crepant resolution X_{res} of the toric degeneration of $\text{Gr}(2, 4)$ has one exceptional quantum parameter q_2 , the divisor class dual to the exceptional curve is denoted by m_2 . The residue of the quantum product matrix of m_2 at $q_2 = 1$ is given by

$$N = \text{Res}_{q_2=1}(m_2\star) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The residue N defines the filtration $0 \subset W = \text{Im}(N) \cap \text{Ker}(N) \subset V = \text{Ker}(N) \subset H^*(X_{\text{res}})$. We have the following theorem:

Theorem 2.4 (see theorem 9.2). *The filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ defined by the residue $N = \text{Res}_{q_2=1}(m_2\star)$ along $q_2 = 1$ matches with the filtration*

$$0 \subset \pi^*(\text{Ker } r^*) \subset \text{Im } \pi^* \subset H^*(X_{\text{res}}).$$

The quantum products of elements in $\text{Im } \pi^$ are regular at $q_2 = 1$ and the map*

$$r^* \circ (\pi^*)^{-1}: \text{Im } \pi^* \rightarrow H^*(\text{Gr}(2, 4))$$

intertwines the quantum product $\star|_{q_2=1}$ on $\text{Im } \pi^* = V$ with the quantum product on $H^*(\text{Gr}(2, 4))$ under the identification $q_1 = q$ of the Novikov variables. This map also preserves the Poincaré pairing.

Remark 2.5. For $\text{Gr}(2, 4)$, the map $r^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{sm}})$ is not surjective and the subquotient $(V/W, \star|_{q_{\text{exc}}=1})$ of $H^*(X_{\text{res}})$ defined by the residue endomorphism N is identified with a *proper* subring $\text{Im } r^* \subsetneq QH^*(\text{Gr}(2, 4))$.

If we consider the weight filtration $\{W_\bullet\}$ associated to N , we can extend the inclusion $(V/W, \star|_{q_{\text{exc}}=1}) \hookrightarrow QH^*(\text{Gr}(2, 4))$ to an isomorphism $W_0/W_{-1} \cong QH^*(\text{Gr}(2, 4))$. The isomorphism $W_0/W_{-1} \cong QH^*(\text{Gr}(2, 4))$ involves an imaginary number.

The crepant resolution X_{res} of toric degeneration of $\text{Gr}(2, 5)$ has two exceptional quantum parameters q_2 and q_3 corresponding to exceptional curve classes β_2 and β_3 . Denote by m_2 and m_3 the divisors dual to β_2 and β_3 . The matrix of quantum multiplication by m_2 has simple poles along $q_2 = 1$ and $q_2 + q_3 = 1$; the matrix of quantum multiplication by m_3 has simple poles along $q_3 = 1$ and $q_2 + q_3 = 1$. We define

$$N_2 := \text{Res}_{q_2=1}(m_2 \star) \frac{dq_2}{q_2} \Big|_{(q_2, q_3)=(1, 1)}$$

$$N_3 := \text{Res}_{q_3=1}(m_3 \star) \frac{dq_3}{q_3} \Big|_{(q_2, q_3)=(1, 1)}$$

These are nilpotent endomorphisms. Thus the monodromy of the quantum connection around the normal crossing divisors $(q_2 = 1)$, $(q_3 = 1)$ is unipotent. The endomorphisms N_2, N_3 define the filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ by:

$$V := \text{Ker}(N_2) \cap \text{Ker}(N_3), \quad W := V \cap (\text{Im}(N_2) + \text{Im}(N_3)).$$

We have the following theorem for $\text{Gr}(2, 5)$:

Theorem 2.6 (see theorem 10.1). *The quantum product on $H^*(X_{\text{res}})$ at $q_2 = q_3 = 1$ descends to a well-defined product structure on V/W . The linear isomorphism $\theta: V/W \cong H^*(\text{Gr}(2, 5))$ intertwines the quantum product $\star|_{q_2=q_3=1}$ on V/W with the quantum product on $H^*(\text{Gr}(2, 5))$. Moreover θ preserves the Poincaré pairing.*

Remark 2.7. For $\text{Gr}(2, 5)$, the subquotient $(V/W, \star|_{q_{\text{exc}}=1})$ defined by the residue endomorphisms N_2, N_3 is isomorphic to $QH^*(\text{Gr}(2, 5))$. On the other hand, the quotient space W_0/W_{-1} associated to the weight filtration $\{W_\bullet\}$ of $aN_2 + bN_3$ ($a \neq 0, b \neq 0$) is of dimension bigger than $\dim H^*(\text{Gr}(2, 5))$.

This thesis is organized as follows. It can be divided into two parts. The first part consists of section 3, 4, 5 and 6. In the section 3, we shall review Gromov-Witten theory and small quantum cohomology. In the section 4, we shall recall preliminaries on toric varieties and the method for computing the small quantum cohomology of toric varieties using mirror symmetry theorem [16]. In section 5, we shall review Aaron Bertram's work [4] on quantum cohomology

of Grassmannian. We shall use his result to compute the quantum cohomology of $\text{Gr}(2, 4)$ in section 9 and that of $\text{Gr}(2, 5)$ in section 10. In section 6, we shall review basic facts about partial flag varieties and their degenerations [2] [7], at the end of section 6 we recall the LG B-model of partial flag varieties following [14] and [2]. Then we compare the LG B-model of partial flag varieties Fl and that of the crepant resolution X_{res} of toric degeneration of Fl. We find that the LG B-model of Fl can be embedded into the LG B-model of X_{res} . The superpotential of LG B-model determines Jacobi ring and residue pairing, under mirror map, Jacobi ring and residue pairing are identified with information of quantum cohomology and Poincaré pairing on the A-model side. Consequently, such a relationship between the superpotentials of LG B-model is an evidence that there should be some relationship between quantum cohomology of Fl and the quantum cohomology of X_{res} . This constitutes the motivation of this thesis. The second part of this thesis consists of sections 7, 8, 9, 10 and 11. In section 7, we study the extremal transition from $\mathbb{F}_2$ to $\mathbb{P}^1 \times \mathbb{P}^1$. In section 8, we study $\text{Fl}(1, 2, 3)$ and the crepant resolution of its toric degeneration. In section 9, we shall study the example of $\text{Gr}(2, 4)$ and crepant resolution of its singular toric degeneration X_{res} . In section 10, we study 6-dimensional example of $\text{Gr}(2, 5)$ and crepant resolution of its toric degeneration. In section 11, we study the 3-fold conifold transition using Li-Ruan's result [28]. Finally in the concluding section 12, we shall talk about unsolved problems related to extremal transitions in higher dimensions and our future work. This thesis is partially based on the joint work with Hiroshi Iritani [24].

3. GROMOV-WITTEN INVARIANTS AND SMALL QUANTUM COHOMOLOGY

In this section, we briefly review preliminaries on stable maps and Gromov-Witten invariants, see [8] and [20] for a comprehensive introduction. Gromov-Witten invariants can be regarded as intersection numbers of cycles on the moduli space of stable maps. If the cycles are given by constraint conditions in enumerative geometry, the corresponding Gromov-Witten invariants have a clear enumerative interpretation. However, the moduli space of stable maps are not always well-behaved, sometimes it is very singular and might have several components of different dimensions, it is difficult to develop a meaningful intersection theory on such moduli space. Fortunately, we can construct a fundamental cycle of expected dimension on the moduli space of stable maps, and define the Gromov-Witten invariants as integrals against this cycle. For the details of construction of virtual cycles, we refer the readers to [3][29]. The idea of virtual fundamental classes originally comes from [26].

3.1. Stable curves and Stable maps. First of all, let us recall some standard facts about curves.

Definition 3.1. An n -marked genus- g pre-stable curve is a tuple $(C, p_1, \dots, p_n)$ in which

- (1) C is a compact connected curve with at worst nodes as singularities,
- (2) $p_1, \dots, p_n$ are distinct smooth points of C which are called the marked points.

We call the nodes and marked points $p_1, \dots, p_n$ on C special points. An isomorphism f between two pre-stable curve $(C, p_1, \dots, p_n)$ and $(C', p'_1, \dots, p'_n)$ is an isomorphism between C and C' taking p_i to p'_i .

Definition 3.2. A n -marked genus- g , stable curve $(C, p_1, \dots, p_n)$ is a pre-stable curve with only finite automorphisms.

It is well-known that the finiteness of automorphism is equivalent to the following conditions:

- (1) every irreducible component of geometric genus 0 has at least three special points,
- (2) every irreducible component of geometric genus 1 has at least one special points.

The moduli space of n -marked genus- g stable curves is a Deligne-Mumford stack, which we denote by $\overline{\mathcal{M}}_{g,n}$. The open substack of smooth stable curves is denoted by $\mathcal{M}_{g,n}$. Both $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$ are connected, irreducible, smooth Deligne-Mumford stack of complex dimension $3g - 3$.

3.2. Gromov-Witten invariants. Let X be a non-singular complex projective variety.

Definition 3.3. An n -marked pre-stable map from n -marked genus- g pre-stable curve to X is a tuple $(f, C, p_1 \dots p_n)$ in which

- (1) $(C; p_1, \dots, p_n)$ is a n -marked genus- g pre-stable curve,
- (2) f is a morphism from C to X .

Definition 3.4. An isomorphism $\phi : (f, C, p_1, \dots, p_n) \longrightarrow (f', C', p'_1, \dots, p'_n)$ is an isomorphism from C to C' such that $f' \circ \phi = f$ and $\phi(p_i) = p'_i$ ($i = 1, \dots, n$)

Definition 3.5. A pre-stable map of degree $d \in H_2(X, \mathbb{Z})$ from a n -marked genus- g pre-stable curve to X is stable if the automorphism group of $(f, C, p_1, \dots, p_n)$ is finite and $[f_*(C)] = d \in H_2(X, \mathbb{Z})$.

The finiteness of automorphism group is equivalent to the following conditions

- (1) every irreducible contracted component of genus zero has at least 3 special points,
- (2) every irreducible contracted component of genus one has at least 1 special points.

Denote by $\overline{\mathcal{M}}_{g,n}(X, d)$ the moduli space of stable maps of degree d from n -marked genus- g pre-stable curves to X . It is a Deligne-Mumford stack. But it is much complicated than the moduli space of stable curves. It might be very singular and have several components of different dimensions. However, there is a cycle $[\overline{\mathcal{M}}_{g,n}(X, d)]_{\text{vir}} \in A_{\text{vir dim}}(\overline{\mathcal{M}}_{g,n}(X, d))$. We call this cycle virtual fundamental

class. Here, vir dim is the expected dimension of the moduli $\overline{\mathcal{M}}_{g,n}(X, d)$. The deformation theory tells us that it is $(3 - \dim(X))(g - 1) + \langle c_1(X), d \rangle + n$. Gromov-Witten invariants are defined as integrals against this cycle.

We won't delve into the details of virtual fundamental class. We refer the interested readers to [3]. There are several natural maps from $\overline{\mathcal{M}}_{g,n}(X, d)$ to X . The i -th valuation map

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, d) \longrightarrow X, i = 1, \dots, n.$$

takes the i -th marked point $p_i \in (f, C, p_1, \dots, p_n)$ to $f(p_i)$. There are also several tautological bundles on $\overline{\mathcal{M}}_{g,n}(X, d)$. The i -th cotangent line bundle L_i over $\overline{\mathcal{M}}_{g,n}(X, d)$ is a line bundle such that the fiber over $(f, C, p_1, \dots, p_n)$ is the cotangent line to the curve C at the i -th marked point p_i . Conventionally, the first Chern class of L_i is denoted by ψ_i . Now, we are ready to define descendant Gromov-Witten invariants of X .

Definition 3.6. Given cohomology classes $\gamma_1, \dots, \gamma_n \in H^*(X, \mathbb{C})$ and non-negative integers $k_1, \dots, k_n$. The descendant Gromov-Witten invariant $\langle \tau_{k_1}(\gamma_1), \dots, \tau_{k_n}(\gamma_n) \rangle_{g,d}^X$ is defined as integral

$$\int_{[\overline{\mathcal{M}}_{g,n}(X, d)]_{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \psi_i^{k_i}$$

If the non-negative integers k_i are all zero, we call such Gromov-Witten invariants primary Gromov-Witten invariants.

There are some nice properties about Gromov-Witten invariants we will use frequently in this thesis. We list them as follows:

String equation: when $n \geq 3$ and $d \neq 0$,

$$\langle 1, \gamma_1, \dots, \gamma_n \rangle_{0, n+1, d} = 0,$$

here $1 \in H^*(X)$ is the Poincaré dual of the fundamental class $[X]$.

Divisor equation: for $p \in H^2(X, \mathbb{C}), n \geq 3$ and $d \neq 0$,

$$\langle p, \gamma_1, \dots, \gamma_n \rangle_{0, n+1, d} = \langle p, d \rangle \langle \gamma_1, \dots, \gamma_n \rangle_{0, n, d}$$

WDVV equation: for $\eta_1, \dots, \eta_n \in H^*(X, \mathbb{C}), \alpha, \beta, \gamma, \delta \in H^*(X, \mathbb{C})$ we have

$$\Sigma_{d=d_1+d_2, I \cup J = \{1, \dots, n\}} \langle \alpha, \beta, \{\eta_i\}_{i \in I}, \phi_k \rangle_{0, 3+|I|, d_1} \langle \phi^k, \gamma, \delta, \{\eta_j\}_{j \in J} \rangle_{0, 3+|J|, d_2}$$

is symmetric with respect to $\alpha, \beta, \gamma, \delta$. Here $\{\phi_k\}_k$ is a basis of $H^*(X, \mathbb{C})$, and $\{\phi^k\}_k$ is the dual basis with respect to Poincaré pairing.

Remark 3.7. In some nice situations, Gromov-Witten invariants have clear enumerative significance. Given n cycles $\Gamma_1, \dots, \Gamma_n$ in X and move them to generic position. Denote by γ_i the Poincaré dual of Γ_i . The primary Gromov-Witten invariant

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0, n, d}^X$$

is the number of rational curves in X of degree d passing through the given cycles Γ_i transversely. For example, Kontsevich derived a recursive formula for the number N_d of rational curves in $\mathbb{P}^2$ of degree d passing through $3d - 1$ given points in generic position. Although this formula has nothing to do with this thesis, it is so beautiful that I cannot help writing it down.

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1^2 d_2 \left(d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right),$$

with $N_1 = 1$. The geometric meaning of $N_1 = 1$ means there is only one line in $\mathbb{P}^2$ passing through two given points. This formula is an implicit manifestation of the associativity of quantum product. It can also be proved by degeneration argument see [1]. However, in most situations, the enumerative significance of these invariants are not obvious. The Gromov-Witten invariants might be rational number or even negative number. This is related to the properties of virtual fundamental class.

3.3. Quantum cohomology and Quantum D-Modules. As above, let X be a non-singular projective variety. Using Gromov-Witten invariants, we can define a new ring structure on the cohomology group $H^*(X, \mathbb{C})$. In the following, unless otherwise specified, we only consider the even cohomology groups. We will use $H^*(X, \mathbb{C})$ to denote the even cohomology ring of X . We will recall the definition of quantum cohomology and quantum D -module. We assume the total cohomology ring of $H^*(X, \mathbb{C})$ is generated by two dimensional cohomology classes. For example, the cohomology ring of a toric variety satisfies this condition. Let $\{\phi_i\}_{i=0}^N$ be a basis of total cohomology ring $H^*(X, \mathbb{C})$, such that:

- (1) ϕ_0 is the identity element of $H^*(X, \mathbb{C})$;
- (2) $\phi_1, \dots, \phi_r$ form a nef integral basis for $H^2(X, \mathbb{Z})/\text{torsion}$, where r is the rank of $H^2(X, \mathbb{Z})$;
- (3) ϕ_i is homogeneous.

Let $(\alpha, \beta) = \int_X \alpha \cup \beta$ denote the Poincaré pairing. Let $\{\phi^0, \dots, \phi^N\}$ denote the basis dual to $\{\phi_0, \dots, \phi_N\}$ with respect to the Poincaré pairing: $(\phi_i, \phi^j) = \delta_i^j$. Notice that (2) above implies that the cone spanned by the dual basis $\{\phi^1, \dots, \phi^r\}$ in $H^{2 \dim X - 2}(X, \mathbb{R}) \cong H_2(X, \mathbb{R})$ contains the Mori cone $\overline{\text{NE}}(X)$ of effective curves. Denote by $q_1, \dots, q_r$ the Novikov variables dual to the basis $\{\phi_1, \dots, \phi_r\}$ of $H^2(X, \mathbb{C})$. For $d \in H_2(X, \mathbb{Z})$, we write

$$q^d = q_1^{\phi_1 \cdot d} q_2^{\phi_2 \cdot d} \cdots q_r^{\phi_r \cdot d}.$$

Note that if d is an effective class, the right-hand side only contains non-negative powers of $q_1, \dots, q_r$. We define the Novikov ring to be $\Lambda := \mathbb{C}[[q_1, \dots, q_r]]$. The small quantum product $\star$ on $H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$ is defined by

$$(u \star v, w) = \sum_{d \in \text{Eff}(X)} \langle u, v, w \rangle_{0,3,d}^X q^\beta.$$

The product $\star$ defines an associative and commutative ring structure on $H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$. The commutativity of quantum product $\star$ follows from the definition immediately. The associativity follows from **WDVV** equation. Moreover, the quantum cohomology is graded with respect to the grading defined as follows:

$$\deg(\alpha q_i) = 2\langle c_1(X), \phi^i \rangle + \deg(\alpha),$$

where α is some homogeneous cohomology element in $H^*(X, \mathbb{C})$. $H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$ together with the product $\star$ is called small quantum cohomology of X , which we denote by $QH^*(X)$. The quantum product $\star$ is a deformation of the usual cup product in the sense that if we set the quantum parameters to zero, the quantum product reduces to the usual cup product. The structure constants of small quantum cohomology are not necessarily convergent in general (as power series in $q_1, \dots, q_r$); however, the coefficients are convergent for all examples in this thesis.

The quantum D-module of X is constructed as follows. First of all, let us recall the dual Givental connection. It is a connection defined on the trivial bundle $H^*(X, \mathbb{C}) \times H^2(X, \mathbb{C}^*) \rightarrow H^2(X, \mathbb{C}^*)$ by

$$\nabla^z := d + \frac{1}{z} \sum_{i=1}^r (\phi_i \star) \frac{dq_i}{q_i}$$

in which z is a degree two parameter. This is a formal connection. It defines a map

$$\nabla^z : H^*(X, \mathbb{C}) \otimes \mathbb{C}((z^{-1}))[[q]] \longrightarrow \sum_i H^*(X, \mathbb{C}) \otimes \mathbb{C}((z^{-1}))[[q]] \frac{dq_i}{q_i}.$$

For convenience, we define:

$$\nabla_i^z := \nabla_{q_i \frac{\partial}{\partial q_i}}^z = q_i \frac{\partial}{\partial q_i} + \frac{1}{z} \phi_i \star.$$

The connection ∇^z is flat. Consequently, there exists a fundamental solution matrix L satisfying $\nabla^z L = 0$. A fundamental solution is given in [22] explicitly as follows:

$$L(T_i) := e^{-\phi \log q/z} \phi_i - \sum_{d \in \Lambda \setminus \{0\}, k=0, \dots, N} q^d \phi^k \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]^{\text{vir}}} \text{ev}_1^*(e^{-\phi \log q/z} T_i) \text{ev}_2^*(\phi_k) \frac{1}{z + \psi_1}$$

Where ψ_1 denotes the first Chern class of the first cotangent line bundle on $\overline{\mathcal{M}}_{0,2}(X, d)$ and $\phi \log q/z = \frac{1}{z} \sum_{i=1}^r \phi_i \log q_i$. The expression $1/(z + \psi_1)$ is understood as:

$$\sum_{i=0}^{\infty} z^{-i-1} (-\psi_1)^i$$

Let D denote the Heisenberg algebra

$$D := \mathbb{C}[z][[q_1, \dots, q_r]][\mathbf{p}_1, \dots, \mathbf{p}_r],$$

in which $\deg \mathbf{p}_i = \deg z = 2$. The commutation relations of D are as follows:

$$[\mathbf{p}_i, q_j] = \delta_{i,j} q_j, [\mathbf{p}_i, \mathbf{p}_j] = [q_i, q_j] = 0.$$

We define an action of D on the module $H^*(X, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[z][[q]]$ by

$$q_i \mapsto \text{multiplication by } \phi_i, \quad \mathbf{p}_i \mapsto z \nabla_i^z.$$

We call this D -module the small quantum D -module of X . We are in the position to introduce the J -function of the quantum D -module. The fundamental solution L can be decomposed into

$$L = S \circ \exp(-\phi \log q/z),$$

where S is defined as:

$$S(T_i) := T_i - \sum_{d \in \Lambda \setminus \{0\}} \sum_{k=0, \dots, N} q^d \phi^k \int_{[\overline{\mathcal{M}}_{0,2}]_{\text{vir}}} \text{ev}_1^*(T_i) \text{ev}_2^*(\phi_k) \frac{1}{z + \psi_1}.$$

We define the J -function by

$$J(q, z) := L^{-1}(1) = \exp(\phi \log q/z) S^{-1}(1).$$

The J -function is a cohomology valued formal function. In order to calculate J -function, we have to calculate $S^{-1}(1)$. We need the following lemmas:

Lemma 3.8. [22] *Let μ be the constant matrix defined by $\mu(\phi_i) = \deg(\phi_i) \phi_i$. The matrix function S satisfy the following differential equations:*

$$(z q^i \frac{\partial}{\partial q^i} S + p_i \star) \circ S - S \circ (p_i \cup) = 0,$$

$$[2z \frac{\partial}{\partial z} + \sum_{i=1}^r (\deg q_i) q_i \frac{\partial}{\partial q_i}] S + [\mu, S] = 0.$$

Proof. The first equation follows from $\nabla^z L = 0$ immediately. The second equation follows from the fact that the matrix S preserves the degree. These two equations imply the unitarity of S which is stated in [15] $\square$

Lemma 3.9. [18, 22] *For $\alpha, \beta \in H^*(X)$, we have*

$$(S(\alpha)(q, -z), S(\beta)(q, z)) = (\alpha, \beta).$$

Proof. Firstly, if we set quantum parameters q_i to zero, the left hand side equals to (α, β) . If we differentiate the left hand side, we get:

$$\begin{aligned} z q_i \frac{\partial}{\partial q_i} (S(\alpha)(q, -z), S(\beta)(q, z)) = \\ - (S(\phi_i \cup \alpha)(q, -z), S(\beta)(q, z)) + (S(\alpha)(q, -z), S(\phi_i \cup \beta)(q, z)). \end{aligned}$$

Since ϕ_i^n is zero for sufficiently large n , we have $(z \frac{\partial}{\partial q_i})^n (S(\alpha)(q, -z), S(\beta)(q, z)) = 0$ for some n . This implies the lemma. $\square$

Lemma 3.10. [22] *The action of inverse matrix of S on $H^*(X)$ is given by the following formula:*

$$S^{-1}(\Phi) := \Phi + \sum_{d \in \Lambda \setminus \{0\}} \sum_{k=0, \dots, N} q^d \phi^k \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]_{\text{vir}}} \text{ev}_1^*(\phi_k) \text{ev}_2^*(\Phi) \frac{1}{z - \psi_1}.$$

Proof. Since S is unitary, that is ${}^t S(-z)S(z) = 1$, we get $S^{-1}(z) = {}^t S(-z)$. This lemma is equivalent to saying that for any α and β in $H^*(X)$, we have:

$$({}^t S(-z)(\alpha), \beta) = (S^{-1}(\alpha), \beta),$$

now let us compute $({}^t S(-z)(\alpha), \beta) = (\alpha, S(-z)(\beta))$.

$$\begin{aligned} (\alpha, S(-z)(\beta)) &= (\alpha, \beta) + \sum_{d \in \Lambda \setminus \{0\}, j=0, \dots, N} q^d (\phi^j, \alpha) \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]_{\text{vir}}} \frac{\text{ev}_1^*(\beta) \text{ev}_2^*(\phi_j)}{z - \psi_1} \\ &= (\alpha, \beta) + \sum_{d \in \Lambda \setminus \{0\}} \int_{\overline{\mathcal{M}}_{0,2}(X,d)} \frac{\text{ev}_1^*(\beta) \text{ev}_2^*(\alpha)}{z - \psi_1} \\ &= (\alpha, \beta) + \sum_{d \in \Lambda \setminus \{0\}, j=0, \dots, N} q^d (\phi^j, \beta) \int_{[\overline{\mathcal{M}}_{0,2}(X,d)]_{\text{vir}}} \frac{\text{ev}_1^*(\phi_j) \text{ev}_2^*(\alpha)}{z - \psi_1} \end{aligned}$$

The last line equals

$$(\alpha + \sum_{d \in \Lambda \setminus \{0\}, j=0, \dots, N} q^d \phi^j \int_{[\overline{\mathcal{M}}_{(0,2)}(X,d)]_{\text{vir}}} \frac{\text{ev}_1^*(\phi_j) \text{ev}_2^*(\alpha)}{z - \psi_1}, \beta).$$

□

Corollary 3.11. *We have*

$$J(\phi, z) = e^{\phi \log / z} \left(1 + \sum_{d \in \Lambda \setminus \{0\}, j=0, \dots, N} q^d \phi^j \int_{[\overline{\mathcal{M}}_{(0,2)}(X,d)]_{\text{vir}}} \frac{\text{ev}_1^*(\phi_j)}{z - \psi_1} \right).$$

The J -function contains information of quantum product. Since L is a fundamental solution, we have $z \nabla^z \circ L = L \circ z d$. It implies $P(z \nabla_i^z) \circ L = L \circ P(z q_i \frac{\partial}{\partial q_i})$. If we act $P(z \nabla_i^z) \circ L$ on the J -function, we have the following:

$$P(z \nabla_i^z) \circ L \circ L^{-1}(1) = L \circ P(z \nabla_i^z)(1).$$

Setting $P(z \nabla_i^z) = z \nabla_i^a z \nabla_j^z$, we have

$$\phi_i \star \phi_j = L \circ (z q_i \frac{\partial}{\partial q_i} z q_j \frac{\partial}{\partial q_j} J)$$

It is not difficult to see that the expansion of L takes the following form:

$$L = \text{Id} + O\left(\frac{1}{z}\right),$$

which gives us:

$$z q_i \frac{\partial}{\partial q_i} z q_j \frac{\partial}{\partial q_j} J = \phi_i \star \phi_j + O\left(\frac{1}{z}\right).$$

Now, we explain how to compute the small quantum cohomology of a weak-Fano toric manifold using Givental's mirror theorem [16].

Let X_{res} be the toric variety in section 10 below, which is a crepant resolution of a toric degeneration of $\text{Gr}(2, 5)$. The I -function of X_{res} is a cohomology-valued hypergeometric function given by:

$$I(q, z) = e^{m \log q / z} \sum_{\beta \in H_2(X_{\text{res}}, \mathbb{Z})} q^\beta \prod_{i=1}^9 \frac{\prod_{c=-\infty}^0 (R_i + cz)}{\prod_{c=-\infty}^{R_i \cdot \beta} (R_i + cz)},$$

where we set $m \log q := \sum_{i=1}^3 m_i \log q_i$. In the case at hand, the mirror map is trivial and the mirror theorem of Givental [16] says that $I(q, z)$ equals the J -function:

$$J(q, z) = e^{m \log q / z} \left(1 + \sum_{i=0}^N \sum_{\beta \neq 0} \left\langle \frac{\phi_i}{z(z - \psi)} \right\rangle_{0,1,\beta} \phi^i q^\beta \right),$$

where $\{\phi_i\}_{i=0}^N, \{\phi^i\}_{i=0}^N$ are mutually dual basis of the cohomology. The class ψ is the first Chern class of the universal cotangent line bundle over $\overline{M}_{0,1}(X_{\text{res}}, \beta)$. More generally, the I -function and the J -function match under a change of coordinates (mirror map).

The method for evaluating the quantum product is as follows: first of all, we find differential operators $\mathcal{D}_i(z\partial_1, z\partial_2, z\partial_3, z, q_1, q_2, q_3)$, which are polynomials in $z\partial_i := zq_i \frac{\partial}{\partial q_i}$ and z such that we have the asymptotic expansion:

$$\mathcal{D}_i I(q, z) = e^{m \log q / z} (\phi_i + O(z^{-1})) \quad 0 \leq i \leq N = 19.$$

Then the quantum product by m_j , $j = 1, 2, 3$ is determined by the asymptotics:

$$z\partial_j(\mathcal{D}_i I(q, z)) = e^{m \log q / z} (m_j \star \phi_i + O(z^{-1})).$$

In our case, for the choice of a basis in (14), we can take $\mathcal{D}_i$ as follows:

$$\begin{aligned} \mathcal{D}_0 &= 1, \quad \mathcal{D}_1 = z\partial_1, \quad \mathcal{D}_2 = z\partial_2, \quad \mathcal{D}_3 = z\partial_3, \quad \mathcal{D}_4 = (z\partial_1)^2, \quad \mathcal{D}_5 = z\partial_1 z\partial_2, \\ \mathcal{D}_6 &= z\partial_1 z\partial_3, \quad \mathcal{D}_7 = z\partial_2 z\partial_3, \quad \mathcal{D}_8 = (z\partial_1)^3, \quad \mathcal{D}_9 = (z\partial_1)^2 z\partial_2, \quad \mathcal{D}_{10} = (z\partial_1)^2 z\partial_3, \\ \mathcal{D}_{11} &= z\partial_1 z\partial_2 z\partial_3, \quad \mathcal{D}_{12} = (z\partial_1)^4, \quad \mathcal{D}_{13} = (z\partial_1)^3 z\partial_2, \quad \mathcal{D}_{14} = (z\partial_1)^3 z\partial_3, \\ \mathcal{D}_{15} &= (z\partial_1)^2 z\partial_2 z\partial_3, \quad \mathcal{D}_{16} = (z\partial_1)^5 - q_1(1 + q_2 + q_3), \\ \mathcal{D}_{17} &= (z\partial_1)^4 z\partial_2 - q_1 q_2, \quad \mathcal{D}_{18} = (z\partial_1)^4 z\partial_3 - q_1 q_3, \\ \mathcal{D}_{19} &= (z\partial_1)^6 - zq_1(1 + q_2 + q_3) - q_1(1 + 3q_2 + 3q_3 + q_2 q_3)z\partial_1 \\ &\quad - q_1(2 + q_3)(1 - q_2)z\partial_2 - q_1(2 + q_2)(1 - q_3)z\partial_3. \end{aligned}$$

4. TORIC VARIETIES AND LG B-MODELS

An algebraic variety X is called a toric variety if it contains an algebraic torus as an open dense subset, and the natural action of the algebraic torus on itself extends to the whole variety X . The structure of a toric variety can be nicely

encoded in combinatorial data. In this section, we shall give a brief introduction to toric varieties. Then we will recall Landau-Ginzburg B-model of toric varieties. There are many classical references for toric varieties. For details, see [12], [33] and the comprehensive book by D. Cox et al. [9].

Definition 4.1. A toric variety is a complex variety X containing a torus $\mathbb{T} := (\mathbb{C}^*)^r$ as an open dense subset, and the action of $\mathbb{T}$ on itself by multiplication extends to an action on the whole variety X .

Example 4.2. Complex projective space $\mathbb{P}^n$ is a toric variety. The open dense torus is $\{[x_0 : \cdots : x_n] | x_i \neq 0\}$.

From the definition, the combinatorial nature of toric variety is not so obvious. When the toric variety is normal, it can be constructed from a combinatorial object called fan.

Let N be a lattice, that is, a free abelian group of finite rank r . Denote $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$

Definition 4.3. A convex cone $\sigma \in N_{\mathbb{R}}$ is a subset of the form

$$(2) \quad \sigma = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \geq 0 \quad \forall v_1, \dots, v_k \in N_{\mathbb{R}}\}.$$

We say σ is generated by the vectors $\{v_1, \dots, v_k\}$. A convex cone is called rational if $v_i \in N_{\mathbb{Z}}$. It is called strongly convex if $\sigma \cap -\sigma = 0$.

In this thesis, unless otherwise specified, all of the cones are strongly convex rational cones. A face of a cone σ is defined as the subset in (2) given by setting some a_i to zero.

Definition 4.4. A fan is a collection Σ of strongly convex rational polyhedral cones satisfying:

- (1) Each face of a cone in Σ is also a cone in Σ .
- (2) The intersection of any two cones in Σ is a face of each of them.

We will use $\Sigma(r)$ to denote the set of r -dimensional fans in Σ . For instance, $\Sigma(1)$ is the set of 1-dimensional cones (or rays) in Σ .

Example 4.5. Let $N \cong \mathbb{Z}^2$. We consider a complete fan Σ in N such that the primitive generators of the cones in $\Sigma(1)$ are given as follow:

$$\begin{aligned} v_1 &= (1, 0), \\ v_2 &= (0, 1), \\ v_3 &= (-1, -1). \end{aligned}$$

The fan Σ has three 2-dimensional cones which are generated by $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ and the 0-dimensional cone $\{0\}$. We will denote this fan by $\Sigma(\mathbb{P}^2)$ throughout this thesis. The toric variety associated to this fan is the two dimensional projective space.

Example 4.6. Let $N \cong \mathbb{Z}^2$. We consider a complete fan Σ in N such that the primitive generators of cones in $\Sigma(1)$ are given as follows:

$$\begin{aligned} v_1 &= (1, 0), \\ v_2 &= (0, 1), \\ v_3 &= (-1, 2), \\ v_4 &= (0, -1). \end{aligned}$$

The fan Σ has four two dimensional cones which are generated by $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$ and 0-dimensional cone $\{0\}$. The toric variety corresponds to this fan is the Hirzebruch surface H_2 , namely the projectivization of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

We will use $\Sigma(\mathbb{P}^2)$ and H_2 to illustrate the construction of LG-model of toric varieties.

There are several approaches to constructing a toric variety from a fan. The first approach, maybe the most standard one is to define an affine variety for each cone in the fan and glue the cones together by using the intersections of the cones. In this thesis, we don't use this method. We don't delve into the details. We describe in detail the quotient construction of toric variety. Let Σ be a given fan in $N_{\mathbb{R}} \cong \mathbb{R}^r$. Define an affine space $\mathbb{C}^k := \{(x_\rho) \mid x_\rho \in \mathbb{C}\}_{\rho \in \Sigma(1)}$. Here k is the number of 1-dimensional rays in Σ . Inside this affine space, the discriminant locus is defined to be

$$\mathbf{Z}(\Sigma) = \bigcup_{S \in \overline{C}(\Sigma)} \mathbb{C}^S,$$

where $\mathbb{C}^S := \{(x_\rho) \in \mathbb{C}^n \mid x_\rho = 0, \forall \rho \in S\}$, $\overline{C}(\Sigma)$ is the set of anti-cones. An anti-cone of Σ is a subset of rays that don't span a cone in Σ . Next, we define a group action on $\mathbb{C}^n \setminus \mathbf{Z}(\Sigma)$. For convenience, let us index the 1-dimensional cones ρ in Σ by integers from 1 to $k = \#\Sigma(1)$. Let the primitive generators of 1-dimensional cones be $v_i = (v_{i,1}, \dots, v_{i,r})$, $i = 1, \dots, k$. Then we define the following group homomorphism ϕ_Σ from $(\mathbb{C}^*)^k$ to $(\mathbb{C}^*)^r$:

$$\phi_\Sigma : (\mathbb{C}^*)^{\#\Sigma(1)} \ni (t_1, \dots, t_k) \mapsto \left(\prod_{i=1}^n t_i^{v_{i,1}}, \dots, \prod_{i=1}^n t_i^{v_{i,r}} \right),$$

Denote by G_Σ the kernel of the map ϕ_Σ . Then we have a natural action of G_Σ on $\mathbb{C}^k$. It is easy to see that the action of G preserves $\mathbf{Z}(\Sigma)$. The toric variety X_Σ is defined to be:

$$X_\Sigma := (\mathbb{C}^n \setminus \mathbf{Z}(\Sigma)) / G_\Sigma$$

Example 4.7. Let the fan Σ be that of Example 4.5 then $r = 2$ and $k = 3$. The discriminant locus is

$$\mathbf{Z}(\Sigma) = \{0\},$$

this is because the only subset of the vectors v_1, v_2, v_3 that does not span a cone is $\{v_1, v_2, v_3\}$. The group morphism ϕ is as follows:

$$\phi_\Sigma := (\mathbb{C}^*)^3 \rightarrow (\mathbb{C}^*)^2,$$

$$(t_1, t_2, t_3) \mapsto (t_1 t_3^{-1}, t_2 t_3^{-1}).$$

Consequently, we have $\text{Ker}(\phi_\Sigma) = \{(t, t, t) \mid t \in \mathbb{C}^*\}$. We have

$$X_\Sigma = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^2.$$

Example 4.8. Let the fan Σ be that in Example 4.6. Then $r = 2$ and $k = 4$. We have

$$\mathbf{Z}(\Sigma) = \mathbb{C}^{\{1,3\}} \cup \mathbb{C}^{\{2,4\}}.$$

The map ϕ_Σ is given as follows:

$$(t_1, t_2, t_3, t_4) \mapsto (t_1 t_3^{-1}, t_2 t_3^{-1} t_4^{-1}),$$

whose kernel is

$$G_\Sigma = \{(t_1, t_2, t_1, t_1^2 t_2)\} \cong (\mathbb{C}^*)^2.$$

We have

$$X_\Sigma = (\mathbb{C}^4 \setminus (\{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\}))/G_\Sigma.$$

4.1. Chow ring and cohomology ring of toric varieties. Since we shall compute the cohomology of toric varieties frequently in this thesis, in this subsection, we shall recall some facts the Chow ring and cohomology of toric varieties. In algebraic geometry, Chow ring $A_*(X)$ of an algebraic manifold X contains information how cycles on X intersect. Given two cycles A and B on X , if they intersect transversely, intuitively, the intersection $[A] \cap [B]$ should be the intersection loci of A and B . However, two given cycles are not necessary in a transverse position. A natural idea is to move cycles to a transverse position. Consequently, we have to identify the cycles which can be moved to each other. In Chow theory, the notion of rational equivalence is introduced to move cycles. To be precise, a cycle is allowed to move in a family of varieties parametrized by $\mathbb{P}^1$. Two algebraic cycles on X can always be moved to a transverse position by rational equivalence. We refer the readers to [13] for a systematic exposition of intersection theory and the details of proofs.

Let A_k denote the group of k -dimensional Chow group. Namely, the free group of k -dimensional algebraic cycles on X modulo rational equivalence. Let $A_* = \bigoplus_k A_k(X)$, as discussed above, we can define a product on A_* to make it a ring. We call this ring the Chow ring of X . Suppose we have a proper morphism $\pi : Y \longrightarrow X$. This morphism induces a homomorphism:

$$\pi_* := A_*(Y) \longrightarrow A_*(X).$$

Suppose Y is a closed subvariety in X . Then we have the following exact sequence:

$$(3) \quad A_*(Y) \longrightarrow A_*(X) \longrightarrow A_*(X \setminus Y) \longrightarrow 0.$$

Now, let us return to the discussion of toric varieties. We still use Σ to denote a fan in $N_{\mathbb{R}}$ which is isomorphic to $\mathbb{R}^r$. As discussed in the last section, we can associate every cone σ in Σ a toric subvariety of codimension $\dim \sigma$. We denote this subvariety by X_σ . We have the following theorem:

Theorem 4.9. [10] *The cycles $\{X_\sigma\}_{\sigma \in \Sigma}$ generate the Chow ring of X .*

Proof. In the exact sequence (3), we choose $X \setminus Y$ to be the open dense torus $\mathbb{T}$ in X , $Y = X \setminus \mathbb{T}$. That is, we have:

$$(4) \quad A_d(X \setminus \mathbb{T}) \longrightarrow A_d(X) \longrightarrow A_d(\mathbb{T}) \longrightarrow 0$$

Since $A_*(\mathbb{T}) = \mathbb{Z}$ is generated by the fundamental cycle $\mathbb{T}$, combined with the exact sequence (4), we know that for $d < r$, the map $A_d(X \setminus \mathbb{T}) \longrightarrow A_d(X)$ is surjective. This is equivalent to saying that any cycle on X can be moved into $X \setminus \mathbb{T}$. Since X_σ is toric variety for any $\sigma \neq \{0\}$, by induction method, we are done. $\square$

Let D_ρ be the toric invariant divisor associated to a 1-dimensional cone ρ in Σ . Take a regular function f on the open dense torus $\mathbb{T}$, which extends to a meromorphic function on X . All the poles and zeros are supported on the toric invariant divisors. Hence f gives a linear relation between D_ρ . We denote by R_{lin} all of these relation between D_ρ . If the intersection of $D_{\rho_1}, \dots, D_{\rho_k}$ are empty, the product $[D_{\rho_1}] \cap \dots \cap [D_{\rho_k}]$ in $A_*(X)$ is zero. We denote by R_{int} all such relations between D_ρ . It is clear that there is a surjective ring homomorphism $\text{pr} : \mathbb{C}[D_\rho] / \langle R_{\text{lin}}, R_{\text{int}} \rangle \rightarrow A_*(X)$. In fact, for a smooth toric variety X , pr is an isomorphism. For any smooth complete algebraic variety X , there is a natural homomorphism from $A_*(X)$ to $H^{\text{ev}}(X)$. It is known that $A_*(X) \cong H^{\text{ev}}(X)$ for any smooth complete toric variety. In summary, we can compute the cohomology ring of any smooth complete toric variety:

Theorem 4.10. *Suppose that Σ is a complete fan and that X_Σ is smooth. Then*

$$H^{\text{ev}}(X_\Sigma) = \mathbb{C}[D_\rho] / \langle R_{\text{lin}}, R_{\text{int}} \rangle.$$

Proof. Details of the proof can be found in [10]. $\square$

Example 4.11. Let Σ be the fan in 4.5. There are two linear relations between D_1, D_2 , and D_3 :

$$D_1 = D_3, \quad D_2 = D_3.$$

In fact, $D_1 = D_2 = D_3 = H$, where H is the hyperplane class. Since the intersection of D_1, D_2 , and D_3 is empty, $D_1 D_2 D_3 = H^3 = 0$. We get the well-know presentation of $H^*(\mathbb{P}^2)$:

$$H^*(\mathbb{P}^2) = \mathbb{C}[H] / \langle H^3 \rangle.$$

Example 4.12. Let Σ be the fan in 4.6. We have two linear relations:

$$D_1 = D_3, \quad D_2 + D_3 = D_4.$$

The intersection of D_1 and D_3 is empty, and the intersection of D_2 and D_4 is also empty. As a result, we have:

$$D_1 D_3 = 0, \quad D_2 D_4 = 0.$$

We get the following presentation of $H^*(H_2)$:

$$H^*(H_2) = \mathbb{C}[D_1, D_4] / \langle D_1^2, (D_4 - D_1)D_4 \rangle.$$

4.2. Landau-Ginzburg B-Model. In this section, we shall describe the Landau-Ginzburg (LG) models, which are mirror to compact toric manifolds. The LG mirrors for toric manifolds have been proposed by Givental in [16], [17] and Hori and Vafa [21]. This construction can be easily adapted to toric orbifolds [23]. In this section, we will review briefly the Landau-Ginzburg B-Model for a toric Fano variety X . Let Σ be a fan in $\mathbf{N}$ for X , where $\mathbf{N}$ is a free abelian group of rank r . The fan Σ gives an exact sequence as follows:

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^k \xrightarrow{\beta} \mathbf{N} \cong \mathbb{Z}^r \rightarrow 0,$$

where k is the number of 1-dimensional rays in Σ . If we choose a coordinate for N , the morphism β can be written as a $(r \times k)$ -matrix whose columns are the coordinates of primitive generators of rays. The variety X is defined as the quotient $(\mathbb{C}^k \setminus \mathbf{Z}(\Sigma))/G_\Sigma$. Tensoring the short exact sequence dual to 4.2, we get the following exact sequence:

$$1 \rightarrow \text{Hom}(\mathbf{N}, \mathbb{C}^*) \rightarrow (\mathbb{C}^*)^k \xrightarrow{\text{pr}} \text{Hom}(\mathbb{L}, \mathbb{C}^*) \rightarrow 1.$$

Denote the co-ordinates of the algebraic torus $(\mathbb{C}^*)^k$ by $(w_1, w_2, \dots, w_k)$. The superpotential for toric variety X is given by

$$W = w_1 + \dots + w_k.$$

Any fiber of the morphism pr is an algebraic torus. The restriction of W on the fibers of pr vary with the fibers. We call the parameter space of the superpotential, that is the space $\text{Hom}(\mathbb{L}, \mathbb{C}^*)$, the moduli space of B-model which we denote by $\mathcal{M}_q$. We denote by q the co-ordinates on $\mathcal{M}_q$. Then for a point $q \in \mathcal{M}_q$, we get an algebraic torus $Y_q := \text{pr}(q)$ and a function $W_q = W|_{Y_q}$. We call the tuple $(\mathcal{M}_q, Y_q, W_q)$ the LG Model of the toric variety X . We choose a basis for $N \cong \mathbb{Z}^n$. Then co-ordinates $(x_1, x_2, \dots, x_k)$ on $\text{Hom}(N, \mathbb{C}^*)$ are determined. We choose a basis for $\mathbb{L}$. Then co-ordinates $(q_1, \dots, q_{k-r})$ on $\text{Hom}(\mathbb{L}, \mathbb{C}^*) = \mathcal{M}_q = (\mathbb{C}^*)^{k-r}$ are determined. Here it is better to choose co-ordinates $(q_1, \dots, q_{k-r})$ corresponding to a nef basis of $\text{Pic}(X)$, because such co-ordinates will guarantee that no monomials of q_i with negative power exponent would be involved.

Next, we use our favorite two examples to illustrate how to construct LG B-models for toric varieties.

Example 4.13. Let toric diagram Σ be given by Example 4.5. The toric diagram can be encoded in the following short exact sequence:

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^m \rightarrow \mathbf{N} \rightarrow 0.$$

The map $\text{pr} : (\mathbb{C}^*)^3 \rightarrow \text{Hom}(\mathbb{L}, \mathbb{C}^*) \cong \{q\}$ is given by $q = w_1 w_2 w_3$. The superpotential W_q is given by

$$W_q = w_1 + w_2 + \frac{q}{w_1 w_2}.$$

Example 4.14. Let the toric diagram Σ be given by Example 4.6. We choose the coordinates on $\mathcal{M}_q$ as follows. An easy calculation tells us that the nef cone

of H_2 is spanned by D_1 and D_4 . We choose dual coordinates (q_1, q_2) on M_q . Then the map $\text{pr} : (\mathbb{C}^*)^4 \rightarrow \text{Hom}(\mathbb{L}, \mathbb{C}^*) \cong \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{C}^*\}$ is given by:

$$\begin{aligned} q_1 &= w_1 w_2^{-2} w_3, \\ q_2 &= w_2 w_4. \end{aligned}$$

The superpotential is given as follows:

$$W_q = w_1 + w_2 + q_1 \frac{w_2^2}{w_1} + q_2 \frac{1}{w_2}.$$

5. GRASSMANNIAN AND QUANTUM SCHUBERT CALCULUS

In this section, we shall review Aaron Bertram's work on quantum Schubert calculus. The main reference is [4]. The classical Schubert calculus tells us how the Schubert classes intersect on Grassmannian. In [4], the author computed the quantum product matrix of Schubert divisors. In what follows, we will give a brief review of [4].

In [4], Bertram used a compactification different from Kontsevich's moduli space of stable maps. He used the Grothendieck quot scheme as a compactification of the space of holomorphic maps of a fixed degree from $\mathbb{P}^1$ to a Grassmannian. Then the author considered intersection theory on Quot schemes. In order to fix notation, we begin with an overview of the classical Schubert calculus.

Let V be an n -dimensional vector space over $\mathbb{C}$. Fix a full flag of V :

$$0 = V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n = V.$$

Grassmannian $\text{Gr}(n-k, n)$ is the space of $(n-k)$ -dimensional linear subspaces of V . Let x be a point of the Grassmannian $\text{Gr}(n-k, n)$. We denote by Λ_x the linear subspace corresponding to $x \in \text{Gr}(n-k, n)$. Given a $(n-k)$ -tuple of integers $\vec{a} := (a_1, \dots, a_{n-k})$ satisfying inequalities $k \geq a_1 \geq \cdots \geq a_{n-k} \geq 0$, we define the Schubert cycle $\sigma_{\vec{a}}$ as follows:

$$\sigma_{\vec{a}} = \{x \in \text{Gr}(n-k, n) \mid \dim(\Lambda_x \cap V_{k+i-a_i}) \geq i\}.$$

The subvariety $\sigma_{\vec{a}}$ is of complex codimension $|\vec{a}| = \sum_{i=1}^{n-k} a_i$. We denote by $\omega_{\vec{a}}$ the Poincaré dual of $\sigma_{\vec{a}}$. It is well known that the set of cohomology classes associated to $\vec{a}$ constitute an additive basis of the cohomology group $H^*(\text{Gr}(n-k, n))$. We call the Schubert varieties $\sigma_a = \sigma_{(a, 0, \dots, 0)}$ special Schubert varieties. The Poincaré dual ω_a of σ_a coincides with the Chern classes $c_a(Q)$ in which Q is the universal quotient bundle on $\text{Gr}(n-k, n)$.

We have the following classical result:

Theorem 5.1 (Giambelli's Formula). *The cohomology class $\omega_{\vec{a}}$ can be expressed as the determinant:*

$$\omega_{\vec{a}} = \begin{vmatrix} \omega_{a_1} & \omega_{a_1+1} & \cdots & \omega_{a_1+n-k-1} \\ \omega_{a_2-1} & \omega_{a_2} & \cdots & \omega_{a_2+n-k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{a_{n-k}-(n-k)+1} & \omega_{a_{n-k}-(n-k)+2} & \cdots & \omega_{a_{n-k}} \end{vmatrix}.$$

The following formula tells us how to compute the intersections of a special schubert class and a general one:

Theorem 5.2 (Pieri's Formula). *The product of ω_a and $\omega_{\vec{a}}$ is given by*

$$\omega_a \cdot \omega_{\vec{a}} = \sum_{\vec{b}} \omega_{\vec{b}},$$

in which the $\vec{b}$ is summed over all $(n-k)$ -tuples satisfying:

$$|\vec{b}| = |\vec{a}| + a, \quad k \geq b_1 \geq a_1 \geq \cdots \geq b_{n-k} \geq a_{n-k} \geq 0$$

The above two formulas determine completely the cohomology ring structure of Grassmannian. We set $\vec{a}^c := (k - a_{n-k}, k - a_{n-k-1}, \dots, k - a_1)$. Then the Poincaré pairing of Grassmannian takes a very nice shape:

$$(5) \quad \langle \omega_{\vec{a}}, \omega_{\vec{b}} \rangle = \begin{cases} 1, & \text{if } \vec{a} = \vec{b}^c \\ 0, & \text{otherwise} \end{cases}$$

In [4], Bertram computed the small quantum cohomology of Grassmannian. The following two theorems are the main results in [4].

Theorem 5.3 (Quantum Giambelli [4]). *The cohomolgy class $\omega_{\vec{a}}$ can be expressed as the determinant:*

$$\omega_{\vec{a}} = \begin{vmatrix} \omega_{a_1} & \omega_{a_1+1} & \cdots & \omega_{a_1+n-k-1} \\ \omega_{a_2-1} & \omega_{a_2} & \cdots & \omega_{a_2+n-k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{a_{n-k}-(n-k)+1} & \omega_{a_{n-k}-(n-k)+2} & \cdots & \omega_{a_{n-k}} \end{vmatrix},$$

in which the determinant is evaluated in $QH^*(\text{Gr})$, the product is quantum product. In other words, no higher order terms arise from the Giambelli determinant.

Theorem 5.4 (Quantum Pieri [4]). *The quantum product of ω_a and $\omega_{\vec{a}}$ is given by*

$$\omega_a \star \omega_{\vec{a}} = \omega_a \cdot \omega_{\vec{a}} + q \left(\sum_{\vec{b}} \omega_{\vec{b}} \right),$$

in which the $\vec{b}$ range over all $(n-k)$ -tuples satisfying:

$$|\vec{b}| = a + |\vec{a}| - n, \quad a_1 - 1 \geq c_1 \geq a_2 - 1 \geq c_2 \geq \cdots \geq a_{n-k} - 1 \geq c_{n-k} \geq 0.$$

The quantum Giambelli and quantum Pieri theorems determine the quantum cohomology ring of Grassmannian variety. In the following, we will calculate two examples we will use later.

Example 5.5. Quantum Cohomology of $\text{Gr}(2, 4)$:

The Schubert classes of $\text{Gr}(2, 4)$ are indexed by 2-tuples:

$$\begin{array}{lll} (0, 0), & (1, 0), & (1, 1), \\ (2, 0), & (2, 1), & (2, 2). \end{array}$$

According to Quantum Pieri, we have:

$$\begin{aligned} \omega_{(1,0)} \star \omega_{(0,0)} &= \omega_{(1,0)}, \\ \omega_{(1,0)} \star \omega_{(1,0)} &= \omega_{(2,0)} + \omega_{(1,1)}, \\ \omega_{(1,0)} \star \omega_{(1,1)} &= \omega_{(2,1)}, \\ \omega_{(1,0)} \star \omega_{(2,0)} &= \omega_{(2,1)}, \\ \omega_{(1,0)} \star \omega_{(2,1)} &= \omega_{(2,2)}, \\ \omega_{(1,0)} \star \omega_{(2,2)} &= q\omega_{(1,0)}. \end{aligned}$$

So the quantum multiplication by $\omega_{(1,0)}$ is given as follows:

$$\omega_{(1,0)} \star = \begin{pmatrix} 0 & 0 & 0 & 0 & q & 0 \\ 1 & 0 & 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Example 5.6. Quantum Cohomology of $\text{Gr}(2, 5)$: the Schubert cycles are indexed by the following 2-tuples:

$$\begin{array}{lllll} (0, 0), & (1, 0), & (1, 1), & (2, 0), & (2, 1), \\ (3, 0), & (3, 1), & (2, 2), & (3, 2), & (3, 3). \end{array}$$

According to Quantum Pieri, we can compute the quantum product of $\omega_{(1,0)}$ with any other Schubert cycles:

$$\begin{aligned}
\omega_{(1,0)} \star \omega_{(0,0)} &= \omega_{(1,0)}, \\
\omega_{(1,0)} \star \omega_{(1,0)} &= \omega_{(2,0)} + \omega_{(1,1)} & \omega_{(1,0)} \star \omega_{(1,1)} &= \omega_{(2,1)}, \\
\omega_{(1,0)} \star \omega_{(2,0)} &= \omega_{(2,1)} + \omega_{(3,0)}, \\
\omega_{(1,0)} \star \omega_{(2,1)} &= \omega_{(2,2)} + \omega_{(3,1)}, \\
\omega_{(1,0)} \star \omega_{(3,0)} &= q\omega_{(3,1)}, \\
\omega_{(1,0)} \star \omega_{(3,1)} &= \omega_{(3,2)} + q, \\
\omega_{(1,0)} \star \omega_{(2,2)} &= \omega_{(3,2)}, \\
\omega_{(1,0)} \star \omega_{(3,2)} &= \omega_{(3,3)} + q\omega_{(1,0)}, \\
\omega_{(1,0)} \star \omega_{(3,3)} &= q\omega_{(2,0)}.
\end{aligned}$$

Then quantum product matrix of $\omega_{(1,0)}$ is given as follows:

$$\omega_{(1,0)} \star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 1 & 0 & 0 \end{pmatrix}$$

6. PARTIAL FLAG MANIFOLDS AND THEIR LG MIRROR

6.1. LG B-model of $\text{Fl}(n_1, \dots, n_l, n)$. In this section, first of all, we recall Givental's construction of LG B-model for partial flag varieties. Then we recall toric degenerations of partial flag varieties without proofs. We refer the interested reader to [2].

Let us begin with recalling some basic notations about partial flag varieties. Let $s_1, s_2, \dots, s_{l+1}$ be a fixed sequence of positive integers. We set $n_0 = 0, n_i := s_1 + \dots + s_i$ ($i = 1, \dots, l+1$), and $n := n_{l+1}$. A partial flag in $\mathbb{C}^n$ of type $(n_1, \dots, n_l)$ is a sequence of linear subspaces $V_i \in \mathbb{C}^n$ satisfying:

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_l \subset \mathbb{C}^n,$$

where the dimension of V_i is n_i . We denote by $\text{Fl}(n_1, \dots, n_l, n)$ the moduli space of partial flags in $\mathbb{C}^n$ of type $(n_1, \dots, n_l)$. It's a compact algebraic variety. The dimension of partial flag variety $\text{Fl}(n_1, \dots, n_l, n)$ is given by

$$\dim \text{Fl}(n_1, \dots, n_l, n) = \sum_{i=1}^l (n_i - n_{i-1})(n - n_i).$$

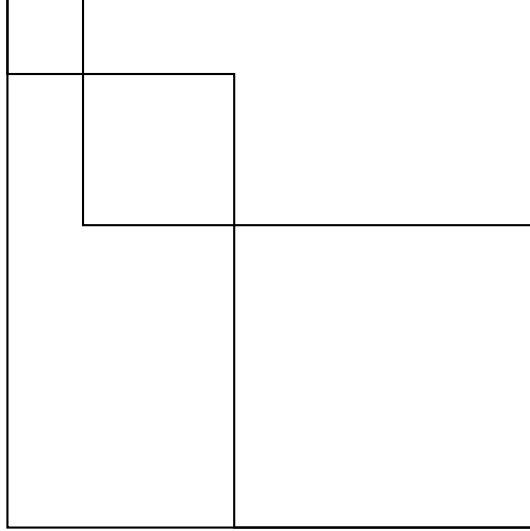


FIGURE 1.

For brevity, we will use Fl to denote the flag variety $\text{Fl}(n_1, \dots, n_l, n)$, if there is no ambiguity about the numbers $n_1, \dots, n_l, n$. By a classical result, a natural basis for integral cohomology of Fl is given by Schubert classes. By Schubert classes, we mean the Poincaré dual to the fundamental classes of the closed Schubert cells $C_w \subset \text{Fl}$, which are indexed by permutations $w \in S_n$ modulo the subgroup

$$W(s_1, \dots, s_{l+1}) := S(s_1) \times \dots \times S(s_{l+1}) \subset S_n,$$

where the subgroup $S(s_i)$ is the permutation group on numbers $\{s_{i-1} + 1, \dots, s_{i-1} + s_i\}$. The Picard group of Fl , which is isomorphic to $H^2(\text{Fl}, \mathbb{Z})$, is generated by l divisors $C_1, \dots, C_l$, where C_i corresponds to the simple transpositions $t_i \in S_n$ exchanging n_i and $n_i + 1$. To be precise, the divisors C_i can be described as follows. Let P be the maximal parabolic subgroup of $GL(n, \mathbb{C})$ fixing a given partial flag in $\text{Fl}(n_1, \dots, n_l, n)$. Let B be the Borel subgroup of $GL(n, \mathbb{C})$. Let $w_0 := (n, n-1, \dots, 2, 1)$, the order-reversing permutation. Then C_i is closure of the schubert cell $C_{w_0(n_i, n_{i+1})P} := Bw_0(n_i, n_{i+1})P/P$.

We draw a Graph G for Fl as follows: First of all, draw a square with sides of length n units. Then we draw a sequence of squares $Q_i (i \in 1, \dots, l+1)$ with sides of length s_i units along the diagonal from upper-left to lower-right (See Figure 1). Then we divide the lower-left part into unit squares as shown in Figure 2. Put a dot at the center of every unit square in the lower-left part and a cross star at the lower-left corner of diagonal boxes (see Figure 3). Then we draw edges connecting the neighboring dots and cross stars. Furthermore, we orient vertical edges from top to bottom and orient horizontal edges from left to right (as shown in Figure 4). Up to this point, we get a directed graph as shown in Figure 5, which we denote by $G(n_1, \dots, n_l, n)$.

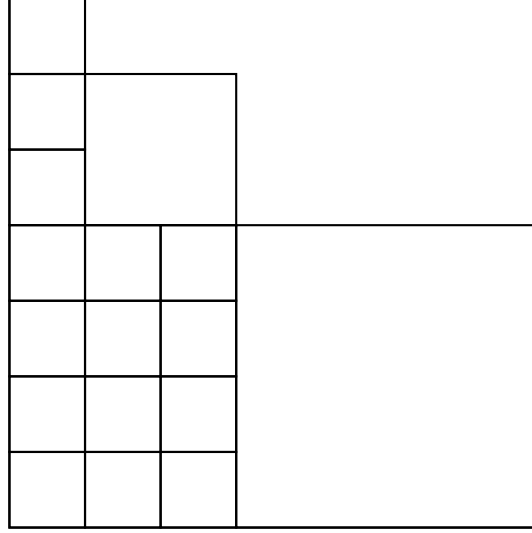


FIGURE 2.

For convenience, we will use G to denote the graph $G(n_1, \dots, n_l, n)$, if there is no ambiguity. We associate with G the following data:

- $D = D(n_1, \dots, n_l, n)$, the set of dots at the centers of unit squares in G . As shown in Figure 5, there are 14 dots
- $S = S(n_1, \dots, n_l, n)$, the set consisting of $(l+1)$ cross stars: an element of S is obtained by placing cross star at the lower-left corner of each diagonal squares $Q_i (i \in 1, \dots, l+1)$. In Figure 5, there are 3 cross stars.
- $E = E(n_1, \dots, n_l, n)$, the set of oriented horizontal and vertical segments connecting adjacent elements of $D \cup S$: the vertical segments are oriented downwards, and the horizontal segments are oriented to the right. In Figure 5, there are 23 oriented edges.

Definition 6.1. [2] A roof R_i , $i \in \{1, 2, \dots, l\}$ is the set of $s_i + s_{i+1}$ edges of G forming the oriented path that runs along the upper right "boundary" of G between the i -th and the $(i+1)$ -th cross stars in S . For our example, we have two roofs, see Figure 6.

We need the following definitions:

Definition 6.2. [2] A box b in G is a subset of 4 edges $\{e, f, g, h\}$ of E which form together with their end-points a connected subgraph G_b , as shown in Figure 7. The oriented sides e, f, g, h form a unit box in G . The set of boxes in G will be denoted by B .

Definition 6.3. [2] The corner $\mathcal{C}_b$ of a box $b = \{e, f, g, h\} \in B$ is the pair of oriented edges $\{e, f\} \in b$ meeting at the lower left vertex of G_b . So a corner $\mathcal{C}_b$

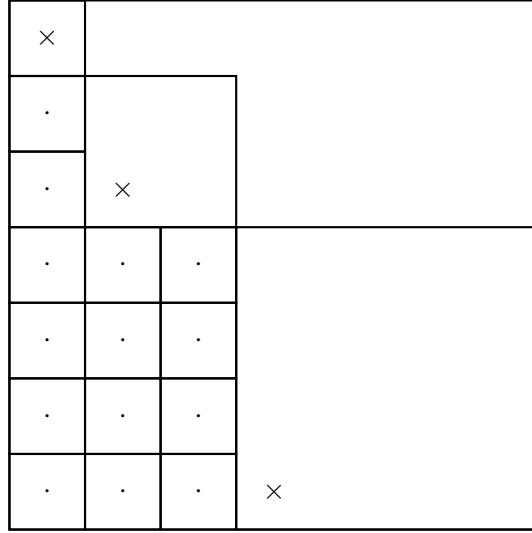


FIGURE 3.

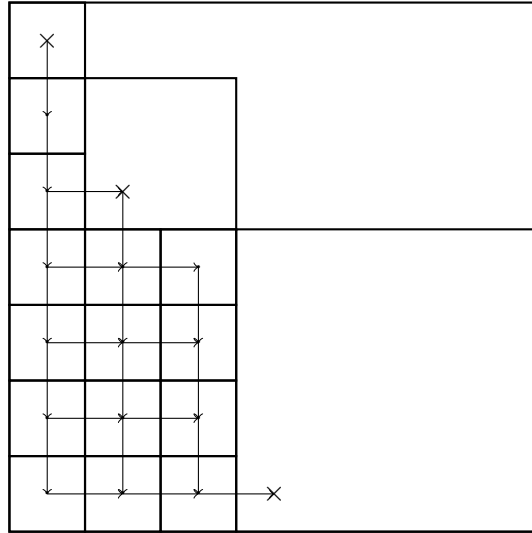


FIGURE 4.

contains one vertical edge e and one horizontal edge f such that $h(e) = t(f)$ as shown in Figure 7.

Definition 6.4. [2] Anti-corner $\mathcal{C}_b^-$ of a box $b = \{e, f, g, h\} \in B$ is the pair of edges $\{g, h\}$ meeting at the upper right vertex of G_b . See Figure 7.

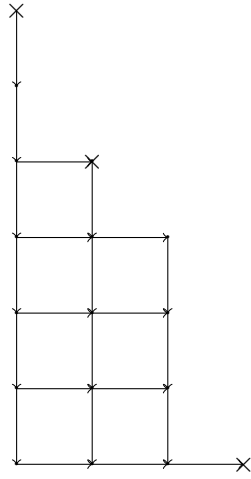


FIGURE 5.

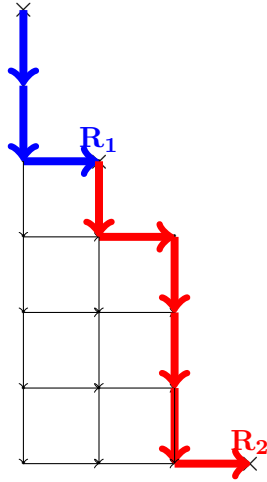


FIGURE 6.

The roofs and corners give a decomposition of the set E of edges of the graph G into a disjoint union of subsets:

$$E = R_1 \cup \cdots \cup R_l \cup \bigcup_{b \in B} C_b$$

In [14][17], Givental and Kim proposed a LG B-model for complete flag varieties. Lately, it is generalized to partial flag varieties in [2]. The LG B-model can be described as follows: to each edge e , we associate a variable X_e , and to each

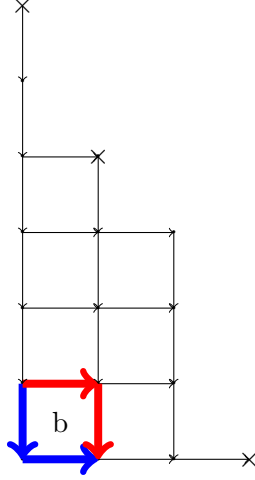


FIGURE 7.

roof R_i , we associate a quantum parameter q_i , and a relation:

$$(7) \quad \prod_{e \in R_i} X_e = q_i$$

and for each box b , we have a relation between the four variables :

$$(8) \quad \prod_{e \in \mathcal{C}_b} X_e = \prod_{f \in \mathcal{C}_b^-} X_f$$

The super-potential is

$$W := \sum_{e \in E} X_e$$

with relations (7) and (8). The LG mirror dual of a complete flag variety was first proposed by Givental in [14], and generalized to partial flag varieties in [2]. The LG B-model of the partial flag variety Fl is the following tuples: the algebraic variety M_q parametrized by quantum parameters $\vec{q} = (q_1, \dots, q_l)$ is defined as:

$$M_q = \{(\dots, X_e, \dots) \in \mathbb{C}^{\#E} \mid \prod_{e \in R_i} X_e = q_i, \prod_{e \in \mathcal{C}_b} X_e = \prod_{f \in \mathcal{C}_b^-} X_f\}.$$

The super-potential is given by $W_q = \sum_{e \in E} X_e|_{M_q}$. The holomorphic volume form on M_q is defined as follows:

$$\omega_q := \frac{\prod_{e \in E} \wedge d \log(X_e)}{\prod \wedge d \log(q_i)}.$$

We have the following mirror conjecture due to Givental [14] and [2].

Conjecture 6.5. *For any flag variety Fl , the tuple (M_q, W_q, ω_q) is as above. The $\mathcal{D}$ -module generated by the integral*

$$\int_{\Gamma \subset M_q} e^{W_q/z} \omega_q,$$

where Γ are suitable Morse-theoretic middle dimensional cycles of the function $\Re(W_q)$, is equivalent to the quantum $\mathcal{D}$ -module of Fl .

6.2. Toric degeneration of partial flag varieties. In this subsection, we construct a singular toric variety X_Γ from a diagram Γ . This singular toric variety turns out to be a degeneration of Fl . Recall that D , S , and E were defined in the last section.

Definition 6.6. [2] We denote by $L(D) := \mathbb{Z}^{|D|}$, $L(S) := \mathbb{Z}^{|S|}$, and $L(E) := \mathbb{Z}^{|E|}$ the free Abelian groups (or lattices) generated by the sets D , S and E .

Then, we have a linear map from $L(E)$ to $L(D) \oplus L(S)$ defined as follows:

$$\partial : L(E) \rightarrow L(D) \oplus L(S), e \mapsto h(e) - t(e),$$

where $h, t : E \rightarrow D \cup S$ are the maps that associate to an oriented edge $e \in E$ its head and its tail respectively. We also consider the projection $\rho : L(D) \oplus L(S) \rightarrow L(D)$ and the composed map $\delta := \rho \circ \partial : L(E) \rightarrow L(D)$.

Now, we are in a position to define the singular toric variety $T(n_1, \dots, n_l, n)$. We denote again by δ the $\mathbb{R}$ -scalar extension $L(E) \otimes \mathbb{R} \rightarrow L(D) \otimes \mathbb{R}$ of the homomorphism $\delta : L(E) \rightarrow L(D)$.

Definition 6.7. [2] The polyhedron $\Delta := \Delta(n_1, \dots, n_l, n)$ associated to Fl is the convex hull of the set

$$\delta(E) \subset L(D) \otimes \mathbb{R},$$

where the set E is identified with the standard basis of $L(E) \otimes \mathbb{R} = \mathbb{R}^{|E|}$.

Definition 6.8. [2] The complete rational polyhedral fan $\Sigma = \Sigma(n_1, \dots, n_l, n)$ is the fan defined as the collection of cones over all faces of Δ . The toric variety $\mathbb{P}_\Sigma$ associated to the fan Σ will be denoted by $\mathbb{P} = \mathbb{P}(n_1, \dots, n_l, n)$.

$\mathbb{P}(n_1, \dots, n_l, n)$ is a Gorenstein toric variety. We can divide the fan Σ into a simplicial fan Σ^{simp} . The simplicial fan Σ^{simp} gives a smooth toric variety X^{res} . The LG model of X^{res} can be constructed as follows. We associate to every edge e a variable X_e . The superpotential of X^{res} is still the sum of all variables X_e .

$$(9) \quad W = \sum X_e.$$

We have the following constraint relations on X_e . For each roof $r \in R$, we have a constraint relation as follows:

$$(10) \quad \prod_{e \in r} X_e = q_r.$$

And for each box $b \in B$, we have a constraint relation as follows:

$$(11) \quad \prod_{e \in \mathcal{C}_b} X_e \prod_{f \in \mathcal{C}_b^-} X_f^{-1} = q_b.$$

We find that the relation (8) is just (11) with quantum parameters $q_b = 1$. As a result, if we set the quantum parameters q_b in the superpotential of X_{res} to be 1, we get the superpotential of Fl. We have proved the following theorem:

Theorem 6.9. *The superpotential $W^{\text{Fl}}(\vec{q}_r)$ of a partial flag variety is equal to the superpotential $W^{\text{res}}(\vec{q}_r, \vec{q}_b)$ of X_{res} restricted to the locus $\vec{q}_b = 1$.*

7. EXAMPLE: $\mathbb{F}_2$

In this section, we study an example of 2-dimensional extremal transition. The Hirzebruch surface $\mathbb{F}_2$ is a toric variety. Its fan is given in Example 4.6. On $\mathbb{F}_2$, there is a (-2) rational curve. Blowing it down, we get a singular variety X_{sing} with an A_1 singular point. The singular variety X_{sing} is isomorphic to the weighted projective space $\mathbb{P}(1, 1, 2)$. The weighted projective space $\mathbb{P}(1, 1, 2)$ can be embedded into $\mathbb{P}^3$ as a hypersurface. We can construct a family of varieties in $\mathbb{P}^3$ as follow:

$$\mathfrak{X}_t = \{Z_0 Z_2 = Z_1^2 + t Z_3^2; [Z_0, \dots, Z_3] \in \mathbb{P}^3\}.$$

The fiber $\mathfrak{X}_0$ is $\mathbb{P}(1, 1, 2)$ and the fiber over non-zero t is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The family $\mathfrak{X}_t$ is a smoothing of $\mathbb{P}(1, 1, 2)$. We denote by π the blowing-down map from $\mathbb{F}_2$ to $\mathbb{P}(1, 1, 2)$ and denote by r the continuous contracting map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}(1, 1, 2)$. We study how the quantum cohomology change under this extremal transition.

7.1. Quantum cohomology of $\mathbb{F}_2$. Hirzebruch surface $\mathbb{F}_2$ is a toric variety. Its fan was given in subsection 4.6. Following conventions in section 4, we denote the four toric invariant divisors corresponding to the four rays by D_i $i = 1, \dots, 4$. We have the following linear relations between D_i :

$$D_1 = D_3, \quad D_4 = D_2 + 2D_1.$$

The intersection of D_2 and D_4 is empty and the intersection of D_1 and D_3 is also empty. Consequently, we have:

$$D_1 D_3 = 0, \quad D_2 D_4 = 0.$$

The nef cone is spanned by D_1 and D_4 . The effective cone of curves is spanned by α_1 and α_2 which are dual to D_1 and D_2 respectively. The Poincaré duals of α_1 and α_2 are given by $D_2 - 2D_1$ and D_1 respectively. For convenience, we change notation as follows:

$$u_1 := D_1, u_2 = D_4.$$

In this notation, we can present $H^*(\mathbb{F}_2)$ as follows:

$$H^*(\mathbb{F}_2) = \mathbb{C}[u_1, u_2] / \langle u_1^2, u_2(u_2 - 2u_1) \rangle$$

Givental's I -function for $\mathbb{F}_2$ is given as follows:

$$I_{\mathbb{F}_2} = \exp(u \log Q/z) \times \left(\sum_{d_1 \geq 0, d_2 \geq 0} Q_1^{d_1} Q_2^{d_2} \frac{1}{\prod_{m=1}^{d_1} (u_1 + mz)^2 \prod_{m=1}^{d_2} (u_2 + mz)} \frac{\prod_{m=-\infty}^0 (u_2 - 2u_1 + mz)}{\prod_{m=-\infty}^{d_2-2d_1} (u_2 - 2u_1 + mz)} \right)$$

The I -function has the asymptotical expansion:

$$\exp(u \log Q/z) \left(1 + (2u_1 - u_2)m(Q_1)/z + o\left(\frac{1}{z}\right) \right),$$

where

$$m(Q_1) = \sum_{d_1=1}^{\infty} \frac{(2d_1-1)!}{(d_1!)^2} Q_1^{d_1}.$$

According to Givental's mirror theorem [16], the mirror transformation is given by

$$q_1 = Q_1 \exp(2m(Q_1)), \quad q_2 = Q_2 \exp(-m(Q_1)).$$

The inverse mirror transformation is given as follows:

$$Q_1 = \frac{q_1}{(1+q_1)^2}, \quad Q_2 = q_2(1+q_1).$$

From the cohomology valued function $I_{\mathbb{F}_2}$ and the inverse mirror map, we know that $J_{\mathbb{F}_2}(q_1, q_2)$ has an asymptotical expansion as follows:

$$J_{\mathbb{F}_2} = \exp(u \log q/z) \left(1 + \frac{q_1 q_2 + q_2}{z^2} + o\left(\frac{1}{z^2}\right) \right)$$

From above expansion, we can calculate the quantum product as follow:

$$u_2 \star u_2 = 2[\text{pt}] + q_1 q_2 + q_2, \quad u_2 \star u_1 = [\text{pt}] + q_1 q_2, \quad u_1 \star u_1 = q_1 q_2.$$

We choose the following basis of $H^*(\mathbb{F}_2)$:

$$\{1, \quad u_1, \quad u_2, \quad u_1 u_2 = \text{PD}([\text{pt}])\}.$$

In this basis, the quantum product matrices of u_1, u_2 are given as follows:

$$u_1 \star = \begin{pmatrix} 0 & q_1 q_2 & q_1 q_2 & 0 \\ 1 & 0 & 0 & -q_1 q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad u_2 \star = \begin{pmatrix} 0 & q_1 q_2 & q_1 q_2 + q_2 & 0 \\ 0 & 0 & 0 & q_2 - q_1 q_2 \\ 1 & 0 & 0 & q_1 q_2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

7.2. Quantum cohomology of X_{res} . The Hirzebruch surface $\mathbb{F}_2$ can be degenerated to $\mathbb{P}(1, 1, 2)$. The weighted projective space $\mathbb{P}(1, 1, 2)$ is smoothable to $\mathbb{P}^1 \times \mathbb{P}^1$. We denote by p_1 the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the first factor $\mathbb{P}^1$, and denote by p_2 the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the second factor $\mathbb{P}^1$. We denote by h_1 a fiber of p_2 , and by h_2 a fiber of p_1 . We denote the Poincaré dual of h_1 by H_1 , and denote the Poincaré dual of h_2 by H_2 . H_1 and H_2 span the nef cone

of $\mathbb{P}^1 \times \mathbb{P}^1$. We denote the curve classes dual to H_1 and H_2 by β_1 and β_2 . The I -function of $\mathbb{P}^1 \times \mathbb{P}^1$ is as follows:

$$I_{\mathbb{P}^1 \times \mathbb{P}^1} = \exp(H \log \tilde{q}/z) \left(\sum_{d_1 \geq 0, d_2 \geq 0} \tilde{q}_1^{d_1} \tilde{q}_2^{d_2} \frac{1}{\prod_{m=1}^{d_1} (H_1 + mz)^2} \frac{1}{\prod_{m=1}^{d_2} (H_2 + mz)^2} \right)$$

Since the asymptotical expansion of $I_{\mathbb{P}^1 \times \mathbb{P}^1}$ has the form:

$$\exp(H \log \tilde{q}/z) \left(1 + o\left(\frac{1}{z^2}\right) \right),$$

the mirror transformation is trivial. So we have $J_{\mathbb{P}^1 \times \mathbb{P}^1} = I_{\mathbb{P}^1 \times \mathbb{P}^1}$. We can compute the quantum products as follows:

$$H_1 \star H_1 = \tilde{q}_1, \quad H_1 \star H_2 = \text{PD}([\text{pt}]), \quad H_2 \star H_2 = \tilde{q}_2.$$

We choose the following basis of $H^*(\mathbb{P}^1 \times \mathbb{P}^1)$:

$$\{1, \quad H_1, \quad H_2, \quad H_1 H_2 = \text{PD}([\text{pt}])\}.$$

In this basis, the quantum product by H_1, H_2 are given as follows:

$$H_1 \star = \begin{pmatrix} 0 & \tilde{q}_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{q}_1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad H_2 \star = \begin{pmatrix} 0 & 0 & \tilde{q}_2 & 0 \\ 0 & 0 & 0 & \tilde{q}_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

7.3. Comparison of quantum cohomology. We have natural maps:

$$\mathbb{F}_2 \xrightarrow{\pi} \mathbb{P}(1, 1, 2) \xleftarrow{r} \mathbb{P}^1 \times \mathbb{P}^1.$$

The map $\pi^* : H^*(\mathbb{P}(1, 1, 2), \mathbb{C}) \longrightarrow H^*(\mathbb{F}_2, \mathbb{C})$ is injective and has the image:

$$S(\mathbb{F}_2) := \mathbb{C}1 \oplus \mathbb{C}u_2 \oplus \mathbb{C}\text{PD}([\text{pt}]).$$

The map $r^* : H^*(\mathbb{P}(1, 1, 2), \mathbb{C}) \longrightarrow H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C})$ is also injective and has the image:

$$S(\mathbb{P}^1 \times \mathbb{P}^1) := \mathbb{C}1 \oplus \mathbb{C}(H_1 + H_2) \oplus \mathbb{C}\text{PD}([\text{pt}]).$$

On the second homology groups, the maps π and r induce a map

$$(r_*)^{-1} \pi_* : H_2(\mathbb{F}_2) \longrightarrow H_2(\mathbb{P}^1 \times \mathbb{P}^1), \alpha_1 \mapsto 0, \alpha_2 \mapsto \beta_2$$

This map induces a map between $\rho : \mathbb{C}[q_1, q_2] \rightarrow \mathbb{C}[\tilde{q}_1, \tilde{q}_2]$ which sends q_1 to 1 and sends q_2 to $\tilde{q}_2$. We see that if we set $\tilde{q}_1 = \tilde{q}_2$, the map $r^*(\pi^*)^{-1}$ restricted on $S(\mathbb{F}_2)$ establishes not only an isomorphism between the subcohomology rings $S(\mathbb{F}_2)$ and $S(\mathbb{P}^1 \times \mathbb{P}^1)$ but also preserves their quantum products. Namely, we have the following theorem:

Theorem 7.1. *For the extremal transition from $\mathbb{F}_2$ to $\mathbb{P}^1 \times \mathbb{P}^1$, the pullback map r^* and π^* are both injective. The image of r^* and π^* are $S(\mathbb{F}_2)$ and $S(\mathbb{P}^1 \times \mathbb{P}^1)$ respectively. If we send the quantum parameters q_1 to 1, and q_2 to $\tilde{q}_2$, and set $\tilde{q}_1 = \tilde{q}_2$, then the map $r^*(\pi^*)^{-1}$ restricted on the subspace $S(\mathbb{F}_2)$ establishes not only an isomorphism between the subcohomology rings $S(\mathbb{F}_2)$ and $S(\mathbb{P}^1 \times \mathbb{P}^1)$ but also preserves their quantum products.*

Remark 7.2. We know that $\mathbb{P}^1 \times \mathbb{P}^1$ is symplectomorphic to $\mathbb{F}_2$, so quantum cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$ and quantum cohomology of $\mathbb{F}_2$ are isomorphic. To be precise, the linear map Φ from $H^*(\mathbb{P}^1 \times \mathbb{P}^1)$ to $H^*(\mathbb{F}_2)$:

$$\begin{aligned}\Phi(1) &= 1, \\ \Phi(u_2 - u_1) &= H_2, \\ \Phi(u_1) &= H_1, \\ \Phi(u_1 u_2) &= H_1 H_2,\end{aligned}$$

together with the homomorphism Ξ from $\mathbb{C}[q_1, q_2]$ to $\mathbb{C}[\tilde{q}_1, \tilde{q}_2]$:

$$\begin{aligned}\Xi(q_1 q_2) &= \tilde{q}_1, \\ \Xi(q_2) &= \tilde{q}_2,\end{aligned}$$

defines an isomorphism between $QH^*(\mathbb{F}_2)$ and $QH^*(\mathbb{P}^1 \times \mathbb{P}^1)$. The map $r^*(\pi^*)^{-1}$ in subsection 7.1 is the restriction of Φ^{-1} to the subcohomology ring $S(\mathbb{F}_2)$.

8. EXAMPLE: $\text{Fl}(1, 2, 3)$

In the last section, we see that the superpotential of $\text{Fl}(1, 2, 3)$ is the restriction of superpotential of the resolution of its toric degeneration. This phenomenon leads us to speculate that after translating the information on the LG B-model side into the information on the A-model side, we might find some relationship between the quantum cohomology of $\text{Fl}(1, 2, 3)$ and that of the resolution of its toric degeneration. In this section, we shall verify our speculation by computing the quantum cohomology. The flag manifold $\text{Fl}(1, 2, 3)$ is the space of complete flags in $\mathbb{C}^3$. Using the degeneration described in [2], we can construct a singular toric degeneration. The degeneration can be described as follow: Consider a family $\{F_t\}_{t \in \mathbb{C}}$ of $(1, 1)$ -hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$:

$$F_t = \{([z_1, z_2, z_3], [Z_1, Z_2, Z_3]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid tz_1 Z_1 + z_2 Z_2 + z_3 Z_3 = 0\}.$$

Then $F_t \cong \text{Fl}(1, 2, 3)$ for $t \neq 0$ and the central fiber $X_{\text{sing}} := F_0$ is a singular toric variety with only one ordinary double point. This singular toric variety X_{sing} admits a small toric crepant resolution $X_{\text{res}} \rightarrow X_{\text{sing}}$. In the following, we shall compute the quantum cohomologies of $\text{Fl}(1, 2, 3)$ and X_{res} . Then we verify the expected relationship.

8.1. Quantum cohomology of $\text{Fl}(1, 2, 3)$. The quantum cohomology ring of a flag variety is well-known. See e.g. [17, 5, 7]. Let L_1, L_2, L_3 be the line bundles on $\text{Fl}(1, 2, 3)$ whose fibers at a flag $0 \subset L \subset V \subset \mathbb{C}^3$ are given by L , V/L and $\mathbb{C}^3/V$ respectively. The cohomology ring of $\text{Fl}(1, 2, 3)$ is generated by the Chern classes $c_i := -c_1(L_i)$, $i = 1, 2, 3$:

$$H^*(\text{Fl}(1, 2, 3)) \cong \mathbb{C}[c_1, c_2, c_3] / \langle \sigma_1, \sigma_2, \sigma_3 \rangle$$

where σ_i is the i -th elementary symmetric polynomial of c_1, c_2, c_3 . A basis of $H^2(\text{Fl}(1, 2, 3))$ is given by

$$p_1 := c_1 = -c_1(L_1), \quad p_2 := c_1 + c_2 = c_1(L_3).$$

These classes span the nef cone of $\text{Fl}(1, 2, 3)$ and satisfy the relations $p_1^2 + p_2^2 - p_1 p_2 = 0$, $p_2^3 = p_1^3 = 0$, $p_1^2 p_2 = p_1 p_2^2$. Its dual basis in $H_2(\text{Fl}(1, 2, 3))$ is given by:

$$\beta_1 = \text{PD}(p_2^2), \quad \beta_2 = \text{PD}(p_1^2).$$

These classes span the Mori cone; they are represented by fibers of the natural maps $\text{Fl}(1, 2, 3) \rightarrow \text{Fl}(2, 3) \cong (\mathbb{P}^2)^*$ and $\text{Fl}(1, 2, 3) \rightarrow \text{Fl}(1, 3) \cong \mathbb{P}^2$ respectively. For an effective class $d = n_1 \beta_1 + n_2 \beta_2 \in H_2(\text{Fl}(1, 2, 3))$, we write $q^d = q_1^{n_1} q_2^{n_2}$ with $q_i = q^{\beta_i}$. Since $c_1(\text{Fl}(1, 2, 3)) = 2p_1 + 2p_2$, we have $\deg q_1 = \deg q_2 = 4$. Consider the basis of $H^*(\text{Fl}(1, 2, 3))$ given by

$$\{1, \quad p_1, \quad p_2, \quad p_1^2 = \text{PD}(\beta_2), \quad p_2^2 = \text{PD}(\beta_1), \quad p_1^2 p_2 = \text{PD}([\text{pt}])\}.$$

In this basis, the quantum multiplication by p_1 and p_2 are given by the following matrices:

$$p_1 \star = \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & q_1 q_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad p_2 \star = \begin{pmatrix} 0 & 0 & q_2 & 0 & 0 & q_1 q_2 \\ 0 & 0 & 0 & 0 & q_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & q_2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

8.2. Quantum cohomology of X_{res} . Now, we study the toric degeneration of $\text{Fl}(1, 2, 3)$. The singular fiber $X_{\text{sing}} = F_0$ is a toric variety and the corresponding fan is given by the following data: one-dimensional cones are spanned by:

$$\begin{aligned} r_1 &= (0, 0, 1), & r_2 &= (1, 1, -1), & r_3 &= (0, 1, 0), \\ r_4 &= (0, -1, 0), & r_5 &= (1, 0, 0), & r_6 &= (-1, 0, 0), \end{aligned}$$

and the full-dimensional cones are given by:

$$\begin{aligned} \langle r_1, r_3, r_6 \rangle, \quad \langle r_1, r_4, r_5 \rangle, \quad \langle r_1, r_4, r_6 \rangle, \quad \langle r_1, r_2, r_3, r_5 \rangle, \\ \langle r_2, r_4, r_5 \rangle, \quad \langle r_2, r_4, r_6 \rangle, \quad \langle r_2, r_3, r_6 \rangle. \end{aligned}$$

A small resolution X_{res} of X_{sing} is given by dividing the cone $\langle r_1, r_2, r_3, r_5 \rangle$ into the two simplicial cones $\langle r_1, r_3, r_5 \rangle$ and $\langle r_2, r_3, r_5 \rangle$. Let $R_1, \dots, R_6 \in H^2(X_{\text{res}})$ be the classes of the prime toric divisors corresponding to the one-dimensional cones $\langle r_1 \rangle, \dots, \langle r_6 \rangle$. The cohomology ring of X_{res} is generated by $R_1, \dots, R_6$ with the relations $R_1 = R_2$, $R_2 + R_3 = R_4$, $R_2 + R_5 = R_6$, $R_1 R_2 = R_3 R_4 = R_5 R_6 = 0$. We choose a basis $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ of $H^2(X_{\text{res}})$ as

$$\mathfrak{p}_1 := R_4, \quad \mathfrak{p}_2 := R_6, \quad \mathfrak{p}_3 := R_2.$$

They span the nef cone of X_{res} and satisfy the relations $\mathbf{p}_1(\mathbf{p}_1 - \mathbf{p}_3) = \mathbf{p}_2(\mathbf{p}_2 - \mathbf{p}_3) = \mathbf{p}_3^2 = 0$. The dual basis in $H_2(X_{\text{res}})$ is given by:

$$\beta_1 := \text{PD}(\mathbf{p}_2\mathbf{p}_3), \quad \beta_2 := \text{PD}(\mathbf{p}_1\mathbf{p}_3), \quad \beta_3 := \text{PD}(R_3R_5) = \text{PD}(\mathbf{p}_1\mathbf{p}_2 - \mathbf{p}_1\mathbf{p}_3 - \mathbf{p}_2\mathbf{p}_3).$$

They span the Mori cone of X_{res} . The class β_3 is represented by the exceptional curve in X_{res} .

We can compute the quantum product of X_{res} by using Givental's mirror theorem [16]. The computation was illustrated in section 6 for the example in §10. For $d = n_1\beta_1 + n_2\beta_2 + n_3\beta_3 \in H_2(X_{\text{res}})$, we write $q^d = q_1^{n_1}q_2^{n_2}q_3^{n_3}$, where we set $q_i = q^{\beta_i}$. Since $c_1(X_{\text{res}}) = 2\mathbf{p}_1 + 2\mathbf{p}_2$, we have $\deg q_1 = \deg q_2 = 4$ and $\deg q_3 = 0$. Consider the following basis of $H^*(X_{\text{res}})$:

$$\{1, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_1\mathbf{p}_2 - \mathbf{p}_1\mathbf{p}_3 - \mathbf{p}_2\mathbf{p}_3, \mathbf{p}_1\mathbf{p}_3, \mathbf{p}_2\mathbf{p}_3, \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\}.$$

In this basis, the quantum multiplication by $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are represented by the following matrices:

$$\begin{aligned} \mathbf{p}_1\star &= \begin{pmatrix} 0 & q_1 & 0 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1(1-q_3) & q_1q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q_1(1-q_3) & q_1(1-q_3) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ \mathbf{p}_2\star &= \begin{pmatrix} 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 0 & 0 & 0 & 0 & q_2(1-q_3) & 0 & q_2q_3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q_2(1-q_3) & 0 & q_2(1-q_3) & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & q_2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \mathbf{p}_3\star &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2q_3 \\ 0 & 0 & 0 & 0 & -q_2q_3 & 0 & q_2q_3 & 0 \\ 0 & 0 & 0 & 0 & -q_1q_3 & q_1q_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & q_3(q_1+q_2) & -q_1q_3 & -q_2q_3 & 0 \\ 0 & 0 & 0 & \frac{q_3}{1-q_3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

8.3. Comparison of quantum cohomology. We write X_{sm} for $\text{Fl}(1, 2, 3)$. The complete flag variety $\text{Fl}(1, 2, 3)$ can be degenerated to a singular toric variety X_{sing} .

X_{sing} admits a crepant resolution X_{res} . We have the following natural maps:

$$X_{\text{res}} \xrightarrow{\pi} X_{\text{sing}} \xleftarrow{r} X_{\text{sm}}.$$

The map $\pi^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{res}})$ is injective and has the image:

$$\pi^*(H^*(X_{\text{sing}})) = \langle 1, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_3, \mathfrak{p}_2\mathfrak{p}_3, \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_2 \rangle.$$

The map $r^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{sm}})$ is surjective with kernel:

$$\pi^*(\text{Ker}(r^*)) = \langle \mathfrak{p}_1\mathfrak{p}_2 - \mathfrak{p}_1\mathfrak{p}_3 - \mathfrak{p}_2\mathfrak{p}_3 \rangle = \langle \text{PD}(\beta_3) \rangle.$$

On the second homology groups, the maps π, r induce a map¹

$$(r_*)^{-1}\pi_*: H_2(X_{\text{res}}) \rightarrow H_2(X_{\text{sm}}), \quad \beta_1 \mapsto \beta_1, \beta_2 \mapsto \beta_2, \beta_3 \mapsto 0.$$

This gives rise to the map $\lim_{q_3 \rightarrow 1}: \mathbb{C}[q_1, q_2, q_3] \rightarrow \mathbb{C}[q_1, q_2]$ between Novikov rings. The residue of the quantum multiplication by $\mathfrak{p}_3$ on $H^*(X_{\text{res}})$ along $q_3 = 1$ is given by:

$$N = \text{Res}_{q_3=1}(\mathfrak{p}_3\star) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is nilpotent and induces the weight filtration on $H^*(X_{\text{res}})$:

$$(12) \quad 0 \subset \text{Im } N \subset \text{Ker } N \subset H^*(X_{\text{res}}).$$

The computation in subsection 8.1 and subsection 8.2 shows the following proposition, which confirms the general argument in section 11 below.

Theorem 8.1. *The weight filtration (12) defined by the nilpotent operator $N = \text{Res}_{q_3=1}(\mathfrak{p}_3\star)$ coincides with the filtration*

$$0 \subset \pi^*(\text{Ker } r^*) \subset \text{Im } \pi^* \subset H^*(X_{\text{res}}).$$

The quantum multiplication by $\mathfrak{p}_1, \mathfrak{p}_2$ on $H^(X_{\text{res}})$ are regular at $q_3 = 1$ and the operators induced by $\lim_{q_3 \rightarrow 1} \mathfrak{p}_1\star, \lim_{q_3 \rightarrow 1} \mathfrak{p}_2\star$ on*

$$\text{Ker } N / \text{Im } N \cong H^*(\text{Fl}(1, 2, 3))$$

coincide with the quantum multiplication by p_1, p_2 on $H^(\text{Fl}(1, 2, 3))$. Here note that $\mathfrak{p}_i \in \text{Im } \pi^*$ and $p_i = r^*(\pi^*)^{-1}\mathfrak{p}_i$ for $i = 1, 2$.*

¹Note that r_* on H_2 is an isomorphism.

9. EXAMPLE: $\text{Gr}(2, 4)$

In this section we study an extremal transition of $\text{Gr}(2, 4)$, the space of complex two planes in $\mathbb{C}^4$. By the Plücker embedding, $\text{Gr}(2, 4)$ can be realized as a quadric in $\mathbb{P}^5 = \mathbb{P}(\wedge^2 \mathbb{C}^4)$. Consider a toric degeneration of $\text{Gr}(2, 4)$ given by a family $\{F_t\}_{t \in \mathbb{C}}$ of quadric hyperplanes in $\mathbb{P}^5$:

$$F_t = \{[Z_{12}, Z_{13}, Z_{14}, Z_{23}, Z_{24}, Z_{34}] \in \mathbb{P}^5 \mid Z_{12}Z_{34} - Z_{13}Z_{24} + tZ_{14}Z_{23} = 0\}.$$

Then $F_t \cong \text{Gr}(2, 4)$ for $t \neq 0$ and the central fiber $X_{\text{sing}} := F_0$ is a singular toric variety with transversal A_1 -singularity along $(Z_{12} = Z_{34} = Z_{13} = Z_{24} = 0) \cong \mathbb{P}^1$. This singular variety admits a small toric crepant resolution $X_{\text{res}} \rightarrow X_{\text{sing}}$. We study a relationship between the quantum cohomology of $\text{Gr}(2, 4)$ and X_{res} .

9.1. Quantum cohomology of $\text{Gr}(2, 4)$. Let T^* be the dual tautological bundle of $\text{Gr}(2, 4)$. The cohomology ring of $\text{Gr}(2, 4)$ is generated by the Chern classes $c_1(T^*)$ and $c_2(T^*)$. Fix a complete flag $0 \subset E_1 \subset E_2 \subset E_3 \subset E_4 = \mathbb{C}^4$ in $\mathbb{C}^4$. Consider the following cycles:

$$D = \{V \in \text{Gr}(2, 4) : \dim(V \cap E_2) = 1\}$$

$$\Delta = \{V \in \text{Gr}(2, 4) : V \subset E_3\}$$

$$C = \{V \in \text{Gr}(2, 4) : E_1 \subset V \subset E_3\}$$

Their Poincaré duals are denoted respectively by d, δ, c . We know that $d = c_1(T^*)$ and $\delta = c_2(T^*)$ and $c = d\delta = d^3/2$. The cohomology ring of $\text{Gr}(2, 4)$ is given by

$$H^*(\text{Gr}(2, 4)) \cong \mathbb{C}[d, \delta] / \langle d^3 - 2d\delta, d^2\delta - \delta^2 \rangle.$$

We choose an additive basis of $\text{Gr}(2, 4)$ as follows:

$$1, \quad d, \quad d^2, \quad d^2 - 2\delta, \quad d^3, \quad d^4 \quad .$$

Let q be the Novikov variable dual to $d \in H^2(\text{Gr}(2, 4))$. We have $\deg q = 8$. We use the quantum Schubert calculus [4, 7] to compute the quantum multiplication by d . Under the above basis, the quantum product matrix of d is given by

$$d\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 2q & 0 \\ 1 & 0 & 0 & 0 & 0 & 2q \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the quantum product matrix of δ is given by:

$$\delta\star = \begin{pmatrix} 0 & 0 & q & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 2q & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 2q \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

9.2. Quantum cohomology of X_{res} . The fan for the singular toric variety X_{sing} is as follows: it is a 4-dimensional fan whose 1-dimensional cones are spanned by

$$\begin{aligned} r_1 &= (1, 0, 0, 0), & r_2 &= (-1, 0, 1, 0), & r_3 &= (0, 0, -1, 1), \\ r_4 &= (-1, 1, 0, 0), & r_5 &= (0, -1, 0, 1), & r_6 &= (0, 0, 0, -1). \end{aligned}$$

This is a complete fan whose top dimensional cones are:

$$\begin{aligned} \langle r_1, r_3, r_5, r_6 \rangle, & \quad \langle r_1, r_2, r_4, r_6 \rangle, & \quad \langle r_2, r_3, r_4, r_5, r_6 \rangle, \\ \langle r_1, r_2, r_5, r_6 \rangle, & \quad \langle r_1, r_3, r_4, r_6 \rangle, & \quad \langle r_1, r_2, r_3, r_4, r_5 \rangle. \end{aligned}$$

Note that there are two non-simplicial 4-dimensional cones. We divide these cones as follows:

- divide $\langle r_2, r_3, r_4, r_5, r_6 \rangle$ into $\langle r_2, r_3, r_4, r_6 \rangle$ and $\langle r_2, r_3, r_5, r_6 \rangle$.
- divide $\langle r_1, r_2, r_3, r_4, r_5 \rangle$ into $\langle r_1, r_2, r_3, r_4 \rangle$ and $\langle r_1, r_2, r_3, r_5 \rangle$.

Then we get a smooth fan. This fan corresponds to a smooth toric variety which we denote by X_{res} . Let R_i denote the class of the toric divisor corresponding to the ray $\langle r_i \rangle$. There are linear relations: $R_1 = R_2 + R_4$, $R_4 = R_5$, $R_2 = R_3$, $R_3 + R_5 = R_6$. The cohomology ring of X_{res} is given by

$$H^*(X_{\text{sm}}) = \mathbb{C}[R_1, R_4] / \langle R_4^2, R_1^4 - 2R_1^3 R_4 \rangle.$$

We choose a basis $\{m_1, m_2\}$ of $H^2(X_{\text{res}})$ as $m_1 = R_1$, $m_2 = R_4$. They span the nef cone of X_{res} . The dual basis in $H_2(X_{\text{res}})$ is given by $\beta_1 = \text{PD}(R_1 R_2 R_4)$ and $\beta_2 = \text{PD}(R_1 R_2 R_3)$. They span the Mori cone of X_{res} . The class β_2 is represented by an exceptional curve.

We compute the quantum product of X_{res} using Givental's mirror theorem [16]. For $d = n_1 \beta_1 + n_2 \beta_2 \in H_2(X_{\text{res}})$, we write $q^d = q_1^{n_1} q_2^{n_2}$, where $q_i = q^{\beta_i}$. We have $\deg q_1 = 8$ and $\deg q_2 = 0$. We choose the following bases for the cohomology ring of X_{res} :

$$\{1, m_1, m_1 - 2m_2, m_1^2, m_1^2 - 2m_1 m_2, m_1^3, m_1^3 - 2m_1^2 m_2, m_1^4 = 2[\text{pt}]\}.$$

Under this basis, the quantum product matrices of the divisors m_1 and m_2 are given as follows:

$$m_1 \star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q_1(1+q_2) & q_1(1-q_2) & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & q_1(1+q_2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_1(1-q_2) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$m_2\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q_1(1-q_2) & 2q_1q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1(1-q_2) \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & -q_1(1+q_2) \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2(1+q_2)}{1-q_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2(1+q_2)}{1-q_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

9.3. Comparison of quantum cohomology. The residue of the quantum product matrix of m_2 at $q_2 = 1$ is given by

$$N = \text{Res}_{q_2=1}(m_2\star) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The residue N defines the filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$, where

$$(13) \quad \begin{aligned} V &:= \text{Ker } N = \text{Span}\{1, m_1, m_1^2, m_1^3, m_1^4, m_1^3 - 2m_1^2m_2\}, \\ W &:= \text{Ker } N \cap \text{Im } N = \mathbb{C}(m_1^3 - 2m_1^2m_2). \end{aligned}$$

This filtration arises from the correspondence $X_{\text{res}} \rightarrow X_{\text{sing}} \leftarrow X_{\text{sm}} := \text{Gr}(2, 4)$ as follows:

Proposition 9.1. *Let $\pi: X_{\text{res}} \rightarrow X_{\text{sing}}$ and $r: X_{\text{sm}} = \text{Gr}(2, 4) \rightarrow X_{\text{sing}}$ be natural maps associated to the resolution and the smoothing.*

- (1) *The singular cohomology group of X_{sing} is given by the table:*

degree p	0	1	2	3	4	5	6	7	8
$H^p(X_{\text{sing}})$	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}^2$	0	$\mathbb{C}$

- (2) *The map $\pi^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{res}})$ is injective and $\text{Im } \pi^* = V$.*
(3) *The map $r^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{sm}})$ is neither surjective nor injective; we have $\pi^*(\text{Ker } r^*) = W$ and $\text{Im } r^* = \text{Span}\{1, d, d^2, d^3, d^4\}$.*
(4) *The map $r^* \circ (\pi^*)^{-1}: V \rightarrow H^*(X_{\text{sm}})$ sends m_1^i to d^i for $0 \leq i \leq 4$ and $m_1^3 - 2m_1^2m_2$ to zero.*

Proof. Note that the non-singular locus Y of X_{sing} is isomorphic to the total space of $\mathcal{O}(1, 1)^{\oplus 2}$ over $\mathbb{P}^1 \times \mathbb{P}^1$. We consider the Mayer-Vietoris exact sequence associated to Y and a neighbourhood ν of the singular locus $\mathbb{P}^1$. The intersection

$\nu \cap Y$ is homotopic to the 3-sphere bundle associated to $\mathcal{O}(1, 1)^{\oplus 2} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and the cohomology of $\nu \cap Y$ can be easily computed by the Gysin sequence:

$$H^*(N \cap Y) = \mathbb{C}, 0, \mathbb{C}^2, 0, 0, \mathbb{C}^2, 0, \mathbb{C} \quad \text{for } * = 0, 1, 2, 3, 4, 5, 6, 7, \quad \text{respectively.}$$

Then the Mayer-Vietoris sequence gives the result for $H^*(X_{\text{sing}})$. To prove the statement about π^* , we consider the hypercohomology spectral sequence for $\mathbb{H}^*(X_{\text{sing}}, \mathbb{R}\pi_*\underline{\mathbb{C}}) = H^*(X_{\text{res}})$. Since we have

$$R^j\pi_*\underline{\mathbb{C}} = \begin{cases} \mathbb{C} & j = 0; \\ \iota_*\underline{\mathbb{C}}_{\mathbb{P}^1} & j = 2; \\ 0 & \text{otherwise,} \end{cases}$$

where $\iota: \mathbb{P}^1 \rightarrow X_{\text{sing}}$ is the inclusion of the singular locus, the spectral sequence degenerates at the E_2 term $H^j(R^i\pi_*\underline{\mathbb{C}})$; this shows that π^* is injective. Since the image of π^* contains the pull-back m_1 of the ample class $\alpha := c_1(\mathcal{O}(1))$ on X_{sing} , it follows that $\text{Im } \pi^* = V$. On the other hand, r^* also sends the ample class α to $d = c_1(\mathcal{O}(1)) \in H^2(X_{\text{sm}})$ and it follows that $\text{Im } r^* = \text{Span}\{1, d, d^2, d^3, d^4\}$. Let $x \in H^6(X_{\text{sing}})$ be a generator of the kernel of r^* . Then we have $\alpha \cup x = 0$ in $H^8(X_{\text{sing}})$ (otherwise we would have $0 \neq r^*(\alpha \cup x) = r^*(\alpha) \cup r^*(x) = 0$). Therefore $0 = \pi^*(\alpha \cup x) = m_1 \cup \pi^*(x)$. This shows that $\pi^*(x)$ is a multiple of $m_1^3 - 2m_1^2m_2$. $\square$

The computation in §9.1–9.2 implies the following theorem:

Theorem 9.2. *The filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ (13) defined by the residue $N = \text{Res}_{q_2=1}(m_2\star)$ along $q_2 = 1$ matches with the filtration*

$$0 \subset \pi^*(\text{Ker } r^*) \subset \text{Im } \pi^* \subset H^*(X_{\text{res}}).$$

The quantum products of elements in $\text{Im } \pi^$ are regular at $q_2 = 1$ and the map*

$$r^* \circ (\pi^*)^{-1}: \text{Im } \pi^* \rightarrow H^*(\text{Gr}(2, 4))$$

intertwines the quantum product $\star|_{q_2=1}$ on $\text{Im } \pi^ = V$ with the quantum product on $H^*(\text{Gr}(2, 4))$ under the identification $q_1 = q$ of the Novikov variables. This map also preserves the Poincaré pairing.*

Remark 9.3. Since N is self-adjoint with respect to the Poincaré pairing, we have $(\text{Ker } N)^\perp = \text{Im } N$. Thus the Poincaré pairing induces a non-degenerate pairing on $V/W = \text{Ker } N/(\text{Ker } N \cap (\text{Ker } N)^\perp)$.

In the above theorem, we identified the subquotient $(V/W, \star|_{q_2=1})$ of $H^*(X_{\text{res}})$ with a *subring* of the quantum cohomology of $\text{Gr}(2, 4)$. We can extend this isomorphism to the whole of $H^*(\text{Gr}(2, 4))$ as follows. The weight filtration $W_{-2} \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 = H^{\text{even}}(X_{\text{res}})$ associated to the nilpotent endomorphism N (see e.g. [6, A.2]) is given as follows:

$$W_{-2} = W_{-1} = \text{Span}\{m_1^3 - 2m_1^2m_2\},$$

$$W_0 = W_1 = \text{Span}\{m_1^3 - 2m_1^2m_2, m_1^2 - 2m_1m_2, 1, m_1, m_1^2, m_1^3, m_1^4\}.$$

This is illustrated by the following table:

$m_1^3 - 2m_1^2m_2$					$W_{-2} = W_{-1}$	
1	m_1	m_1^2	m_1^3	m_1^4	$m_1^2 - 2m_1m_2$	$W_0 = W_1$
$m_1 - 2m_2$					W_2	

Therefore V/W can be regarded as a subspace of W_0/W_{-1} . We define a linear isomorphism $\theta: W_0/W_{-1} \cong H^*(\text{Gr}(2, 4))$ by

$$\begin{aligned}\theta(m_1^i) &= d^i \quad \text{for } 0 \leq i \leq 4, \\ \theta(m_1^2 - 2m_1m_2) &= \sqrt{-1}(d^2 - 2\delta).\end{aligned}$$

This gives an extension of the map $r^* \circ (\pi^*)^{-1}: V/W \rightarrow H^*(\text{Gr}(2, 4))$. We have the following:

Theorem 9.4. *The quantum products of elements in W_0 are regular at $q_2 = 1$ and belong to W_0 . The quantum product $\star|_{q_2=1}$ on W_0 descends to the one on W_0/W_{-1} and θ induces an isomorphism of rings:*

$$\theta: (W_0/W_{-1}, \star|_{q_2=1}) \cong (H^*(\text{Gr}(2, 4)), \star)$$

under the identification $q_1 = q$. Moreover, θ preserves the Poincaré pairing.

Remark 9.5. It is curious that we have imaginary numbers in the isomorphism θ . The assignment $\theta: m_1^2 - 2m_1m_2 \mapsto \sqrt{-1}(d^2 - 2\delta)$ is uniquely determined up to sign if we require that θ coincides with $r^* \circ (\pi^*)^{-1}$ on V/W and intertwines the quantum products.

10. EXAMPLE: $\text{Gr}(2, 5)$

In this section we study an extremal transition of the 6-dimensional Fano variety $\text{Gr}(2, 5)$, the space of complex two planes in $\mathbb{C}^5$. Different from the previous two examples in §8 and §9, the image of the Plücker embedding of $\text{Gr}(2, 5)$ is not a hypersurface or a complete intersection. We use a toric degeneration of $\text{Gr}(2, 5)$ and its crepant resolution studied by Gonciulea-Lakshmibai [19] and Batyrev-Ciocan-Fontanine-Kim-van-Straten [2].

According to [19, 2], the Grassmannian $\text{Gr}(2, 5)$ admits a flat degeneration to the Gorenstein toric variety X_{sing} defined by the following 6-dimensional fan. Primitive generators of the 1-dimensional cones are:

$$\begin{aligned}r_1 &= (1, 0, 0, 0, 0, 0), & r_2 &= (-1, 1, 0, 0, 0, 0), & r_3 &= (-1, 0, 1, 0, 0, 0), \\ r_4 &= (0, -1, 0, 1, 0, 0), & r_5 &= (0, 0, -1, 1, 0, 0), & r_6 &= (0, 0, -1, 0, 1, 0), \\ r_7 &= (0, 0, 0, -1, 0, 1), & r_8 &= (0, 0, 0, 0, -1, 1), & r_9 &= (0, 0, 0, 0, 0, -1).\end{aligned}$$

The top dimensional cones are:

$$\begin{aligned}\langle r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 \rangle, & \quad \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 \rangle, \\ \langle r_1, r_4, r_5, r_6, r_7, r_8, r_9 \rangle, & \quad \langle r_1, r_2, r_5, r_6, r_7, r_8, r_9 \rangle, \quad \langle r_1, r_2, r_3, r_4, r_5, r_8, r_9 \rangle,\end{aligned}$$

$$\begin{aligned} &\langle r_1, r_2, r_3, r_4, r_5, r_6, r_9 \rangle, \quad \langle r_1, r_3, r_4, r_7, r_8, r_9 \rangle, \quad \langle r_1, r_3, r_4, r_6, r_7, r_9 \rangle, \\ &\langle r_1, r_2, r_3, r_7, r_8, r_9 \rangle, \quad \langle r_1, r_2, r_3, r_6, r_7, r_9 \rangle. \end{aligned}$$

In order to obtain a crepant small resolution of X_{sing} , we divide non-simplicial cones as follows:

- divide $\langle r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 \rangle$ into $\langle r_2, r_3, r_5, r_6, r_7, r_9 \rangle$, $\langle r_2, r_3, r_5, r_7, r_8, r_9 \rangle$, $\langle r_3, r_4, r_5, r_6, r_7, r_9 \rangle$, $\langle r_3, r_4, r_5, r_7, r_8, r_9 \rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 \rangle$ into $\langle r_1, r_2, r_3, r_5, r_6, r_7 \rangle$, $\langle r_1, r_2, r_3, r_5, r_7, r_8 \rangle$, $\langle r_1, r_3, r_4, r_5, r_6, r_7 \rangle$, $\langle r_1, r_3, r_4, r_5, r_7, r_8 \rangle$;
- divide $\langle r_1, r_4, r_5, r_6, r_7, r_8, r_9 \rangle$ into $\langle r_1, r_4, r_5, r_6, r_7, r_9 \rangle$, $\langle r_1, r_4, r_5, r_7, r_8, r_9 \rangle$;
- divide $\langle r_1, r_2, r_5, r_6, r_7, r_8, r_9 \rangle$ into $\langle r_1, r_2, r_5, r_6, r_7, r_9 \rangle$, $\langle r_1, r_2, r_5, r_7, r_8, r_9 \rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_8, r_9 \rangle$ into $\langle r_1, r_2, r_3, r_5, r_8, r_9 \rangle$, $\langle r_1, r_3, r_4, r_5, r_8, r_9 \rangle$;
- divide $\langle r_1, r_2, r_3, r_4, r_5, r_6, r_9 \rangle$ into $\langle r_1, r_2, r_3, r_5, r_6, r_9 \rangle$, $\langle r_1, r_3, r_4, r_5, r_6, r_9 \rangle$.

These subdivisions define a smooth toric variety X_{res} . In this section we study a relationship between the quantum cohomologies of $\text{Gr}(2, 5)$ and X_{res} .

10.1. Quantum cohomology of $\text{Gr}(2, 5)$. We refer the reader to [4, 7] for the quantum cohomology of $\text{Gr}(2, 5)$. It is well known that the Poincaré duals of the Schubert cycles form an additive basis of the cohomology ring of $\text{Gr}(2, 5)$. Fix a full flag $0 \subset F_1 \subset F_2 \subset \cdots \subset F_5 = \mathbb{C}^5$. The Schubert cycle $\Omega_{(a_1, a_2)} \subset \text{Gr}(2, 5)$, indexed by a pair (a_1, a_2) of integers satisfying $3 \geq a_1 \geq a_2 \geq 0$, is given by:

$$\Omega_{(a_1, a_2)} = \{V \subset \mathbb{C}^5 : \dim V = 2, \dim(V \cap F_{4-a_1}) \geq 1, V \subset F_{5-a_2}\}.$$

We denote by $\omega_{(a_1, a_2)} \in H^{2(a_1+a_2)}(\text{Gr}(2, 5))$ the Poincaré dual of the Schubert cycle $\Omega_{(a_1, a_2)}$. The dual basis of $\{\omega_{(a_1, a_2)}\}$ is given by $\{\omega_{(3-a_2, 3-a_1)}\}$. We choose the following additive basis of $H^*(\text{Gr}(2, 5))$:

$$\{\omega_{(0,0)}, \omega_{(1,0)}, \omega_{(1,1)}, \omega_{(2,0)}, \omega_{(2,1)}, \omega_{(3,0)}, \omega_{(3,1)}, \omega_{(2,2)}, \omega_{(3,2)}, \omega_{(3,3)}\}.$$

Let q be the Novikov variable dual to the ample class $\omega_{(1,0)} \in H^2(\text{Gr}(2, 5))$. We have $\deg q = 10$. The class $\omega_{(1,0)}$ generates the small quantum cohomology ring of $\text{Gr}(2, 5)$ and its quantum multiplication under the above basis is given by the

following matrix:

$$\omega_{(1,0)}^\star = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

10.2. Quantum cohomology of X_{res} . Let R_i denote the class of the toric divisor corresponding to the ray $\mathbb{R}_{\geq 0}r_i$. We choose a basis $\{m_1, m_2, m_3\}$ of $H^2(X_{\text{res}})$ as $m_1 = R_1$, $m_2 = R_2$, $m_3 = R_6$. Then we have

$$\begin{aligned} R_1 &= m_1, \quad R_2 = m_2, \quad R_3 = m_1 - m_2, \quad R_4 = m_2, \quad R_5 = m_1 - m_2 - m_3, \\ R_6 &= m_3, \quad R_7 = m_1 - m_3, \quad R_8 = m_3, \quad R_9 = m_1. \end{aligned}$$

The cohomology ring of X_{res} is given by:

$$H^*(X_{\text{sm}}) = \mathbb{C}[m_1, m_2, m_3] / \langle m_2^2, m_3^2, m_1^2(m_1 - m_2)(m_1 - m_2 - m_3)(m_1 - m_3) \rangle.$$

The classes m_1, m_2, m_3 span the nef cone of X_{res} . Let $\{\beta_1, \beta_2, \beta_3\} \subset H_2(X_{\text{res}})$ be the dual basis of $\{m_1, m_2, m_3\}$; they span the Mori cone of X_{res} . For $d = n_1\beta_1 + n_2\beta_2 + n_3\beta_3 \in H_2(X_{\text{res}})$, we write $q^d = q_1^{n_1}q_2^{n_2}q_3^{n_3}$, where $q_i = q^{\beta_i}$. We have $\deg q_1 = 10$, $\deg q_2 = \deg q_3 = 0$. We choose the following basis for $H^*(X_{\text{res}})$:

$$(14) \quad \left\{ 1, m_1, m_2, m_3, m_1^2, m_1m_2, m_1m_3, m_2m_3, m_1^3, m_1^2m_2, m_1^2m_3, \right. \\ \left. m_1m_2m_3, m_1^4, m_1^3m_2, m_1^3m_3, m_1^2m_2m_3, m_1^5, m_1^4m_2, m_1^4m_3, m_1^6 \right\}$$

We use Givental's mirror theorem [16] to calculate the quantum product.

The quantum multiplication by m_1 with cohomology classes under the basis (14) are as follows:

$$\begin{aligned} m_1 \star m_1^4 &= m_1^5 + q_1(1 + q_2 + q_3), \\ m_1 \star m_1^3m_2 &= m_1^4m_2 + q_1q_2, \\ m_1 \star m_1^3m_3 &= m_1^4m_3 + q_1q_3, \\ m_1 \star m_1^5 &= m_1^6 + (2m_2 + 2m_3)q_1 + (m_3 + 2m_1 - 2m_2)q_1q_2 \\ &\quad + (m_2 + 2m_1 - 2m_3)q_1q_3 + (m_1 - m_2 - m_3)q_1q_2q_3, \\ m_1 \star m_1^4m_2 &= m_1^5m_2 + m_2q_1 + (m_1 - m_2 + m_3)q_1q_2 + m_2q_1q_3 + (m_1 - m_2 - m_3)q_1q_2q_3, \\ m_1 \star m_1^4m_3 &= m_1^5m_3 + m_3q_1 + (m_1 - m_3 + m_2)q_1q_3 + m_3q_1q_2 + (m_1 - m_2 - m_3)q_1q_2q_3, \\ m_1 \star m_1^6 &= 5m_2m_3q_1 + (5m_1m_3 - 5m_2m_3)q_1q_2 + (5m_1m_2 - 5m_2m_3)q_1q_3 \end{aligned}$$

and all the other quantum products coincide with the cup products.

The quantum multiplication by m_2 with cohomology classes under the basis (14) are as follows:

$$\begin{aligned}
m_2 \star m_2 &= (m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2}, \\
m_2 \star m_1 m_2 &= m_1(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2}, \\
m_2 \star m_2 m_3 &= m_3(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2} \\
&\quad - (m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_2 q_3}{(1 - q_2)(1 - q_2 - q_3)}, \\
m_2 \star m_1^2 m_2 &= m_1^2(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2}, \\
m_2 \star m_1 m_2 m_3 &= m_1 m_3(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2} \\
&\quad - m_1(m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_2 q_3}{(1 - q_2)(1 - q_2 - q_3)}, \\
m_2 \star m_1^4 &= m_1^4 m_2 + q_1 q_2, \\
m_2 \star m_1^3 m_2 &= m_1^3(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2} + q_1 q_2, \\
m_2 \star m_1^2 m_2 m_3 &= m_1^2 m_3(m_1 - m_2)(m_1 - m_2 - m_3) \frac{q_2}{1 - q_2}, \\
m_2 \star m_1^5 &= m_1^5 m_2 + (m_3 + 2m_1 - 2m_2)q_1 q_2 + (m_1 - m_2 - m_3)q_1 q_2 q_3, \\
m_2 \star m_1^4 m_2 &= (m_1 - m_2 + m_3)q_1 q_2 + (m_1 - m_2 - m_3)q_1 q_2 q_3, \\
m_2 \star m_1^4 m_3 &= m_1^4 m_2 m_3 + (m_1 - m_2 - m_3)q_1 q_2 q_3 + m_3 q_1 q_2, \\
m_2 \star m_1^6 &= (5m_1 m_3 - 5m_2 m_3)q_1 q_2.
\end{aligned}$$

All the other quantum products with m_2 are the same as the cup products.

The quantum products of m_3 with cohomology classes under the basis (14) are as follows:

$$\begin{aligned}
m_3 \star m_3 &= (m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3}, \\
m_3 \star m_1 m_3 &= m_1(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3}, \\
m_3 \star m_2 m_3 &= m_2(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3} \\
&\quad - (m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_2 q_3}{(1 - q_3)(1 - q_2 - q_3)}, \\
m_3 \star m_1^2 m_3 &= m_1^2(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3}, \\
m_3 \star m_1 m_2 m_3 &= m_1 m_2(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3},
\end{aligned}$$

$$\begin{aligned}
& -m_1(m_1 - m_2)(m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_2 q_3}{(1 - q_3)(1 - q_2 - q_3)}, \\
m_3 \star m_1^4 &= m_1^4 m_3 + q_1 q_3, \\
m_3 \star m_1^3 m_3 &= m_1^3 (m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3} + q_1 q_3, \\
m_3 \star m_1^2 m_2 m_3 &= m_1^2 m_2 (m_1 - m_3)(m_1 - m_2 - m_3) \frac{q_3}{1 - q_3}, \\
m_3 \star m_1^5 &= m_1^5 m_3 + (m_2 + 2m_1 - 2m_3) q_1 q_3 + (m_1 - m_2 - m_3) q_1 q_2 q_3, \\
m_3 \star m_1^4 m_2 &= m_1^4 m_2 m_3 + m_2 q_1 q_3 + (m_1 - m_2 - m_3) q_1 q_2 q_3, \\
m_3 \star m_1^4 m_3 &= (m_1 - m_3 + m_2) q_1 q_3 + (m_1 - m_2 - m_3) q_1 q_2 q_3, \\
m_3 \star m_1^6 &= (5m_1 m_2 - 5m_2 m_3) q_1 q_3.
\end{aligned}$$

All the other quantum products with m_3 are the same as the cup products.

10.3. Comparison of quantum cohomology. The quantum product of m_2 has simple poles along $q_2 = 1$ and $q_2 + q_3 = 1$; the quantum product of m_3 has simple poles along $q_3 = 1$ and $q_2 + q_3 = 1$. We define

$$\begin{aligned}
N_2 &:= \text{Res}_{q_2=1}(m_2 \star) \frac{dq_2}{q_2} \Big|_{(q_2, q_3)=(1,1)}, \\
N_3 &:= \text{Res}_{q_3=1}(m_3 \star) \frac{dq_3}{q_3} \Big|_{(q_2, q_3)=(1,1)}.
\end{aligned}$$

These are nilpotent endomorphisms. Thus the monodromy of the quantum connection around the normal crossing divisors $(q_2 = 1)$, $(q_3 = 1)$ is unipotent. As before, the endomorphisms N_2 , N_3 define the filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ by:

$$V := \text{Ker}(N_2) \cap \text{Ker}(N_3), \quad W := V \cap (\text{Im}(N_2) + \text{Im}(N_3)).$$

We have $\dim V = 12$ and $\dim W = 2$. The basis of V is given by

$$1, m_1, m_1^2, m_1^3, m_1^4, m_1^5, m_1^6, \alpha, m_1 \alpha, m_1^2 \alpha, m_1^4 m_2, m_1^4 m_3,$$

where $\alpha := m_1 m_2 + m_1 m_3 - m_2 m_3$. Define a linear map $\theta: V \rightarrow H^*(X_{\text{sm}})$ as follows:

$$\begin{aligned}
\theta(m_1^i) &= (\omega_{(1,0)})^i, & 0 \leq i \leq 6, \\
\theta(m_1^i \alpha) &= (\omega_{(1,0)})^i \omega_{(2,0)}, & 0 \leq i \leq 2, \\
\theta(m_1^4 m_2) &= 2\omega_{(3,2)}, \\
\theta(m_1^4 m_3) &= 2\omega_{(3,2)}.
\end{aligned}$$

We have $\text{Ker } \theta = W$ and the map θ induces an isomorphism:

$$\theta: V/W \cong H^*(\text{Gr}(2, 5)).$$

Note that the quantum multiplication by m_1 is regular along $q_2 = q_3 = 1$. Since $(m_1 \star)$ commutes with $(m_2 \star)$ and $(m_3 \star)$, it follows that $(m_1 \star)|_{q_2=q_3=1}$ commutes

with N_2 and N_3 ; thus $(m_1 \star)|_{q_2=q_3=1}$ descends to the quotient space V/W and defines a ring structure on V/W . The following result follows from a direct computation:

Theorem 10.1. *The quantum product on $H^*(X_{\text{res}})$ at $q_2 = q_3 = 1$ descends to a well-defined product structure on V/W . The linear isomorphism $\theta: V/W \cong H^*(\text{Gr}(2, 5))$ intertwines the quantum product $\star|_{q_2=q_3=1}$ on V/W with the quantum product on $H^*(\text{Gr}(2, 5))$. Moreover θ preserves the Poincaré pairing.*

Remark 10.2. When $a \neq 0$ and $b \neq 0$, the nilpotent operator $N = aN_2 + bN_3$ defines a weight filtration $\{W_\bullet\}$ independent of (a, b) . The Jordan normal form of N consists of 10 Jordan blocks of size 1 (one-by-one zero matrices) and 2 Jordan blocks of size 5. Therefore W_0/W_{-1} gives a 12-dimensional space which is bigger than $H^*(X_{\text{sm}})$. The above quotient V/W corresponds to Jordan blocks of size 1.

11. CONIFOLD TRANSITION AND QUANTUM COHOMOLOGY

In the last three sections, §8, §9 and §10, we studied the how the quantum cohomology changes under extremal transition case by case. We find that the quantum cohomology of flag varieties $\text{Fl}(1, 2, 3)$, $\text{Gr}(2, 4)$, and $\text{Gr}(2, 5)$ and the resolution of their toric degenerations have some relations. However, we haven't got a unified understanding. In this section, we shall describe the change of quantum cohomology under conifold transition in dimension three, using a result of Li-Ruan [28]. Our main result in this section is stated in Theorem 11.5. We observe that the quantum cohomology of a smoothing arises as a limit of the quantum cohomology of a resolution when the quantum variables associated to exceptional curves go to one.

11.1. Geometry of conifold transition. In this subsection, we shall study how the geometry changes on the classical level under 3-dimensional extremal transition. The conifold transition in dimension 3 is a surgery which replaces a $(-1, -1)$ -rational curve with a real 3-sphere. In this section we describe topological properties of the conifold transition. See e.g. [31, 34] for more background material.

Let X_{sing} be a three-dimensional projective variety whose only singularities are ordinary double points $p_1, \dots, p_k$. Recall that an ordinary double point (or A_1 -singularity) is a singularity whose neighborhood is analytically isomorphic to a neighborhood of the origin in $\{xy = zw\} \subset \mathbb{C}^4$. Let X_{res} be a small resolution of X_{sing} and suppose that X_{sing} admits a smoothing X_{sm} . The passage from X_{res} to X_{sm} is called the *conifold transition*. Since we are interested in Gromov-Witten theory, we assume that both X_{sing} and X_{sm} are projective. The inverse image E_i of p_i in the small resolution X_{res} is a rational curve whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The vanishing cycle $S_i \subset X_{\text{sm}}$ associated to p_i is a real 3-sphere. In topological terms, the conifold transition replaces a neighborhood $S^2 \times D^4$ of E_i with a neighborhood $D^3 \times S^3 \cong T^*S_i$ of S_i . There are two natural maps:

- a morphism $\pi: X_{\text{res}} \longrightarrow X_{\text{sing}}$ contracting the rational curves $E_1, \dots, E_k$;

- a continuous map $r : X_{\text{sm}} \longrightarrow X_{\text{sing}}$ contracting the real 3-spheres $S_1, \dots, S_k$.

They give the following correspondence between the cohomology groups of the resolution and the smoothing:

$$\begin{array}{ccc} H^*(X_{\text{res}}) & & H^*(X_{\text{sm}}) \\ & \nwarrow \pi^* \quad \nearrow r^* & \\ & H^*(X_{\text{sing}}) & \end{array}$$

Set $E = E_1 \cup E_2 \cup \dots \cup E_k \subset X_{\text{res}}$ and $S = S_1 \cup S_2 \cup \dots \cup S_k \subset X_{\text{sm}}$. The relative cohomology exact sequence gives the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow H^2(X_{\text{res}}, E) \longrightarrow H^2(X_{\text{res}}) \longrightarrow H^2(E) \\ 0 &\longrightarrow H^4(X_{\text{res}}, E) \longrightarrow H^4(X_{\text{res}}) \longrightarrow 0 \\ 0 &\longrightarrow H^2(X_{\text{sm}}, S) \longrightarrow H^2(X_{\text{sm}}) \longrightarrow 0 \\ H^3(S) &\longrightarrow H^4(X_{\text{sm}}, S) \longrightarrow H^4(X_{\text{sm}}) \longrightarrow 0 \end{aligned}$$

Set $P = \{p_1, \dots, p_k\} \subset X_{\text{sing}}$. Then we have $H^*(X_{\text{res}}, E) \cong H^*(X_{\text{sing}}, P) \cong H^*(X_{\text{sm}}, S)$ and $H^i(X_{\text{sing}}, P) \cong H^i(X_{\text{sing}})$ for $i \geq 2$. Therefore we obtain:

$$(15) \quad \begin{aligned} 0 \rightarrow H^2(X_{\text{sing}}) \xrightarrow{\pi^*} H^2(X_{\text{res}}) \rightarrow \mathbb{C}^k, \quad \pi^*: H^4(X_{\text{sing}}) \cong H^4(X_{\text{res}}), \\ \mathbb{C}^k \rightarrow H^4(X_{\text{sing}}) \xrightarrow{r^*} H^4(X_{\text{sm}}) \rightarrow 0, \quad r^*: H^2(X_{\text{sing}}) \cong H^2(X_{\text{sm}}). \end{aligned}$$

Combining these sequences, we obtain:

$$(16) \quad \begin{aligned} 0 &\longrightarrow H^2(X_{\text{sm}}) \longrightarrow H^2(X_{\text{res}}) \longrightarrow \mathbb{C}^k, \\ \mathbb{C}^k &\longrightarrow H^4(X_{\text{res}}) \longrightarrow H^4(X_{\text{sm}}) \longrightarrow 0. \end{aligned}$$

Note that the map $H^2(X_{\text{res}}) \rightarrow \mathbb{C}^k$ in the first sequence sends a class $\alpha \in H^2(X_{\text{res}})$ to the vector $([E_1] \cdot \alpha, \dots, [E_k] \cdot \alpha) \in \mathbb{C}^k$.

Lemma 11.1. *The two sequences in (16) are dual to each other with respect to the Poincaré pairing. In other words, the image of the standard basis vector $e_i \in \mathbb{C}^k$ in $H^4(X_{\text{res}})$ under the map $\mathbb{C}^k \rightarrow H^4(X_{\text{res}})$ in the second sequence of (16) is the Poincaré dual of E_i .*

Proof. The dual of the map $\mathbb{C}^k = H^3(S) \rightarrow H^4(X_{\text{sm}}, S) \cong H^4(X_{\text{sing}}) \cong H^4(X_{\text{res}})$ is identified with the following boundary map:

$$(17) \quad H_4(X_{\text{res}}) \cong H_4(X_{\text{sing}}) \cong H_4(X_{\text{sm}}, S) \xrightarrow{\partial} H_3(S) = \mathbb{C}^k$$

It suffices to show that this is given by the intersection numbers with the exceptional curves $E_1, \dots, E_k$. Take a real 4-cycle $D \subset X_{\text{res}}$ and suppose that D intersects every E_i transversely. Under the conifold transition, each intersection point of D and E_i is replaced by the 3-sphere S_i . Therefore the image of $[D]$ under (17) is given by $(E_i \cdot D)_{i=1}^k$. The lemma follows. $\square$

Note that the map $\pi^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{res}})$ is injective and $r^*: H^*(X_{\text{sing}}) \rightarrow H^*(X_{\text{sm}})$ is surjective. The exact sequences (15), (16) and the above lemma imply the following description of $H^*(X_{\text{sm}})$ as a subquotient of $H^*(X_{\text{res}})$.

Proposition 11.2. *Consider the filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ defined by*

$$W := \sum_{i=1}^k \mathbb{C}[E_i] \subset H^4(X_{\text{res}}), \quad V := \text{Im } \pi^* \cong H^*(X_{\text{sing}}).$$

Then we have $V/W \cong H^(X_{\text{sm}})$. More precisely, the following holds:*

- (1) *W is the annihilator of V with respect to the Poincaré pairing, i.e. $W = V^\perp$. In particular, V/W has a non-degenerate pairing.*
- (2) *The map r^* induces an isomorphism $V/W \cong H^*(X_{\text{sm}})$ which preserves the pairing and the cup product.*

Remark 11.3. It is a subtle problem if X_{sing} admits a smoothing or if the small resolution X_{res} is projective. In the Calabi-Yau case, there is a criterion due to Friedman, Kawamata and Tian [11, 25, 35] about the smoothability of X_{sing} : X_{sing} is smoothable if and only if there exist non-zero rational numbers $\alpha_1, \dots, \alpha_k \in \mathbb{Q}^\times$ such that $\sum_{i=1}^k \alpha_i [E_i] = 0$ in $H_2(X_{\text{res}})$. In the Fano case, X_{sing} is always smoothable [11, 32].

11.2. A theorem of Li and Ruan. We write $\langle \dots \rangle_{g,n,d}^{\text{res}}$ for Gromov-Witten invariants for X_{res} and $\langle \dots \rangle_{g,n,\beta}^{\text{sm}}$ for Gromov-Witten invariants for X_{sm} . Li-Ruan [28] showed the following theorem:

Theorem 11.4 ([28, Theorem B]). *Let $v_1, \dots, v_n$ be elements of $H^*(X_{\text{sing}})$ and let $0 \neq \beta \in H_2(X_{\text{sm}}, \mathbb{Z})$ be a non-zero degree. We have:*

$$\sum_{d: \pi_*(d)=\beta} \langle \pi^*(v_1), \dots, \pi^*(v_n) \rangle_{g,n,d}^{\text{res}} = \langle r^*(v_1), \dots, r^*(v_n) \rangle_{g,n,\beta}^{\text{sm}}$$

The sum in the left-hand side is finite, i.e. $\langle \pi^(v_1), \dots, \pi^*(v_n) \rangle_{g,n,d}$ with $\pi_*(d) = \beta$ with a fixed β vanishes except for finitely many d .*

11.3. Transition of quantum cohomology. We choose a suitable basis of $H_2(X_{\text{res}}, \mathbb{Z})$. Let L be an ample line bundle over X_{sing} . Then the line bundle π^*L is nef on X_{res} , and for any curve $C \subset X_{\text{res}}$, we have $L \cdot C = 0$ if and only if C is one of the exceptional curves $E_1, \dots, E_k$. Therefore the face $F := \{d \in \overline{\text{NE}}(X_{\text{res}}) : d \cdot \pi^*L = 0\}$ of the Mori cone $\overline{\text{NE}}(X_{\text{res}})$ is spanned by the classes of $E_1, \dots, E_k$. We choose an integral basis $d_1, \dots, d_r$ of $H_2(X_{\text{res}}, \mathbb{Z})/\text{torsion}$ such that

- $d_1, \dots, d_e$ span a cone containing the face F , where² $e = \dim F$;
- $d_1, \dots, d_r$ span a cone containing $\overline{\text{NE}}(X_{\text{res}})$.

²We have $e \leq k$. It is possible that $[E_1], \dots, [E_k]$ are linearly dependent.

Let $q_1, \dots, q_r$ be the Novikov variables corresponding to the basis $d_1, \dots, d_r$. For any $d = \sum_{i=1}^r n_i d_i \in H_2(X_{\text{res}}, \mathbb{Z})/\text{torsion}$, we write $q^d = q_1^{n_1} q_2^{n_2} \cdots q_r^{n_r}$. By the exact sequence (16), we have

$$\bigoplus_{i=1}^k \mathbb{C}[E_i] \longrightarrow H_2(X_{\text{res}}) \xrightarrow{\pi_*} H_2(X_{\text{sing}}) \cong H_2(X_{\text{sm}}) \longrightarrow 0.$$

Therefore $\pi_*(d_{e+1}), \dots, \pi_*(d_r)$ form a basis of $H_2(X_{\text{sing}}) \cong H_2(X_{\text{sm}})$ and $q_{e+1}, \dots, q_r$ can be identified with the Novikov parameters for X_{sm} . Notice that, by Li-Ruan's theorem and by the surjectivity of r^* , Gromov-Witten invariants of X_{sm} of degree $\beta \in H_2(X_{\text{sm}}, \mathbb{Z})$ can be non-zero only when β is a linear combination of $\pi_*(d_{e+1}), \dots, \pi_*(d_r)$ with *non-negative* coefficients. Therefore the quantum product of X_{sm} is defined over the ring $\mathbb{C}[[q_{e+1}, \dots, q_r]]$.

Before stating the result, we explain the meaning of analytic continuation. We will consider analytic continuation of the quantum product of X_{res} to the locus $\Delta_{\text{exc}} := \{q_1 = q_2 = \cdots = q_e = 1\} \subset \mathbb{C}^r$, where all the quantum variables associated to exceptional classes equal one. The map $\pi_*: H_2(X_{\text{res}}) \rightarrow H_2(X_{\text{sing}}) \cong H_2(X_{\text{sm}})$ induces a ring homomorphism

$$\lim_{q_{\text{exc}} \rightarrow 1} := \lim_{(q_1, \dots, q_e) \rightarrow (1, \dots, 1)} : \mathbb{C}[q_1, \dots, q_r] \longrightarrow \mathbb{C}[q_{e+1}, \dots, q_r],$$

where q_{exc} stands for the quantum variables associated to exceptional curves. However, this does not extend to a homomorphism between the Novikov rings $\mathbb{C}[[q_1, \dots, q_r]]$ and $\mathbb{C}[[q_{e+1}, \dots, q_r]]$. Instead we have a map

$$\lim_{q_{\text{exc}} \rightarrow 1} : \mathbb{C}[q_1, \dots, q_e][[q_{e+1}, \dots, q_r]] \rightarrow \mathbb{C}[[q_{e+1}, \dots, q_r]].$$

Thus, if $v \star w$ is defined over the ring $\mathbb{C}[q_1, \dots, q_e][[q_{e+1}, \dots, q_r]]$, we have a well-defined limit $\lim_{q_{\text{exc}} \rightarrow 1} v \star w$.

Theorem 11.5. *The quantum cohomology of X_{sm} is a subquotient of the quantum cohomology of X_{res} restricted to the locus $\Delta_{\text{exc}} := \{q_1 = q_2 = \cdots = q_e = 1\}$ where the Novikov variables of exceptional curves equal one. More precisely, we have:*

- (1) *The small quantum product of $v \in H^*(X_{\text{res}})$ is of the form:*

$$(v \star) = \sum_{i=1}^k (v \cdot [E_i]) \frac{q^{E_i}}{1 - q^{E_i}} N_i + R(v),$$

where $R(v) \in \text{End}(H^*(X_{\text{res}})) \otimes \mathbb{C}[q_1, \dots, q_e][[q_{e+1}, \dots, q_r]]$ is regular along Δ_{exc} , $R(v)|_{q_{e+1}=\dots=q_r=0}$ is the cup product by v , and $N_i \in \text{End}(H^*(X_{\text{res}}))$ is a nilpotent endomorphism defined by $N_i(w) = (w \cdot [E_i])[E_i]$.

- (2) *The endomorphisms N_i define the filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ by*

$$V := \bigcap_{i=1}^k \text{Ker}(N_i) \quad W := V \cap \sum_{i=1}^k \text{Im}(N_i).$$

This filtration coincides with the one in Proposition 11.2, i.e. $W = \sum_{i=1}^k \mathbb{C}[E_i]$ and $V = \text{Im}(\pi^) \cong H^*(X_{\text{sing}})$.*

- (3) For $v, w \in V$, the limit $\lim_{q_{\text{exc}} \rightarrow 1} v \star w$ exists and lies in $V \otimes \mathbb{C}[[q_{e+1}, \dots, q_r]]$. Moreover, the map $r^*: V \rightarrow H^*(X_{\text{sm}})$ satisfies:

$$r^* \left(\lim_{q_{\text{exc}} \rightarrow 1} v \star w \right) = r^*(v) \star r^*(w).$$

Therefore, the isomorphism $V/W \cong H^*(X_{\text{sm}})$ in Proposition 11.2 intertwines the quantum product of X_{res} restricted to Δ_{exc} with the quantum product of X_{sm} .

In terms of the quantum connection, we can rephrase the above result as follows.

Corollary 11.6. *The small quantum connection ∇^{res} of X_{res} is of the form:*

$$\nabla^{\text{res}} = \nabla' + \frac{1}{z} \sum_{i=1}^k N_i \frac{dq^{E_i}}{1 - q^{E_i}}$$

where ∇' is a connection regular along $\Delta_{\text{exc}} = \{q_1 = \dots = q_e = 1\}$. The restriction of ∇' to Δ_{exc} induces a flat connection on the vector bundle $(V/W) \times \Delta_{\text{exc}} \rightarrow \Delta_{\text{exc}}$ which is isomorphic to the small quantum connection ∇^{sm} of X_{sm} under the natural isomorphism $r^*: V/W \cong H^*(X_{\text{sm}})$.

Remark 11.7. The filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ is the weight filtration associated to the nilpotent endomorphism $\sum_{i=1}^k a_i N_i$ (see e.g. [6, A.2]) for a generic choice of $a_1, \dots, a_k$. As we shall see later in §10 for $\text{Gr}(2, 5)$, however, the quantum cohomology of a smoothing does not necessarily appear as a subquotient associated with the weight filtration.

Remark 11.8. The monodromy of the quantum connection ∇^{res} around the divisor $\{q^{E_i} = 1\}$ is conjugate to $\exp(2\pi\sqrt{-1}N_i/z)$ and is unipotent.

Remark 11.9. The residue of $(v \star)$ along the divisor $q^{E_i} = 1$ is also computed by Lee-Lin-Wang [27, Lemma 3.12].

11.4. Proof of Theorem 11.5. We set $V = \text{Im } \pi^*$ and $W = \sum_{i=1}^k \mathbb{C}[E_i]$ as in Proposition 11.2. Since $V \cong H^*(X_{\text{sing}})$, we may regard r^* as a map from V to $H^*(X_{\text{sm}})$. Part (2) of Theorem 11.5 follows from part (1) of Theorem 11.5 and the exact sequences (15), (16). Thus it suffices to prove parts (1) and (3) of Theorem 11.5. Part (1) of Theorem 11.5 follows from the following lemma:

Lemma 11.10. *Fix $\beta \in H_2(X_{\text{sm}}, \mathbb{Z}) \cong H_2(X_{\text{sing}}, \mathbb{Z})$ and take $v_1, v_2, v_3 \in H^*(X_{\text{res}})$. Consider the sum*

$$(18) \quad \sum_{d \in \pi_*^{-1}(\beta)} \langle v_1, v_2, v_3 \rangle_{0,3,d}^{\text{res}} q^d.$$

- (1) *If $\beta \neq 0$, then the sum is finite;*
- (2) *If $\beta = 0$, the sum equals:*

$$\int_{X_{\text{res}}} v_1 \cup v_2 \cup v_3 + \sum_{i=1}^k (v_1 \cdot [E_i])(v_2 \cdot [E_i])(v_3 \cdot [E_i]) \frac{q^{E_i}}{1 - q^{E_i}}.$$

Proof. We may assume that v_1, v_2, v_3 are homogeneous. Suppose that $\beta \neq 0$. If $v_1, v_2, v_3 \in V = \text{Im } \pi^*$, the finiteness of the sum (18) follows from Theorem 11.4. If $v_1 \notin V$, then $v_1 \in H^2(X_{\text{res}})$ by homogeneity. Thus we can use the divisor equation to factor out v_1 :

$$\text{equation (18)} = \sum_{d \in \pi_*^{-1}(\beta)} (v_1 \cdot d) \langle v_2, v_3 \rangle_{0,2,d}^{\text{res}} q^d.$$

If in addition $v_2, v_3 \in V$, Theorem 11.4 again shows the finiteness of the sum. The finiteness in the other cases can be similarly shown by using the divisor equation ([8] p.193) and Theorem 11.4.

Next suppose that $\beta = 0$. The $d = 0$ term in (18) gives $\int_{X_{\text{res}}} v_1 \cup v_2 \cup v_3$. The only curves in X_{res} contributing to the sum (18) are multiples of the exceptional curve E_i . By the degree axiom ([8] p.192), we have $\deg v_1 + \deg v_2 + \deg v_3 = 6$. If one of $\deg v_1, \deg v_2, \deg v_3$ is zero, the invariant $\langle v_1, v_2, v_3 \rangle_{0,3,d}$ is zero for $d \neq 0$. Therefore we only need to consider the case where $v_1, v_2, v_3 \in H^2(X_{\text{res}})$. Since the moduli space $M_{0,0}(X_{\text{res}}, d)$ with $\pi_*(d) = 0$ consists of multiple covers of some E_i , we have

$$\begin{aligned} \sum_{d \neq 0: \pi_*(d)=0} \langle v_1, v_2, v_3 \rangle_{0,3,d}^{\text{res}} &= \sum_{d \neq 0: \pi_*(d)=0} (v_1 \cdot d)(v_2 \cdot d)(v_3 \cdot d) \langle \rangle_{0,0,d}^{\text{res}} q^d \\ &= \sum_{i=1}^k \sum_{n=1}^{\infty} (v_1 \cdot nE_i)(v_2 \cdot nE_i)(v_3 \cdot nE_i) \frac{1}{n^3} q^{nE_i} \\ &= \sum_{i=1}^k (v_1 \cdot E_i)(v_2 \cdot E_i)(v_3 \cdot E_i) \frac{q^{E_i}}{1 - q^{E_i}} \end{aligned}$$

by the multiple cover formula [30] for a $(-1, -1)$ -curve (each multiple cover of degree n contributes $1/n^3$). The lemma is proved. $\square$

Finally we prove part (3) of Theorem 11.5. Suppose that $v, w \in V$. The existence of the limit $\lim_{q_{\text{exc}} \rightarrow 1} v \star w$ follows from Lemma 11.10. We claim that

$$(19) \quad \lim_{q_{\text{exc}} \rightarrow 1} (u, v \star w) = (r^*(u), r^*(v) \star r^*(w))$$

for all $u \in V$. The left-hand side of (19) equals

$$(20) \quad \sum_{d: \pi_*(d)=0} \langle u, v, w \rangle_{0,3,d}^{\text{res}} + \sum_{\beta \neq 0} \sum_{d: \pi_*(d)=\beta} \langle u, v, w \rangle_{0,3,d}^{\text{res}} q^\beta.$$

By Lemma 11.10, the first term equals:

$$\int_{X_{\text{res}}} u \cup v \cup w = \int_{X_{\text{sm}}} r^*u \cup r^*v \cup r^*w$$

since $u \cdot [E_i] = v \cdot [E_i] = w \cdot [E_i] = 0$. We also used the fact that r^* preserves the cup product and the pairing (Proposition 11.2). By Theorem 11.4, the second

term of (20) equals:

$$\sum_{\beta \neq 0} \langle r^*(u), r^*(v), r^*(w) \rangle_{0,3,\beta}^{\text{sm}} q^\beta.$$

Therefore, the claim (19) follows. Setting $u = [E_i]$ in equation (19) and using the fact that $r^*([E_i]) = 0$ from Lemma 11.1, we obtain $([E_i], \lim_{q_{\text{exc}} \rightarrow 1} v \star w) = 0$. This means that $\lim_{q_{\text{exc}} \rightarrow 1} v \star w$ lies in V . Using again the fact that r^* preserves the pairing, we obtain from equation (19) that

$$\left(r^*(u), r^* \left(\lim_{q_{\text{exc}} \rightarrow 1} v \star w \right) \right) = (r^*(u), r^*(v) \star r^*(w)).$$

Since r^* is surjective, this shows that $r^*(\lim_{q_{\text{exc}} \rightarrow 1} v \star w) = r^*(v) \star r^*(w)$. Part (3) of Theorem 11.5 is proved.

12. CONCLUSION AND FUTURE WORK

In this thesis, we have studied 3-dimensional extremal transition, and higher dimensional cases $\text{Gr}(2,4)$ and $\text{Gr}(2,5)$. In general, for partial flag variety $\text{Fl}(n_1, \dots, n_l, n)$, the Mori cone of $\text{Fl}(n_1, \dots, n_l, n)$ is generated by curves Δ_i , $i = 1, \dots, l$. The curve Δ_i can be described as follows. We choose two linear subspaces W_1 and W_2 of dimension $n_i - 1$ and $n_i + 1$ respectively such that $W_1 \subset W_2$. Then $\Delta_i = \{[\{0\} \subset V_1 \subset \dots \subset V_l \subset V = \mathbb{C}^n] \in \text{Fl}(n_1, \dots, n_l, n) \mid \{0\} \subset V_1 \subset \dots \subset V_{i-1} \subset W_1 \subset V_i \subset W_2 \subset V_{i+1} \subset \dots \subset V_l, \text{ in which the vector spaces } V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_l \text{ are fixed}\}$. We denote by $\bar{q}_i$ the quantum parameter corresponding to Δ_i . As we said in section 6, the partial flag variety $\text{Fl}(n_1, \dots, n_l, n)$ can be degenerated to a singular toric variety X_{sing} with Gorenstein singularities. The singular toric variety X_{sing} admits a crepant resolution X_{res} . We have the following natural maps:

$$X_{\text{res}} \xrightarrow{\pi} X_{\text{sing}} \xleftarrow{\tau} X_{\text{sm}} = \text{Fl}(n_1, \dots, n_l, n).$$

Setting $e = \sum_{i=1}^l (n_i - 1)(n_{i+1} - n_i - 1) + \sum_{j=1}^{l-1} n_j$, the second homology group $H_2(X_{\text{res}}, \mathbb{Z})$ is of rank $l + e$. We can choose a nef basis $\{\beta_1, \dots, \beta_{l+e}\}$ of $H_2(X_{\text{res}})$ in such a way that $\langle c_1(TX_{\text{res}}), \beta_i \rangle = n_{i+1} - n_{i-1}$, for $i = 1, \dots, l$, and $\langle c_1(TX_{\text{res}}), \beta_j \rangle = 0$, $j \geq l+1$. Denote by q_i the quantum parameter corresponding to β_i . Let $\{p_1, \dots, p_{l+e}\}$ be the basis of $H^2(X_{\text{res}}, \mathbb{C})$ dual to $\{\beta_1, \dots, \beta_{l+e}\}$. Then we have the following conjecture

Conjecture 12.1 (Relations between quantum cohomologies of $\text{Fl}(n_1, \dots, n_l, n)$ and X_{res}).

- (1) The quantum product matrices of p_j are rational functions. And

$$N_j := \text{Res}_{q_j=1} p_j \star \frac{dq_j}{q_j} \Big|_{(q_{l+1}, \dots, q_{l+e})=(1, \dots, 1)}, j \geq l+1,$$

are nilpotent matrices.

- (2) We define the a filtration $\{0\} \subset W \subset V \subset H^*(X_{\text{res}}, \mathbb{C})$ of $H^*(X_{\text{res}}, \mathbb{C})$, where

$$V := \bigcap_{j=l+1}^{l+e} \text{Ker } N_j, \quad W := V \cap (\text{Im } N_{l+1} + \cdots + \text{Im } N_{l+e}).$$

The filtration $0 \subset W \subset V \subset H^*(X_{\text{res}})$ defined by the residue $N_j, j \geq l+1$ along $q_j = 1, j \geq l+1$, matches with the filtration

$$0 \subset \pi^*(\text{Ker } r^*) \subset \text{Im } \pi^* \subset H^*(X_{\text{res}}).$$

- (3) The map $\text{Im } \pi^*$ is injective and the quantum products of elements in $\text{Im } \pi^*$ are regular at $q_j = 1, j \geq l+1$ and the map

$$r^* \circ (\pi^*)^{-1}: \text{Im } \pi^* \rightarrow H^*(\text{Fl}(n_1, \dots, n_l, n), \mathbb{C})$$

intertwines the quantum product $\star|_{q_j=1, \dots, q_{l+e}=1}$ on $\text{Im } \pi^* = V$ with the quantum product on $H^*(\text{Fl}(n_1, \dots, n_l, n), \mathbb{C})$ under the identification $q_i = \bar{q}_i$ of the Novikov variables. The map $r^* \circ (\pi^*)^{-1}$ induces a homomorphism from $(V/W, \star|_{q_l=1, \dots, q_{l+e}=1})$ of $H^*(X_{\text{res}})$ with a subring of the quantum cohomology of $\text{Fl}(n_1, \dots, n_l, n)$. This map also preserves the Poincaré pairing.

In this thesis, we have computed the quantum cohomology of flag varieties and X_{res} explicitly, case by case. It is evident that the complexity of the computation grows more and more difficult when the dimension of varieties grow higher and higher. As a result, computing the quantum cohomology explicitly and the comparing them with eyes seems not a good way to establish relations between quantum cohomology of $\text{Fl}(n_1, \dots, n_l, n)$ and the quantum cohomology of X_{res} . We hope to establish a relationship between the J -function of $\text{Fl}(n_1, \dots, n_l, n)$ and the J -function of X_{res} . All these problems remain to be solved.

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