

有界な台をもつ尺度モデルでの Rényi divergence の漸近展開

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1 はじめに

$p = p(x)$ と $q = q(x)$ をルベーク測度に関する pdf とする. このとき p から q への次数 s の Rényi divergence は

$$D_s(p||q) = \frac{1}{s-1} \log \int p(x)^s q(x)^{1-s} d\mu(x)$$

で定義される ([4]).

次数を変えることで, Rényi divergence から様々な divergence や距離が導出される ([5]). 例えば, Hellinger 距離の 2 乗

$$H^2(p||q) = \frac{1}{2} \int \left(\sqrt{p(x)} - \sqrt{q(x)} \right)^2 d\mu(x)$$

に対して,

$$D_{1/2}(p||q) = -2 \log \left(1 - H^2(p||q) \right)$$

となる. また, 正則条件の下で, $s \rightarrow 1$ とすれば,

$$\lim_{s \rightarrow 1} D_s(p||q) = D_{\text{KL}}(p||q)$$

となる. ただし, $D_{\text{KL}}(p||q)$ は Kullback-Leibler divergence

$$D_{\text{KL}}(p||q) = \int p(x) \log \frac{p(x)}{q(x)} d\mu(x)$$

とする.

また, Rényi divergence は統計的推測において重要な役割を果たすことが知られている (例えば, 非正則な位置母数分布族に対する情報量損失については Akahira (1996), 非正則な位置母数分布族に対する大偏差型の漸近理論については Hayashi (2006, 2010) 等). さらに, 正則な確率分布族においては, Rényi divergence の摂動が Fisher 情報量を与えることが Akahira (1996) により示されている.

本稿では, 台が有界な尺度母数分布族の下で, Rényi divergence の漸近展開を考える.

2 仮定

pdf $f(x | \theta) = \frac{1}{\theta} g\left(\frac{x}{\theta}\right)$ をもつ確率分布族を考える ($\theta > 0$ は尺度母数). ただし, 本稿においては pdf $g(x)$ に次の条件を課す.

条件

$$g(x) \begin{cases} > 0 & (a < x < b), \\ = 0 & (\text{それ以外}) \end{cases}$$

かつ, $a < x < b$ において $g(x)$ は十分滑らかな関数で

$$g(a) = \lim_{x \rightarrow a+0} g(x) > 0, \quad g(b) = \lim_{x \rightarrow b-0} g(x) > 0$$

である.

また, 簡単のため

$$\begin{aligned} g'(a) &= \lim_{x \rightarrow a+0} g'(x), \quad g'(b) = \lim_{x \rightarrow b-0} g'(x), \\ g''(a) &= \lim_{x \rightarrow a+0} g''(x), \quad g''(b) = \lim_{x \rightarrow b-0} g''(x) \end{aligned}$$

のように表記をする.

3 Fisher 情報量の違い

pdf $f(x | \theta) = \frac{1}{\theta} g\left(\frac{x}{\theta}\right)$ をもつ確率分布に従うとする ($\theta > 0$ は尺度母数). いま,

$$\begin{aligned} \frac{\partial}{\partial \theta} \log f(x | \theta) &= \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} = \frac{\partial}{\partial \theta} \left\{ \frac{1}{\theta} g\left(\frac{x}{\theta}\right) \right\} \bigg/ \left\{ \frac{1}{\theta} g\left(\frac{x}{\theta}\right) \right\} = \left\{ -\frac{1}{\theta} - \frac{x g'\left(\frac{x}{\theta}\right)}{\theta^2 g\left(\frac{x}{\theta}\right)} \right\}, \\ \frac{\partial^2}{\partial \theta^2} \log f(x | \theta) &= \frac{\frac{\partial^2}{\partial \theta^2} f(x | \theta)}{f(x | \theta)} - \left\{ \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} \right\}^2 \\ &= \frac{2}{\theta^2} + \frac{4x g'\left(\frac{x}{\theta}\right)}{\theta^3 g\left(\frac{x}{\theta}\right)} + \frac{x^2 g''\left(\frac{x}{\theta}\right)}{\theta^4 g\left(\frac{x}{\theta}\right)} - \left\{ -\frac{1}{\theta} - \frac{x g'\left(\frac{x}{\theta}\right)}{\theta^2 g\left(\frac{x}{\theta}\right)} \right\}^2 \\ &= \frac{1}{\theta^2} + \frac{-x^2 g'\left(\frac{x}{\theta}\right)^2}{\theta^4 g^2\left(\frac{x}{\theta}\right)} + \frac{2x g'\left(\frac{x}{\theta}\right)}{\theta^3 g\left(\frac{x}{\theta}\right)} + \frac{x^2 g''\left(\frac{x}{\theta}\right)}{\theta^4 g\left(\frac{x}{\theta}\right)} \end{aligned}$$

である. Fisher 情報量を

$$\begin{aligned} I_0(\theta) &= \int_{S(\theta)} \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 f(x | \theta) dx, \\ I(\theta) &= - \int_{S(\theta)} \frac{\partial^2 \log f(x | \theta)}{\partial \theta^2} f(x | \theta) dx \end{aligned}$$

とおく¹⁾。ただし、 $S(\theta)$ は $f(x|\theta)$ の台で $S(\theta) = (a\theta, b\theta)$ である。

3.1 $I(\theta)$ と $I_0(\theta)$ の差

まず $I(\theta)$ と $I_0(\theta)$ の差を求める。 $y = x/\theta$ とおくと

$$\begin{aligned} I(\theta) &= - \int_{a\theta}^{b\theta} \frac{\partial^2 \log f(x|\theta)}{\partial \theta^2} f(x|\theta) dx \\ &= \int_{a\theta}^{b\theta} \left[-\frac{2}{\theta^2} - \frac{4xg'(\frac{x}{\theta})}{\theta^3 g(\frac{x}{\theta})} - \frac{x^2 g''(\frac{x}{\theta})}{\theta^4 g(\frac{x}{\theta})} + \left\{ -\frac{1}{\theta} - \frac{xg'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\}^2 \right] \frac{1}{\theta} g\left(\frac{x}{\theta}\right) dx \\ &= \frac{1}{\theta^2} \int_a^b \{-2g(y) - 4yg'(y) - y^2 g''(y)\} dy + I_0(\theta) \end{aligned} \quad (3.1)$$

となる。ここで、部分積分より

$$\begin{aligned} &\int_a^b \{-2g(y) - 4yg'(y) - y^2 g''(y)\} dy \\ &= -2 \int_a^b g(y) dy - 4 \left\{ [yg(y)]_a^b - \int_a^b g(y) dy \right\} - \left\{ [y^2 g'(y)]_a^b - 2[yg(y)]_a^b + 2 \int_a^b g(y) dy \right\} \\ &= -2\{bg(b) - ag(a)\} - \{b^2 g'(b) - a^2 g'(a)\} \end{aligned}$$

である。よって、(3.1) より

$$I(\theta) = \frac{-2\{bg(b) - ag(a)\} - \{b^2 g'(b) - a^2 g'(a)\}}{\theta^2} + I_0(\theta)$$

を得る。

3.2 $I_0(\theta)$ と $I'_0(\theta)$ の差

次に、

$$I'_0(\theta) = \int_S \left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 f(x|\theta) dx$$

において、 $I'_0(\theta)$ と $I_0(\theta)$ の関係式を求める。ただし、 $S = S(\theta) \cap S(\theta + \varepsilon)$ とする。

$$\begin{aligned} I_0(\theta) &= \int_{a\theta}^{b\theta} \left(-\frac{1}{\theta} - \frac{x}{\theta^2} \frac{g'(\frac{x}{\theta})}{g(\frac{x}{\theta})} \right)^2 \frac{1}{\theta} g\left(\frac{x}{\theta}\right) dx \\ &= \int_a^b \left(-\frac{1}{\theta} - \frac{y}{\theta} \frac{g'(y)}{g(y)} \right)^2 g(y) dy \quad (y = x/\theta) \\ &= \frac{1}{\theta^2} \int_a^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy = \frac{1}{\theta^2} I_0(1) \end{aligned}$$

となる。 a, b や ε の正負により S は異なる区間になるので、場合分けをして考える。まず“ $\varepsilon > 0$ かつ $0 < a < b$ ” または “ $\varepsilon > 0$ かつ $a < b < 0$ ” のときを考える。このとき、 $S = (a(\theta + \varepsilon), b\theta)$ となる

1) ただし、本稿のモデルでは pdf $f(\cdot|\theta)$ の台が母数 θ に依存するため、本来は Fisher 情報量は存在せず、正則な場合のときの Fisher 情報量の定義を当てはめたものである。

ので

$$\begin{aligned}
 I'_0(\theta) &= \int_S \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 f(x | \theta) dx = \int_{a(\theta+\varepsilon)}^{b\theta} \left(-\frac{1}{\theta} - \frac{x}{\theta^2} \frac{g'(\frac{x}{\theta})}{g(\frac{x}{\theta})} \right)^2 \frac{1}{\theta} g\left(\frac{x}{\theta}\right) dx \\
 &= \int_{a(1+(\varepsilon/\theta))}^b \left(-\frac{1}{\theta} - \frac{y}{\theta} \frac{g'(y)}{g(y)} \right)^2 g(y) dy \quad (y = x/\theta) \\
 &= \frac{1}{\theta^2} \int_a^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy - \frac{1}{\theta^2} \int_a^{a(1+(\varepsilon/\theta))} \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy \\
 &= I_0(\theta) - \frac{1}{\theta^2} \left\{ \frac{a\varepsilon (a^2 g'(a)^2 + 2ag(a)g'(a) + g(a)^2)}{\theta g(a)} \right. \\
 &\quad + \frac{a^2 \varepsilon^2 (-a^2 g'(a)^3 + 2a^2 g(a)g'(a)g''(a) + 2ag(a)^2 g''(a) + 2ag(a)g'(a)^2 + 3g(a)^2 g'(a))}{2\theta^2 g(a)^2} \\
 &\quad \left. + O(\varepsilon^3) \right\}
 \end{aligned}$$

となる。次に“ $\varepsilon > 0$ かつ $a < 0 < b$ ”のときを考える。このとき、 $S = (a\theta, b\theta) = S(\theta)$ となるので

$$I'_0(\theta) = \int_S \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 f(x | \theta) dx = I_0(\theta)$$

となる。次に“ $\varepsilon > 0$ かつ $a < b < 0$ ”または“ $\varepsilon < 0$ かつ $0 < a < b$ ”のときを考える。このとき、 $S = (a\theta, b(\theta + \varepsilon))$ となるので

$$\begin{aligned}
 I'_0(\theta) &= \int_S \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 f(x | \theta) dx = \int_{a\theta}^{b(\theta+\varepsilon)} \left(-\frac{1}{\theta} - \frac{x}{\theta^2} \frac{g'(\frac{x}{\theta})}{g(\frac{x}{\theta})} \right)^2 \frac{1}{\theta} g\left(\frac{x}{\theta}\right) dx \\
 &= \int_a^{b(1+(\varepsilon/\theta))} \left(-\frac{1}{\theta} - \frac{y}{\theta} \frac{g'(y)}{g(y)} \right)^2 g(y) dy \quad (y = x/\theta) \\
 &= \frac{1}{\theta^2} \int_a^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy - \frac{1}{\theta^2} \int_{b(1+(\varepsilon/\theta))}^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy \\
 &= I_0(\theta) - \frac{1}{\theta^2} \left\{ -\frac{b\varepsilon (b^2 g'(b)^2 + 2bg(b)g'(b) + g(b)^2)}{\theta g(b)} \right. \\
 &\quad - \frac{\varepsilon^2 (b^2 (-b^2 g'(b)^3 + 2b^2 g(b)g'(b)g''(b) + 2bg(b)^2 g''(b) + 2bg(b)g'(b)^2 + 3g(b)^2 g'(b)))}{2\theta^2 g(b)^2} \\
 &\quad \left. + O(\varepsilon^3) \right\} \\
 &= I_0(\theta) + \frac{1}{\theta^2} \left\{ \frac{b\varepsilon (b^2 g'(b)^2 + 2bg(b)g'(b) + g(b)^2)}{\theta g(b)} \right. \\
 &\quad + \frac{\varepsilon^2 (b^2 (-b^2 g'(b)^3 + 2b^2 g(b)g'(b)g''(b) + 2bg(b)^2 g''(b) + 2bg(b)g'(b)^2 + 3g(b)^2 g'(b)))}{2\theta^2 g(b)^2} \\
 &\quad \left. + O(\varepsilon^3) \right\}
 \end{aligned}$$

となる。

次に“ $\varepsilon < 0$ かつ $a < 0 < b$ ”のときを考える．このとき， $S = (a(\theta + \varepsilon), b(\theta + \varepsilon))$ となるので

$$\begin{aligned}
 & I'_0(\theta) \\
 &= \int_S \left(\frac{\partial}{\partial \theta} \log f(x | \theta) \right)^2 f(x | \theta) dx = \int_{a(\theta+\varepsilon)}^{b(\theta+\varepsilon)} \left(-\frac{1}{\theta} - \frac{x}{\theta^2} \frac{g'(\frac{x}{\theta})}{g(\frac{x}{\theta})} \right)^2 \frac{1}{\theta} g\left(\frac{x}{\theta}\right) dx \\
 &= \int_{a(1+(\varepsilon/\theta))}^{b(1+(\varepsilon/\theta))} \left(-\frac{1}{\theta} - \frac{y}{\theta} \frac{g'(y)}{g(y)} \right)^2 g(y) dy \quad (y = x/\theta) \\
 &= \frac{1}{\theta^2} \int_a^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy - \frac{1}{\theta^2} \int_{b(1+(\varepsilon/\theta))}^b \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy \\
 &\quad - \frac{1}{\theta^2} \int_a^{a(1+(\varepsilon/\theta))} \left(1 + \frac{yg'(y)}{g(y)} \right)^2 g(y) dy \\
 &= I_0(\theta) - \frac{1}{\theta^2} \left\{ \varepsilon \left(\frac{a(a^2 g'(a)^2 + 2ag(a)g'(a) + g(a)^2)}{\theta g(a)} - \frac{b(b^2 g'(b)^2 + 2bg(b)g'(b) + g(b)^2)}{\theta g(b)} \right) \right. \\
 &\quad + \varepsilon^2 \left(\frac{a^2(-a^2 g'(a)^3 + 2a^2 g(a)g'(a)g''(a) + 2ag(a)^2 g''(a) + 2ag(a)g'(a)^2 + 3g(a)^2 g'(a))}{2\theta^2 g(a)^2} \right. \\
 &\quad \left. \left. - \frac{b^2(-b^2 g'(b)^3 + 2b^2 g(b)g'(b)g''(b) + 2bg(b)^2 g''(b) + 2bg(b)g'(b)^2 + 3g(b)^2 g'(b))}{2\theta^2 g(b)^2} \right) \right\} \\
 &\quad + O(\varepsilon^3) \Big\}
 \end{aligned}$$

となる．

4 Rényi divergence の漸近展開

$f(x | \theta)$ の台を $S(\theta) = \{x | f(x | \theta) > 0\}$ とする． $0 < t < 1$ とする．pdf $f(x | \theta)$ と $f(x | \theta + \varepsilon)$ に対して

$$H_t(\theta, \theta + \varepsilon) = \int_{S(\theta) \cap S(\theta + \varepsilon)} f^{1-t}(x | \theta) f^t(x | \theta + \varepsilon) dx$$

とおく． $\log f(x | \theta + \varepsilon) = \log \frac{1}{\theta + \varepsilon} g\left(\frac{x}{\theta + \varepsilon}\right) = -\log(\theta + \varepsilon) + \log g\left(\frac{x}{\theta + \varepsilon}\right)$ の θ の周りでの Taylor 展開より， $\varepsilon \rightarrow 0$ のとき

$$\begin{aligned}
 & \log f(x | \theta + \varepsilon) \\
 &= \log f(x | \theta) + \frac{\partial}{\partial \theta} \log f(x | \theta) \varepsilon + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \log f(x | \theta) \varepsilon^2 + O(\varepsilon^3) \\
 &= \log f(x | \theta) + \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} \varepsilon + \frac{1}{2} \left[\frac{\frac{\partial^2}{\partial \theta^2} f(x | \theta)}{f(x | \theta)} - \left\{ \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} \right\}^2 \right] \varepsilon^2 + O(\varepsilon^3) \\
 &= \log f(x | \theta) + \left\{ -\frac{1}{\theta} - \frac{xg'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\} \varepsilon + \frac{1}{2} \left[\frac{2}{\theta^2} + \frac{4xg'(\frac{x}{\theta})}{\theta^3 g(\frac{x}{\theta})} + \frac{x^2 g''(\frac{x}{\theta})}{\theta^4 g(\frac{x}{\theta})} - \left\{ -\frac{1}{\theta} - \frac{xg'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\}^2 \right] \varepsilon^2 + O(\varepsilon^3)
 \end{aligned}$$

となるので,

$$\begin{aligned}
 & H_t(\theta, \theta + \varepsilon) \\
 &= \int_S f^{1-t}(x | \theta) f^t(x | \theta + \varepsilon) dx \\
 &= \int_S \exp \{ (1-t) \log f(x | \theta) + t \log f(x | \theta + \varepsilon) \} dx \\
 &= \int_S f(x | \theta) \left[1 + \frac{\partial}{\partial \theta} \log f(x | \theta) t \varepsilon + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \log f(x | \theta) t \varepsilon^2 \right. \\
 &\quad \left. + \frac{1}{2} \left\{ \frac{\partial}{\partial \theta} \log f(x | \theta) \right\}^2 t^2 \varepsilon^2 \right] dx + O(\varepsilon^3) \\
 &= \int_S f(x | \theta) \left[1 + \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} t \varepsilon + \frac{1}{2} \left[\frac{\frac{\partial^2}{\partial \theta^2} f(x | \theta)}{f(x | \theta)} - \left\{ \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} \right\}^2 \right] t \varepsilon^2 \right. \\
 &\quad \left. + \frac{1}{2} \left\{ \frac{\frac{\partial}{\partial \theta} f(x | \theta)}{f(x | \theta)} \right\}^2 t^2 \varepsilon^2 \right] + O(\varepsilon^3) \\
 &= \int_S \frac{1}{\theta} g\left(\frac{x}{\theta}\right) \left[1 + \left\{ -\frac{1}{\theta} - \frac{x g'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\} t \varepsilon + \frac{1}{2} \left[\frac{2}{\theta^2} + \frac{4x g'(\frac{x}{\theta})}{\theta^3 g(\frac{x}{\theta})} + \frac{x^2 g''(\frac{x}{\theta})}{\theta^4 g(\frac{x}{\theta})} - \left\{ -\frac{1}{\theta} - \frac{x g'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\}^2 \right] t \varepsilon^2 \right. \\
 &\quad \left. + \frac{1}{2} \left\{ -\frac{1}{\theta} - \frac{x g'(\frac{x}{\theta})}{\theta^2 g(\frac{x}{\theta})} \right\}^2 t^2 \varepsilon^2 \right] dx + O(\varepsilon^3) \\
 &= \int_{S_0} g(y) \left[1 - \frac{1}{\theta} \left\{ 1 + \frac{y g'(y)}{g(y)} \right\} t \varepsilon + \frac{1}{2 \theta^2} \left[2 + \frac{4y g'(y)}{g(y)} + \frac{y^2 g''(y)}{g(y)} - \left\{ 1 + \frac{y g'(y)}{g(y)} \right\}^2 \right] t \varepsilon^2 \right. \\
 &\quad \left. + \frac{1}{2 \theta^2} \left\{ 1 + \frac{y g'(y)}{g(y)} \right\}^2 t^2 \varepsilon^2 \right] dy + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0)
 \end{aligned} \tag{4.1}$$

を得る. ただし, $S_0 = S(1) \cap S(1 + (\varepsilon/\theta))$ とする. (4.1) は ε の冪により次のように分解できる.

$$\begin{aligned}
 & H_t(\theta, \theta + \varepsilon) \\
 &= \int_{S_0} g(y) dy - \frac{1}{\theta} \left\{ \int_{S_0} g(y) dy + \int_{S_0} y g'(y) dy \right\} t \varepsilon \\
 &\quad + \frac{1}{2 \theta^2} \left[2 \int_{S_0} g(y) dy + 4 \int_{S_0} y g'(y) dy + \int_{S_0} y^2 g''(y) dy - I'_0(1) \right] t \varepsilon^2 + \frac{1}{2 \theta^2} I'_0(1) t^2 \varepsilon^2 + O(\varepsilon^3) \\
 &= \int_{S_0} g(y) dy - \frac{1}{\theta} \left\{ \int_{S_0} g(y) dy + \int_{S_0} y g'(y) dy \right\} t \varepsilon \\
 &\quad + \frac{1}{2 \theta^2} \left[2 \int_{S_0} g(y) dy + 4 \int_{S_0} y g'(y) dy + \int_{S_0} y^2 g''(y) dy - \theta^2 I'_0(\theta) \right] t \varepsilon^2 + \frac{1}{2} I'_0(\theta) t^2 \varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0).
 \end{aligned}$$

この式は部分積分によって

$$\begin{aligned}
 H_t(\theta, \theta + \varepsilon) &= \int_{S_0} g(y) dy - \frac{1}{\theta} [y g(y)]_{S_0} t \varepsilon \\
 &\quad + \frac{1}{2 \theta^2} \left(2[y g(y)]_{S_0} + [y^2 g'(y)]_{S_0} - \theta^2 I'_0(\theta) \right) t \varepsilon^2 + \frac{1}{2} I'_0(\theta) t^2 \varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \tag{4.2}
 \end{aligned}$$

と書き換えることができる。ただし、 $S_0 = [c, d]$ であるとき、 $[yg(y)]_{S_0} = [yg(y)]_c^d = dg(d) - cg(c)$,
 $[y^2g'(y)]_{S_0} = [y^2g'(y)]_c^d = d^2g'(d) - c^2g'(c)$ とする。

付記より、(4.2) は、“ $\varepsilon > 0$ かつ $0 < a < b$ ” または “ $\varepsilon < 0$ かつ $a < b < 0$ ” のとき

$$\begin{aligned} & H_t(\theta, \theta + \varepsilon) \\ &= 1 + \frac{\varepsilon \{atg(a) - g(a) - btg(b)\}}{\theta} \\ &\quad + \frac{\varepsilon^2 \{a^2tg'(a) - g'(a) + b^2tg'(b) + 2btg(b) + \theta^2 I'_0(\theta)t(t-1)\}}{2\theta^2} + O(\varepsilon^3) \\ &= 1 + \frac{\varepsilon \{atg(a) - g(a) - btg(b)\}}{\theta} \\ &\quad + \frac{\varepsilon^2 \{a^2tg'(a) - g'(a) + b^2tg'(b) + 2btg(b) + \theta^2 I_0(\theta)t(t-1)\}}{2\theta^2} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \end{aligned}$$

となる。同様に、“ $\varepsilon > 0$ かつ $a < 0 < b$ ” のとき

$$\begin{aligned} H_t(\theta, \theta + \varepsilon) &= 1 + \frac{\varepsilon}{\theta} \{atg(a) - btg(b)\} \\ &\quad + \frac{\varepsilon^2}{2\theta^2} \{2btg(b) - 2atg(a) + b^2tg'(b) - a^2tg'(a) + \theta^2 t(t-1)I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \end{aligned}$$

となる。次に “ $\varepsilon > 0$ かつ $a < b < 0$ ” または “ $\varepsilon < 0$ かつ $0 < a < b$ ” のとき

$$\begin{aligned} H_t(\theta, \theta + \varepsilon) &= 1 + \frac{\varepsilon}{\theta} \{atg(a) - btg(b) + g(b)\} \\ &\quad + \frac{\varepsilon^2}{2\theta^2} \{a^2(-t)g'(a) - 2atg(a) - b^2tg'(b) + g'(b) + \theta^2 t(t-1)I'_0(\theta)\} + O(\varepsilon^3) \\ &= 1 + \frac{\varepsilon}{\theta} \{atg(a) - btg(b) + g(b)\} \\ &\quad + \frac{\varepsilon^2}{2\theta^2} \{-a^2tg'(a) - 2atg(a) - b^2tg'(b) + g'(b) + \theta^2 t(t-1)I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \end{aligned}$$

となる。最後に “ $\varepsilon < 0$ かつ $a < 0 < b$ ” のときを考える。

$$\begin{aligned} H_t(\theta, \theta + \varepsilon) &= 1 + \frac{\varepsilon}{\theta} \{atg(a) - g(a) - btg(b) + g(b)\} \\ &\quad + \frac{\varepsilon^2}{2\theta^2} \{2a^2tg'(a) - g'(a) - 2b^2tg'(b) + g'(b) + \theta^2 t(t-1)I'_0(\theta)\} + O(\varepsilon^3) \\ &= 1 + \frac{\varepsilon}{\theta} \{atg(a) - g(a) - btg(b) + g(b)\} \\ &\quad + \frac{\varepsilon^2}{2\theta^2} \{2a^2tg'(a) - g'(a) - 2b^2tg'(b) + g'(b) + \theta^2 t(t-1)I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \end{aligned}$$

となる。

特に、上記の $H_t(\theta, \theta + \varepsilon)$ の漸近展開より、 $f(x | \theta)$ と $f(x | \theta + \varepsilon)$ の間の affinity $D_{1/2}(f(x | \theta) \| f(x | \theta + \varepsilon))$ の漸近展開を得ることもできる。

さらに、 $H_t(\theta, \theta + \varepsilon)$ と Rényi divergence $D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon))$ の間には

$$D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon)) = \frac{1}{s-1} \log H_{1-s}(\theta, \theta + \varepsilon)$$

の関係があるから, Rényi divergence の漸近展開は以下ようになる. “ $\varepsilon > 0$ かつ $0 < a < b$ ” または “ $\varepsilon < 0$ かつ $a < b < 0$ ” のとき

$$\begin{aligned} & D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon)) \\ &= \frac{\varepsilon}{\theta(1-s)} \{-a(1-s)g(a) + g(a) + b(1-s)g(b)\} \\ &\quad - \frac{\varepsilon^2}{2\theta^2(1-s)} \{(g(a) - a(1-s)g(a) + b(1-s)g(b))^2 \\ &\quad + g'(a) - a^2(1-s)g'(a) - b^2(1-s)g'(b) - 2btg(b) + \theta^2 s(s-1)I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0), \end{aligned}$$

“ $\varepsilon > 0$ かつ $a < 0 < b$ ” のとき

$$\begin{aligned} & D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon)) \\ &= \frac{\varepsilon}{\theta} \{bg(b) - ag(a)\} + \frac{\varepsilon^2(1-s)}{2\theta^2} \{2bg(b) - 2ag(a) + b^2g'(b) - a^2g'(a) - s\theta^2 I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0), \end{aligned}$$

“ $\varepsilon > 0$ かつ $a < b < 0$ ” または “ $\varepsilon < 0$ かつ $0 < a < b$ ” のとき

$$\begin{aligned} & D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon)) \\ &= \frac{\varepsilon}{\theta(1-s)} \{-a(1-s)g(a) + b(1-s)g(b) - g(b)\} \\ &\quad + \frac{\varepsilon^2}{2(1-s)\theta^2} \{(a(1-s)g(a) - b(1-s)g(b) + g(b))^2 \\ &\quad + a^2(1-s)g'(a) + 2a(1-s)g(a) + b^2(1-s)g'(b) - g'(b) - \theta^2 s(1-s)\theta^2 I_0(\theta)\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0), \end{aligned}$$

“ $\varepsilon < 0$ かつ $a < 0 < b$ ” のとき

$$\begin{aligned} & D_s(f(\cdot | \theta) \| f(\cdot | \theta + \varepsilon)) \\ &= \frac{\varepsilon}{\theta(1-s)} \{-a(1-s)g(a) + g(a) + b(1-s)g(b) - g(b)\} \\ &\quad + \frac{\varepsilon^2}{2t\theta^2} \{-2a^2(1-s)g'(a) + g'(a) + 2b^2(1-s)g'(b) - g'(b) - s(1-s)\theta^2 I_0(\theta) \\ &\quad + (a(1-s)g(a) - g(a) - b(1-s)g(b) + g(b))^2\} + O(\varepsilon^3) \quad (\varepsilon \rightarrow 0) \end{aligned}$$

となる.

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付記：(4.1) の各項の漸近展開

$g(x) = f(x | 1)$ の台が (a, b) のとき, $f(x | 1 + (\varepsilon/\theta))$ の台は $S(1 + (\varepsilon/\theta)) = (a(1 + (\varepsilon/\theta)), b(1 + (\varepsilon/\theta)))$ となる. $|\varepsilon|$ が十分小さくなるように ε をとると, $\varepsilon > 0$ のとき, a, b の正負によって, $S_0 = S(1) \cap S(1 + (\varepsilon/\theta))$ は, $\varepsilon > 0$ のとき

$$S_0 = \begin{cases} (a(1 + (\varepsilon/\theta)), b) & (0 < a < b), \\ (a, b) & (a < 0 < b), \\ (a, b(1 + (\varepsilon/\theta))) & (a < b < 0) \end{cases}$$

であり, $\varepsilon < 0$ のとき

$$S_0 = \begin{cases} (a, b(1 + (\varepsilon/\theta))) & (0 < a < b), \\ (a(1 + (\varepsilon/\theta)), b(1 + (\varepsilon/\theta))) & (a < 0 < b), \\ (a(1 + (\varepsilon/\theta)), b) & (a < b < 0) \end{cases}$$

である. ここで, 部分積分より

$$\begin{aligned} \int_{S_0} x g'(x) dx &= [x g(x)]_{S_0} - \int_{S_0} g(x) dx, \\ \int_{S_0} x^2 g''(x) dx &= [x^2 g'(x)]_{S_0} - 2 [x g(x)]_{S_0} + 2 \int_{S_0} g(x) dx \end{aligned}$$

である. (4.1) の各項は ε や S_0 の端点の正負に応じて場合分けされる. 例えば, $\varepsilon > 0$ の場合は, $\varepsilon \rightarrow +0$ のとき

$$\begin{aligned} & \int_{S_0} g(x) dx \\ &= \begin{cases} \int_{a(1+(\varepsilon/\theta))}^b g(x) dx = \int_a^b g(x) dx - \int_a^{a(1+(\varepsilon/\theta))} g(x) dx \\ \quad = 1 - g(a)a(\varepsilon/\theta) - \frac{g'(a)}{2} a^2 (\varepsilon/\theta)^2 + o(\varepsilon^2) & (0 < a < b), \\ \int_a^b g(x) dx = 1 & (a < 0 < b), \\ \int_a^{b(1+(\varepsilon/\theta))} g(x) dx = \int_a^b g(x) dx - \int_{b(1+(\varepsilon/\theta))}^b g(x) dx \\ \quad = 1 + g(b)b(\varepsilon/\theta) + \frac{g'(b)}{2} b^2 (\varepsilon/\theta)^2 + o(\varepsilon^2) & (a < b < 0) \end{cases} \end{aligned}$$

となる。また、 $\varepsilon < 0$ の場合は、 $\varepsilon \rightarrow -0$ のとき

$$\begin{aligned} & \int_{S_0} g(x) dx \\ &= \begin{cases} \int_a^{b(1+(\varepsilon/\theta))} g(x) dx = \int_a^b g(x) dx - \int_{b(1+(\varepsilon/\theta))}^b g(x) dx \\ \quad = 1 + g(b)b(\varepsilon/\theta) + \frac{g'(b)}{2} b^2 (\varepsilon/\theta)^2 + o(\varepsilon^2) & (0 < a < b), \\ \int_{a(1+(\varepsilon/\theta))}^{b(1+(\varepsilon/\theta))} g(x) dx = \int_a^b g(x) dx - \int_a^{a(1+(\varepsilon/\theta))} g(x) dx - \int_{b(1+(\varepsilon/\theta))}^b g(x) dx \\ \quad = 1 - g(a)a(\varepsilon/\theta) + g(b)b(\varepsilon/\theta) - \frac{g'(a)}{2} a^2 (\varepsilon/\theta)^2 + \frac{g'(b)}{2} b^2 (\varepsilon/\theta)^2 + o(\varepsilon^2) & (a < 0 < b), \\ \int_{a(1+(\varepsilon/\theta))}^b g(x) dx = \int_a^b g(x) dx - \int_a^{a(1+(\varepsilon/\theta))} g(x) dx \\ \quad = 1 - g(a)a(\varepsilon/\theta) - \frac{g'(a)}{2} a^2 (\varepsilon/\theta)^2 + o(\varepsilon^2) & (a < b < 0) \end{cases} \end{aligned}$$

となる。また、 $\varepsilon \rightarrow +0$ のとき

$$\begin{aligned} & [xg(x)]_{S_0} \\ &= \begin{cases} bg(b) - a(1+(\varepsilon/\theta))g(a(1+(\varepsilon/\theta))) \\ \quad = (bg(b) - ag(a)) - a(\varepsilon/\theta)(ag'(a) + g(a)) + a(\varepsilon/\theta)^2 \left(-\frac{1}{2}a^2g''(a) - ag'(a) \right) + O(\varepsilon^3) & (0 < a < b), \\ bg(b) - ag(a) & (a < 0 < b), \\ b(1+(\varepsilon/\theta))g(b(1+\varepsilon)) - ag(a) \\ \quad = (bg(b) - ag(a)) + b(\varepsilon/\theta)(bg'(b) + g(b)) + b(\varepsilon/\theta)^2 \left(\frac{1}{2}b^2g''(b) + bg'(b) \right) + O(\varepsilon^3) & (a < b < 0), \end{cases} \\ & [x^2g'(x)]_{S_0} \\ &= \begin{cases} b^2g'(b) - a^2(1+(\varepsilon/\theta))^2g'(a(1+(\varepsilon/\theta))) \\ \quad = (b^2g'(b) - a^2g'(a)) + a^2(\varepsilon/\theta)(-ag''(a) - 2g'(a)) \\ \quad + a^2(\varepsilon/\theta)^2 \left(-\frac{1}{2}a^2g^{(3)}(a) - 2ag''(a) - g'(a) \right) + O(\varepsilon^3) & (0 < a < b), \\ b^2g'(b) - a^2g'(a) & (a < 0 < b), \\ b^2(1+(\varepsilon/\theta))^2g'(b(1+(\varepsilon/\theta))) - a^2g'(a) \\ \quad = (b^2g'(b) - a^2g'(a)) + b^2\varepsilon(bg''(b) + 2g'(b)) \\ \quad + b^2(\varepsilon/\theta)^2 \left(\frac{1}{2}b^2g^{(3)}(b) + 2bg''(b) + g'(b) \right) + O(\varepsilon^3) & (a < b < 0) \end{cases} \end{aligned}$$

となる. $\varepsilon \rightarrow -0$ のとき

$$\begin{aligned}
 & [xg(x)]_{S_0} \\
 & \left\{ \begin{aligned} & b(1+\varepsilon)g(b(1+(\varepsilon/\theta))) - ag(a) \\ & = (bg(b) - ag(a)) + b(\varepsilon/\theta)(bg'(b) + g(b)) + b(\varepsilon/\theta)^2 \left(\frac{1}{2}b^2g''(b) + bg'(b) \right) + O(\varepsilon^3) \quad (0 < a < b), \\ & b(1+(\varepsilon/\theta))g(b(1+(\varepsilon/\theta))) - a(1+(\varepsilon/\theta))g(a(1+(\varepsilon/\theta))) \\ & = (bg(b) - ag(a)) + (\varepsilon/\theta)(b(bg'(b) + g(b)) - a(ag'(a) + g(a))) \\ & \quad + \frac{1}{2}(\varepsilon/\theta)^2 (a^3(-g''(a)) - 2a^2g'(a) + b^3g''(b) + 2b^2g'(b)) + O(\varepsilon^3) \quad (a < 0 < b), \\ & bg(b) - a(1+(\varepsilon/\theta))g(a(1+(\varepsilon/\theta))) \\ & = (bg(b) - ag(a)) - a(\varepsilon/\theta)(ag'(a) + g(a)) + a(\varepsilon/\theta)^2 \left(-\frac{1}{2}a^2g''(a) - ag'(a) \right) + O(\varepsilon^3) \quad (a < b < 0), \end{aligned} \right. \\
 & [x^2g'(x)]_{S_0} \\
 & \left\{ \begin{aligned} & b^2(1+(\varepsilon/\theta))^2g'(b(1+(\varepsilon/\theta))) - a^2g'(a) \\ & = (b^2g'(b) - a^2g'(a)) + b^2(\varepsilon/\theta)(bg''(b) + 2g'(b)) \\ & \quad + b^2(\varepsilon/\theta)^2 \left(\frac{1}{2}b^2g^{(3)}(b) + 2bg''(b) + g'(b) \right) + O(\varepsilon^3) \quad (0 < a < b), \\ & b^2(1+(\varepsilon/\theta))^2g'(b(1+(\varepsilon/\theta))) - a^2(1+(\varepsilon/\theta))^2g'(a(1+(\varepsilon/\theta))) \\ & = (b^2g'(b) - a^2g'(a)) + (\varepsilon/\theta)(a^2(-ag''(a) - 2g'(a)) + b^2(bg''(b) + 2g'(b))) \\ & \quad + (\varepsilon/\theta)^2 \left(a^2 \left(-\frac{1}{2}a^2g^{(3)}(a) - 2ag''(a) - g'(a) \right) \right. \\ & \quad \left. + b^2 \left(\frac{1}{2}b^2g^{(3)}(b) + 2bg''(b) + g'(b) \right) \right) + O(\varepsilon^3) \quad (a < 0 < b), \\ & b^2g'(b) - a^2(1+(\varepsilon/\theta))^2g'(a(1+(\varepsilon/\theta))) \\ & = (b^2g'(b) - a^2g'(a)) + a^2(\varepsilon/\theta)(-ag''(a) - 2g'(a)) \\ & \quad + a^2(\varepsilon/\theta)^2 \left(-\frac{1}{2}a^2g^{(3)}(a) - 2ag''(a) - g'(a) \right) + O(\varepsilon^3) \quad (a < b < 0) \end{aligned} \right.
 \end{aligned}$$

となる.