

ON THE EXISTENCE OF HETEROCLINIC SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS ON A STRIP-LIKE DOMAIN

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1. Introduction. In this note, we consider the existence of heteroclinic solutions of semilinear elliptic equations on a strip-like domain. Let S be a strip-like domain, i.e., $S = R \times \Omega$ where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$. We study the existence of solutions of the problem:

$$(P_0) \quad \begin{cases} -\Delta u = g(u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

where $g \in C(L^2(\Omega), L^2(\Omega))$ with $g'(\cdot) \in C(L^2(\Omega), L^2(\Omega))$.

The problem (P_0) appears in several problems in mechanics and physics. For example, the problem (P_0) describes waves in density-stratified channels[8]. It also considered as a model equation for viscous fluid flow between concentric cylinders(cf.[3],[7],[9]). The problem (P_0) can be rewritten as

$$(P) \quad \begin{cases} -u_{tt} - \Delta_x u = g(u) & \text{in } R \times \Omega \\ u(t, x) = 0 & \text{for } x \in \partial\Omega \text{ and } t \in R \end{cases}$$

The t-stationally solutions of problem (P) is the solutions of the problem:

$$(P_s) \quad \begin{cases} -\Delta_x u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The problem (P_s) is variational. That is the solutions of (P_s) are critical points of the functional

$$F(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla_x u|^2 - G(u) \right) dx \quad \text{for } u \in H_0^1(\Omega) \quad (1.1)$$

where $G(t) = \int_0^t g(s) ds$.

We are interested in the existence of heteroclinic solutions of (P). Existence of heteroclinic solutions is deeply related to the critical points (critical levels) of the functional F . Our purpose in this note is to show the existence of heteroclinic solutions of the problem (P). More precisely, we seek for a solution of (P) which converges to critical points of F as t tends to $\pm\infty$. The existence of non-periodic solutions for (P) has been proved by Amick[3], Bona et al[4], Kirchgassner[7] and Turner[9] for the case that g is odd. In [6], Esteban have shown the existence of non-periodic solutions without assuming oddness of g . In [5], Cannino have shown that if g is odd and (P_s) has a positive(negative) solution $v^+(v^-)$ which is a global minimum of F , there is a heteroclinic orbit of (P)(cf. also Kirchgassner[7]). We prove that if F has two global minumums, there exists a heteroclinic solution connecting the two points.

In the following, we denote by $\|\cdot\|$ and $|\cdot|_2$ the norms of the Sobolev space $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. For each $x, y \in L^2(\Omega)$, $\langle x, y \rangle$ denotes the inner product of x, y in $L^2(\Omega)$. We denote by $\lambda_1 < \lambda_2 \leq \dots$, the eigenvalues of the problem

$$-\Delta u = \lambda u, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.4)$$

Let $E(\lambda)$ denote the finite dimensional subspace of $H_0^1(\Omega)$ spanned by the eigenfunctions of the problem (1.4) corresponding to the eigenvalues smaller than or equal to λ . We denote by $M_{loc}(M_{glb})$ the set of local(global) minimums of F in $H_0^1(\Omega)$. We impose the following conditions on F :

(F1) *There exist $a_1, a_2 > 0$ such that*

$$\langle \nabla F(u), u \rangle \geq a_1 \|u\|^2 \quad \text{for all } u \text{ with } |u|_2 > a_2; \quad (1.2)$$

(F2) *there exists $\lambda > 0, \alpha_1 > 0$ and $\alpha_2 > 0$ satisfying that*

$$\langle \nabla F(u) - \nabla F(v), u - v \rangle \geq -\alpha_1 \|u - v\|^2 \quad (1.3)$$

for each $u, v \in H_0^1(\Omega)$, and

$$\langle \nabla F(v + z_1) - \nabla F(v + z_2), z_1 - z_2 \rangle \geq \alpha_2 \|z_1 - z_2\|^2 \quad (1.4)$$

for each $v \in H_0^1(\Omega)$ and $z_1, z_2 \in E(\lambda)^\perp$;

(F3) M_{loc} consists of finite number of points $v_0, v_1, \dots, v_n$: Moreover, F is nondegenerate at each v_i , $0 \leq i \leq n$.

The condition (F1) is imposed for the functional J to satisfy Palais-Smale condition(hereafter referred by P-S). That is we may replace (F1) by

(F1') For each $T > 0$, J satisfies P-S in H_T .

It is easy to see that if g satisfies that

$$\limsup_{|t| \rightarrow \infty} g(t)/t < \lambda_1, \quad (1.5)$$

then (F1) holds. The condition (F2) is fulfilled if g satisfies that for some $\lambda > 0$.

$$\sup_{t \in \mathbb{R}} g'(t) < \lambda \quad (1.6)$$

We give an existence result for the case that the set M_{glb} consists of exactly two points. In this case, we can find a heteroclinical solution(orbit) without assuming any condition on critical points whose critical level are greater than the global minimal value.

Theorem 1. Suppose that (F1) holds and that F has exact two global minimal points $v_{\pm}$ which are nondegenerate. Then there exists a solution u of (P) such that

$$\lim_{t \rightarrow \pm \infty} u(t) = v_{\pm} \quad \text{in } L^2(\Omega).$$

Moreover, there exist sequences $\{t_n^{\pm}\}$ with $\lim_{n \rightarrow \infty} t_n^{\pm} = \pm \infty$ such that

$$\lim_{n \rightarrow \infty} u(t_n^{\pm}) = v_{\pm} \quad \text{in } H_0^1(\Omega).$$

By imposing (F2) and (F3), we can remove that condition that F has exactly two local minimals. That is we have

Theorem 2. Suppose that (F1)-(F3) hold. Moreover suppose that There exists at least two global minimals of F . Then there exists a solution u of (1.1) such that

$$\lim_{t \rightarrow \pm \infty} u(t) = v_{\pm} \quad \text{in } L^2(\Omega),$$

where $v_{\pm} \in H_0^1(\Omega)$ such that $F(v_{\pm}) = \min\{F(v) : v \in H_0^1(\Omega)\}$.

Moreover, there exist sequences $\{t_n^{\pm}\}$ with $\lim_{n \rightarrow \infty} t_n^{\pm} = \pm\infty$ such that

$$\lim_{n \rightarrow \infty} u(t_n^{\pm}) = v_{\pm} \quad \text{in } H_0^1(\Omega).$$

Remark 1. We may replace (F3) with the following condition which is slightly weaker than (F3):

(F3') (a): M_{loc} consists of finite points ;

(b): each point in $M_{loc} \setminus M_{glb}$ is nondegenerate, and for each $v \in M_{glb}$, there exists a neighborhood U of v such that

$$\langle \nabla F(v), u - v \rangle \geq 0 \quad \text{for all } u \in U, \quad (1.7)$$

2. Proof of Theorems . In the following, we assume for simplicity that $\min\{F(v) : v \in H_0^1(\Omega)\} = 0$. Let

$$\Gamma = \{u \in L^2(R; H_0^1(\Omega)) \cap H^1(R; L^2(\Omega)) : \lim_{t \rightarrow \pm\infty} u(t) = v_{\pm} \text{ in } L^2(\Omega)\}$$

and

$$\Gamma_0 = \{u \in \Gamma : |u_t(t)|_2^2 = F(u(t)), \text{ for a.e. } t \in R\}.$$

We also set

$$J_{\infty} = \int_{-\infty}^{\infty} \int_{\Omega} |u_t|^2 dx dt + \int_{-\infty}^{\infty} F(u) dt.$$

Then we have

Lemma 2.1. $m = \inf\{J_{\infty}(u) : u \in \Gamma_0\} = \inf\{J_{\infty}(u) : u \in \Gamma\} < \infty$.

where J_{∞} is the functional defined in (3.8).

Proof. Since $v_{\pm}$ are nondegenerate, there exist neighborhoods $U_{\pm}$ of $v_{\pm}$ and homeomorphisms $\varphi_{\pm} : U_{\pm} \rightarrow H_0^1(\Omega)$ satisfying that

$$F(\varphi_{\pm}(v)) = \|v\|^2 \quad \text{for all } v \in U_{\pm}.$$

For simplicity, we assume that $\varphi_{\pm}(v) = v - v_{\pm}$. That is

$$F(v) = \|v - v_{\pm}\|^2 \quad \text{for all } v \in U_{\pm}. \quad (2.1)$$

We first see that $m = \inf\{J_\infty(u) : u \in \Gamma\} < \infty$. Let $u : (-\infty, \infty) \rightarrow H_0^1(\Omega)$ be a function defined by

$$u(t) = \left(\frac{1}{2} - \beta(t)\right)v_- + \left(\frac{1}{2} + \beta(t)\right)v_+$$

where $\beta(t) = -t/2(|t| + 1)$ for $t \in R$. Then from the definition of β , we have that

$$\int_{-\infty}^{\infty} (|u_t|^2 + \|u\|^2) dt < \infty.$$

Therefore $u \in \Gamma$. By (2.1), it is easy to check that $J_\infty(u) < \infty$. Then we have shown that $m < \infty$. Let P_0 be the set of normalized pathes connecting v_- and v_+ . That is

$$P_0 = \{p(\cdot) \in C([-1, 1]; H_0^1(\Omega)) : |p'(s)| = \text{const. in } (-1, 1), \\ p(\pm 1) = v_\pm \text{ and } p(t) \notin \{v_-, v_+\} \text{ for } t \in (-1, 1)\} \quad (2.2)$$

Let V be a set of mappings $\tau(\cdot) : [-1, 1] \rightarrow (-\infty, \infty)$ satisfying that

$$t(\cdot) \text{ is strictly monotone increasing, and } \lim_{s \rightarrow \pm 1} \tau(s) = \pm \infty.$$

We note that if $\tau(\cdot) \in V$, τ is differentiable a.e. in $(-1, 1)$. Let $\tau(\cdot) \in V$ and $p \in P_0$. For simplicity, we assume that $|p'(s)| = 1$ for all $s \in (-1, 1)$. Now we set $u(t) = p(\tau^{-1}(t))$ for $t \in R$. Then from the relation that $t = \tau(s)$ for $s \in [-1, 1]$, we have

$$\frac{dt}{ds} = | \frac{dp}{dt} |^{-1}, \quad \left(\frac{dp}{dt} = \frac{dp}{ds} \cdot \frac{ds}{dt} \right).$$

Then it follows that

$$J_\infty(u(t)) = \int_{-\infty}^{\infty} (|u_t|^2 + F(u(t))) dt = \int_{-1}^1 \left(\frac{ds}{dt} + F(u(s)) \frac{dt}{ds} \right) ds.$$

Here we set

$$J_p(\tau) = \int_{-1}^1 \left(\frac{ds}{d\tau} + F(v(s)) \frac{d\tau}{ds} \right) ds \quad \text{for each } \tau \in V. \quad (2.3)$$

We claim that for each $p \in P_0$, there exists $\tau_0 \in V$ such that $J_p(\tau_0) = m_p = \inf\{J_p(\tau) : \tau \in V\}$. Let $p \in P_0$ such that $m_p < \infty$. Then there exists a sequence $\{\tau_n\} \subset V$ such that $\lim_{n \rightarrow \infty} J_p(\tau_n) = m_p$. We may assume without any loss of generality that $\tau_n(0) = 0$ for all $n \geq 1$. We first see that there exist $\tau_0 \in V$ and a subsequence of $\{\tau_n\}$ (again denoted by $\{\tau_n\}$) such that $\tau_n(t) \rightarrow \tau_0(t)$ for all $t \in [-1, 1]$. Let t be an arbitrary number in $(-1, 1)$. We may suppose by extracting subsequences that $\tau_n(t) \rightarrow c$, as $n \rightarrow \infty$. Since $p(s) \notin \{v_-, v_+\}$ for $s \in (-1, 1)$, we have

$$c_0 = \inf\{F(p(s)) : 0 \leq s \leq |t|\} > 0. \quad (2.4)$$

Then

$$J_p(\tau_n) \geq \left| \int_0^t F(v(s)) \frac{d\tau_n}{ds} ds \right| \geq c_0 |\tau_n(t)| \quad \text{for each } n \geq 1.$$

Then since $\{J_p(\tau_n)\}$ is bounded, we find that $\{\tau_n(t)\}$ is also bounded. Thus we obtain that $|c| < \infty$. Since τ_n is monotone increasing, it follows that $\tau_n(s)$ is convergent for all $s \in [0, t]$. Since t is arbitrary, we have by repeating the argument above that there exists a subsequence of $\{\tau_n\}$ (denoted by $\{\tau_n\}$) such that $\lim \tau_n(s)$ exists for all $s \in [-1, 1]$. Here we put $\tau_0(s) = \lim \tau_n(s)$ for all $s \in [-1, 1]$. Then τ_0 is monotone increasing. We next see that τ_0 is strictly monotone. Suppose that $\tau_0(a) = \tau_0(b)$ for some $a < b$. By (2.3), we have

$$J_p(\tau_n) \geq \int_{-1}^1 \left(\frac{d\tau_n}{ds}\right)^{-1} ds \geq \int_a^b \left(\frac{d\tau_n}{ds}\right)^{-1} ds.$$

Then since $\lim_{n \rightarrow \infty} \tau_n(a) = \lim_{n \rightarrow \infty} \tau_n(b)$, the right hand side of the inequality above tends to infinity as $n \rightarrow \infty$. Since $\{J_p(\tau_n)\}$ is bounded, this is a contradiction. Therefore we have shown that $\tau_0 \in V$. Since τ_n and τ are monotone increasing, we also obtain that $d\tau_n/ds \rightarrow d\tau/ds$ a.e. on R . Then recalling that $d\tau_n/ds > 0$ for all $n \geq 1$, we obtain from Fatou's lemma that

$$J_p(\tau_0) = \int_{-1}^1 \left(\frac{ds}{d\tau_0} + F(p(s)) \frac{d\tau_0}{ds}\right) ds \leq \liminf \int_{-1}^1 \left(\frac{ds}{d\tau_n} + F(p(s)) \frac{d\tau_n}{ds}\right) ds$$

Thus we have shown that $J_p(\tau_0) = \lim_{n \rightarrow \infty} J_p(\tau_n) = m_0$. This implies that for each $\sigma \in V$,

$$\int_{-1}^1 (J_p(\tau_0))'(\sigma)(s) ds = \int_{-1}^1 \left(-\left(\frac{d\tau_0}{ds}\right)^2 + F(p(s)) \right) \left(\frac{d\sigma}{ds}\right) ds \geq 0.$$

Since $\sigma \in V$ is arbitrary, it is easy to see that

$$\left(\frac{d\tau_0}{ds}\right)^2 = F(p(s)) \quad \text{a.e. on } (-1, 1). \quad (2.5)$$

Now we put $u(t) = p(\tau^{-1}(t))$ for $t \in R$. Then,

$$|u_t|^2 = \left| \frac{dp}{ds} \cdot \frac{d\tau}{dt} \right|^2 = F(u(t)) \quad \text{a.e. on } R.$$

Therefore, $u(\cdot) \in \Gamma_0$. Thus we have shown that the assertion holds. ■

Lemma 2.2. *If $u \in \Gamma$ satisfies $J_\infty(u) = m$, then u is a solution of (P).*

Proof. Let $f(t) \in C_0^1([0, 1])$, $\varphi \in H_1^0(\Omega)$ and $a, b \in R$ with $a < b$. We define a function $v \in \Gamma$ by

$$v(t) = \begin{cases} 0 & \text{on } (-\infty, a) \cup (b, \infty), \\ f((t-a)/(b-a))\varphi & \text{on } [a, b] \end{cases}$$

Then since $u + sv \in \Gamma$ for $s \in [0, 1]$, we find that

$$\langle J'_\infty(u), v \rangle = \langle u_{tt} + \Delta u + g(u), v \rangle \geq 0.$$

From the definition of v , we have that

$$\langle \varphi, \int_a^b (u_{tt} + \Delta u + g(u)) f((t-a)/(b-a)) dt \rangle \geq 0.$$

Since $\varphi \in H_0^1(\Omega)$, $f \in C_0^1([0, 1])$ and the interval $[a, b]$ are arbitrary, we find that u is a solution of (P). ■

Proof of Theorem 1. Let $\{u_n\} \subset \Gamma$ be a sequence such that

$\lim_{n \rightarrow \infty} J_\infty(u_n) = m$. By Lemma 4.1, we may assume that $\{u_n\} \subset \Gamma_0$. Since $\lim_{\|v\| \rightarrow \infty} F(v) = \infty$ by (F1), we may assume that for some $r > 0$,

$$\|u_n(t)\| \leq r \quad \text{for all } n \geq 1 \text{ and } t \in R \quad (2.6)$$

Then since $\{u_n\} \subset \Gamma_0$, we also have that there exists $r_0 > 0$ such that

$$|u_{nt}(t)| < r_0 \quad \text{for all } n \geq 1 \text{ and } t \in R \quad (2.7)$$

Let $B(d)_\pm$ be open balls in $L^2(\Omega)$ centered $v_\pm$ with radius $d > 0$. Let $d_0 > 0$ such that $B(d_0)_- \cap B(d_0)_+ = \emptyset$. Since $u_n \in P_0$, we have $\{u_n(t) : t \in R\} \not\subset B(d_0)_- \cup B(d_0)_+$. Then we may assume without any loss of generality that

$$u_n(0) \notin B(d_0)_- \cap B(d_0)_+ \quad \text{for all } n \geq 1.$$

On the other hand, we have from the assumption that for each $d > 0$, there exists $\epsilon(d) > 0$ such that

$$F(v) > \epsilon(d) \quad \text{for all } v \notin B(d)_- \cup B(d)_+$$

Recalling that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we obtain by (2.6) and (2.7) that u_n converges to $u \in C(-\infty, \infty; L^2(\Omega))$ uniformly on each bounded interval $I \subset R$. It also follows that u_{nt} converges to u_t weakly in $L^2(-\infty, \infty; L^2(\Omega))$ and that $u_n(t)$ converges to $u(t)$ weakly in $H_0^1(\Omega)$, for a.e. $t \in R$. Then we obtain from the upper semicontinuity of J_∞ that $J_\infty(u) \leq \lim_{n \rightarrow \infty} J_\infty(u_n)$. We next show that $u \in \Gamma$. That is we see that $\lim_{t \rightarrow \pm\infty} u(t) = v_\pm$ in $L^2(\Omega)$. Let $0 < d < d_0$. Then since F is greater than $\epsilon(d)$ outside of $B(d)_- \cup B(d)_+$ and $\{J_\infty(u_n)\}$ is bounded, we have that there exists t_0 such that

$$u_n(t) \in B(d)_\pm \quad \text{for all } n \geq 1 \text{ and } |t| \geq t_0. \quad (2.8)$$

Then it follows from (2.8) that $\lim_{t \rightarrow \pm\infty} u(t) = v_\pm$ in $L^2(\Omega)$. Thus we have shown that $u \in \Gamma$. Therefore $J_\infty(u) = m$ and by Lemma 4.2, u is a solution of (P). This completes the proof. ■

Sketch of the proof of Theorem 2. The proof of Theorem 2 is long and complicated. Here we give a sketch of the proof. For each $n \geq 1$, we out

$$J_n(u) = \left(\frac{1}{2} \int_{-nT}^{nT} \int_{\Omega} |u_t|^2 dx dt + \int_{-nT}^{nT} F(u) dt \right), \quad (2.9)$$

for $u \in L^2(0, 2nT; H_0^1(\Omega))$. Then we can find a critical point u_n of J_n such that

$$d_n = (1/nT)J_n(u_n) \rightarrow m, \text{ as } n \rightarrow \infty$$

where m is the global minimal value of F . It then follows that

$$\lim_{n \rightarrow \infty} F(u_n(\pm nT)) = m.$$

It also follows that

$$\sup\{\|u_n(t)\|: n \geq 1, -nT \leq t \leq nT\} < \infty.$$

Then it is not so difficult to see that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and $u \in L^2(R; H_0^1(\Omega))$ such that

$$u_{n_i}(t) \rightarrow u(t) \quad \text{for all } t \in R.$$

Therefore u satisfies the assertion of Theorem 2.

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