

多人数協力型停止問題とパレート最適

Pareto optima in multi-person
cooperative stopping problem

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Abstract. We consider multi-person cooperative stopping problem of Dynkin's type. We are interested in Pareto optimal stopping times. By the method of scalarization we find ε -Pareto optimal stopping times for each player.

1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n \in N}$ an increasing family of sub- σ -fields of $\mathcal{F}$, where $N = \{0, 1, 2, \dots\}$ is a discrete time space.

For each $i, k = 1, 2, \dots, p$, let $(Y_k^i(n) : n \in N)$ be a random sequence defined on $(\Omega, \mathcal{F}, P)$ such that $Y_k^i(n)$ is $\mathcal{F}_n$ -measurable and $\sup_{n \in N} (Y_k^i(n))^+$ and $(Y_k^i(n))^-$ are integrable, where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. Y_k^i means a reward for i th player when k th player stops.

For each $n \in N$, we denote by Λ_n the class of $\tau = (\tau_1, \tau_2, \dots, \tau_p)$ such that each τ_i ($i = 1, 2, \dots, p$) is an $(\mathcal{F}_n)$ -stopping time and $n \leq \min_i \tau_i < \infty$ almost surely.

Now we consider game-theoretically the following cooperative stopping problem. There are p players and each player i ($i = 1, 2, \dots, p$) chooses stopping time τ_i such that $\tau = (\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_0$. We define measurable sets $B(\tau_k)$ by

$$B(\tau, \tau_1) = \{\tau_1 = \min_i \tau_i\},$$

$$B(\tau, \tau_k) = \{\tau_k = \min_i \tau_i\} - \bigcup_{i=1}^{k-1} B(\tau, \tau_i) = \{\tau_k = \min_i \tau_i < \min_{j \leq k-1} \tau_j\}, \quad 2 \leq k \leq p.$$

Then the i th player ($i = 1, 2, \dots, p$) gets the reward

$$X_i(\tau) = \sum_{k=1}^p Y_k^i(\tau_k) I_{B(\tau, \tau_k)}.$$

When $p = 2$, we have

$$X_1(\tau_1, \tau_2) = Y_1^1(\tau_1) I_{(\tau_1 \leq \tau_2)} + Y_2^1(\tau_2) I_{(\tau_2 < \tau_1)},$$

and

$$X_2(\tau_1, \tau_2) = Y_1^2(\tau_1) I_{(\tau_1 \leq \tau_2)} + Y_2^2(\tau_2) I_{(\tau_2 < \tau_1)},$$

which is well known two-person Dynkin's problem, and when $p = 3$, we have

$$X_1(\tau_1, \tau_2, \tau_3) = Y_1^1(\tau_1) I_{(\tau_1 \leq \tau_2, \tau_3)} + Y_2^1(\tau_2) I_{(\tau_2 < \tau_1, \tau_2 \leq \tau_3)} + Y_3^1(\tau_3) I_{(\tau_3 < \tau_1, \tau_2)},$$

and so on. As special cases we can find the following: the first is a case that (Y_k^i) does not depend upon player i , that is, $Y_k^i = Y_k$ (say) for every $i = 1, 2, \dots, p$. Then we have

$$X_i(\tau) = \sum_{k=1}^p Y_k(\tau_k) I_{B(\tau, \tau_k)} \quad (= X(\tau), \text{ say}),$$

that is, every player gets the same reward, and hence this problem is reduced to classical optimal stopping except for finding optimal stopping $(\tau_1, \tau_2, \dots, \tau_p)$ as going into details in section 2. The second is one that (Y_k^i) is independent to whether which player stops, that is, $Y_k^i = Y^i$ (say) for every $k = 1, 2, \dots, p$. Then we have

$$X_i(\tau) = \sum_{k=1}^p Y^i(\tau_k) I_{B(\tau, \tau_k)} = Y^i(\min_k \tau_k).$$

This is a multi-objective stopping problem, which has been investigated in Ohtsubo[1997].

The aim of the i th player is to maximize the expected gain $E[X_i(\tau_1, \tau_2, \dots, \tau_p)]$ with respect to τ_i , cooperating with other players if necessary. However, the stopping time chosen by one of them generally depends upon one decided by other, even if they cooperate. Thus we will use the concept of Pareto optimality as in the usual cooperative game of the game theory or the multi-objective problem of mathematical programming.

We define a conditional expected reward by

$$G_n^i(\tau_1, \tau_2, \dots, \tau_p) = E[X_i(\tau_1, \tau_2, \dots, \tau_p) \mid \mathcal{F}_n]$$

for each player i ($i = 1, 2, \dots, p$).

For $n \in N$ and $\varepsilon \geq 0$, we say that $(\tau_1^\varepsilon, \tau_2^\varepsilon, \dots, \tau_p^\varepsilon)$ in Λ_n is ε -Pareto optimal at n if there is no $(\tau_1, \tau_2, \dots, \tau_p)$ in Λ_n such that

$$G_n^i(\tau_1, \tau_2, \dots, \tau_p) > G_n^i(\tau_1^\varepsilon, \tau_2^\varepsilon, \dots, \tau_p^\varepsilon) + \varepsilon.$$

For the sake of simplicity, without further comments we assume that all inequalities and equalities between random variables hold in the sense of “almost surely”.

2. Special models.

In this section, we consider the first special case given in the introduction and we give fundamental results for properties of shadow (virtual) optimum, which is useful in the next section. We first define shadow optimum α^i for the reward $X_i(\tau_1, \tau_2, \dots, \tau_p)$ as follows:

$$\alpha_n^i = \text{ess sup}_{(\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_n} G_n^i(\tau_1, \tau_2, \dots, \tau_p), \quad n \in N; \quad i = 1, 2, \dots, p.$$

In multi-objective programming, the shadow optima are also called “ideal or utopia point”.

Now, to obtain constructive property of the shadow optima, we generally consider an optimal stopping problem so as to maximize the expected reward

$$G_n(\tau_1, \tau_2, \dots, \tau_p) = E[X(\tau_1, \tau_2, \dots, \tau_p) \mid \mathcal{F}_n]$$

with respect to $(\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_n$, where

$$X(\tau_1, \dots, \tau_p) = \sum_{k=1}^p Y_k(\tau_k) I_{B(\tau, \tau_k)}$$

and (Y_k) satisfies the same conditions as (Y_k^i) . We notice that this is the first special case in section 1. The optimal value process $\beta = (\beta_n)_{n \in N}$ is defined by

$$\beta_n = \operatorname{ess\,sup}_{(\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_n} G_n(\tau_1, \tau_2, \dots, \tau_p), \quad n \in N.$$

For $n \in N$ and $\varepsilon \geq 0$, we say that a pair $(\tau_1^\varepsilon, \tau_2^\varepsilon, \dots, \tau_p^\varepsilon)$ in Λ_n is (ε, β) -optimal at n if $\beta_n \leq G_n(\tau_1^\varepsilon, \tau_2^\varepsilon, \dots, \tau_p^\varepsilon) + \varepsilon$.

Define other process $(\tilde{X}_n)$ by $\tilde{X}_n = \max_k Y_k(n)$.

LEMMA 2.1.

(i) The process $\beta = (\beta_n)$ satisfies the recursive relation:

$$\beta_n = \max(\tilde{X}_n, E[\beta_{n+1} \mid \mathcal{F}_n]), \quad n \in N.$$

(ii) β is the smallest supermartingale dominating the process $(\tilde{X}_n)$.

(iii) $\limsup_n \beta_n = \limsup_n \tilde{X}_n$.

PROOF. The lemma is easily proved as in the classical optimal stopping problem (cf. Chow, Robbins and Siegmund [2] or Neveu [8]). $\square$

From this lemma it is easy to see that the process β coincides with an optimal value process $\hat{\beta} = (\hat{\beta}_n)$ in an optimal stopping problem with a reward $\tilde{X}_n$ of time n , i. e.

$$\hat{\beta}_n = \operatorname{ess\,sup}_{n \leq \tau < \infty} E[\tilde{X}_\tau \mid \mathcal{F}_n].$$

Hence $\beta = \hat{\beta}$ is constructive by the method of the backward induction as in Chow and et. al. [2].

For each $n \in N$ and $\varepsilon \geq 0$, define stopping times $\tau_i^\varepsilon(n) \equiv \tau_i^\varepsilon(n, \beta)$ ($i = 1, 2, \dots, p$) by

$$\tau_i^\varepsilon(n) = \inf\{k \geq n \mid \beta_k \leq Y_i(k) + \varepsilon, \tilde{X}_k = Y_i(k)\},$$

where $\inf(\emptyset) = +\infty$.

THEOREM 2.1. *Let $n \in N$ be arbitrary.*

- (i) *For each $\varepsilon > 0$, $\tau^\varepsilon(n) = (\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is (ε, β) -optimal at n .*
- (ii) *The stopping time $\min_i \tau_i^0(n)$ is a.s. finite, $(\tau_1^0(n), \tau_2^0(n), \dots, \tau_p^0(n))$ is $(0, \beta)$ -optimal at n .*

PROOF. When ε is positive, it follows from Lemma 2.1 (iii) that the stopping time $\min_i \tau_i^\varepsilon(n)$ is a.s. finite. Thus, for $\varepsilon \geq 0$, it suffices to show that inequality $\beta_n \leq G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n)) + \varepsilon$ holds for each $n \in N$. From Lemma 2.1 (i) and the optional sampling theorem, we have

$$\beta_n = E[\beta_{\tau_1^\varepsilon(n) \wedge \tau_2^\varepsilon(n) \wedge \dots \wedge \tau_p^\varepsilon(n)} \mid \mathcal{F}_n] = E[\sum_{k=1}^p \beta_{\tau_k^\varepsilon(n)} I_{B(\tau^\varepsilon(n), \tau_k^\varepsilon(n))} \mid \mathcal{F}_n].$$

Since $\beta_m \leq Y_k(m) + \varepsilon$ on $\{\tau_k^\varepsilon(n) = m\}$, so on $B(\tau^\varepsilon(n), \tau_k^\varepsilon(n))$, we have inequality

$$\beta_n \leq E[\sum_{k=1}^p Y_k(\tau_k^\varepsilon(n)) I_{B(\tau^\varepsilon(n), \tau_k^\varepsilon(n))} \mid \mathcal{F}_n] + \varepsilon \leq G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n)) + \varepsilon. \quad \square$$

3. Scalarization and Pareto optima.

In this section we find Pareto optimal times by the method of the well-known scalarization.

Let S denote the set of vectors $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ in $\mathbf{R}^p$ satisfying $\lambda \geq 0$ and $\sum_i \lambda_i = 1$. For given $\tau = (\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ in S , we define sequences of random variables by

$$X(\tau; \lambda) = \sum_{i=1}^p \lambda_i X_i(\tau) = \sum_{i=1}^p \lambda_i \sum_{k=1}^p Y_k^i(\tau_k) I_{B(\tau, \tau_k)} = \sum_{k=1}^p X_k(\tau_k; \lambda) I_{B(\tau, \tau_k)},$$

where

$$X_k(n_k; \lambda) = \sum_{i=1}^p \lambda_i Y_k^i(n_k), \quad n_k \in N, k = 1, 2, \dots, p,$$

and let

$$G_n(\tau; \lambda) = \sum_{i=1}^p \lambda_i G_n^i(\tau) = E[X(\tau; \lambda) \mid \mathcal{F}_n].$$

Then a maximum value process is defined by

$$V_n(\lambda) = \operatorname{ess\,sup}_{(\tau_1, \tau_2, \dots, \tau_p) \in \Lambda_n} G_n(\tau_1, \tau_2, \dots, \tau_p; \lambda), \quad n \in N.$$

We also define stopping times for the process $V(\lambda) = (V_n(\lambda))$ as follows:

$$\tau_i^\varepsilon(n) = \inf\{k \geq n \mid V_k(\lambda) \leq X_i(k; \lambda) + \varepsilon, \tilde{X}_k(\lambda) = X_i(k; \lambda)\}$$

for $n \in N$ and $\varepsilon \geq 0$, where $\tilde{X}_n(\lambda) = \max_k X_k(n; \lambda)$. The following theorems are immediate results of Lemmas 2.1 and Theorem 2.1.

THEOREM 3.1. *Let λ in S be arbitrary.*

(i) *The process $V(\lambda) = (V_n(\lambda))$ satisfies the recursive relation:*

$$V_n(\lambda) = \max(\tilde{X}_n(\lambda), E[V_{n+1}(\lambda) | \mathcal{F}_n]), \quad n \in N.$$

(ii) *$V(\lambda)$ is the smallest supermartingale dominating $(\tilde{X}_n(\lambda))$.*

(iii) $\limsup_n V_n(\lambda) = \limsup_n \tilde{X}_n(\lambda)$.

THEOREM 3.2. *Let $n \in N$ and $\lambda \in S$ be arbitrary.*

(i) *For each $\varepsilon > 0$, $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is $(\varepsilon, V(\lambda))$ -optimal at n .*

(ii) *The stopping time $\min_i \tau_i^0(n)$ is a. s. finite, $(\tau_1^0(n), \tau_2^0(n), \dots, \tau_p^0(n))$ is $(0, V(\lambda))$ -optimal at n .*

The general lemma below is a well-known result in multi-objective problem.

LEMMA 3.1. *Let $n \in N$, $\varepsilon \geq 0$ and $\lambda \in S$ be arbitrary. If $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ in Λ_n satisfies inequality $V_n(\lambda) \leq G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n); \lambda) + \varepsilon$, then $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is ε -Pareto optimal at n .*

PROOF. We suppose that the pair $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is not ε -Pareto optimal. There then exists $(\tau_1, \tau_2, \dots, \tau_p)$ in Λ_n such that $G_n^i(\tau_1, \tau_2, \dots, \tau_p) > G_n^i(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n)) + \varepsilon$ for every $i = 1, 2, \dots, p$. Thus we have

$$\begin{aligned} G_n(\tau_1, \tau_2, \dots, \tau_p; \lambda) &= \sum_{i=1}^p \lambda_i G_n^i(\tau_1, \tau_2, \dots, \tau_p) \\ &> \sum_{i=1}^p \lambda_i G_n^i(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n)) + \varepsilon \\ &= G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n); \lambda) + \varepsilon, \end{aligned}$$

so that $V_n(\lambda) > G_n(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n); \lambda) + \varepsilon$, which is a contradiction. Hence $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is ε -Pareto optimal. $\square$

Theorem 3.2 and Lemma 3.1 immediately imply the following theorem.

THEOREM 3.3. *Let $n \in N$ and $\lambda \in S$ be arbitrary.*

- (i) *For each $\varepsilon > 0$, $(\tau_1^\varepsilon(n), \tau_2^\varepsilon(n), \dots, \tau_p^\varepsilon(n))$ is ε -Pareto optimal at n .*
- (ii) *If the stopping time $\min_i \tau_i^0(n)$ is a.s. finite, $(\tau_1^0(n), \tau_2^0(n), \dots, \tau_p^0(n))$ is 0-Pareto optimal at n .*

4. Monotone Case and Applications

For the scalarized reward process $(\tilde{X}_n(\lambda))$ defined in Section 2 where $\lambda \in S$, we define subsets of Ω

$$A_n(\lambda) = \{\tilde{X}_n(\lambda) \geq E[\tilde{X}_{n+1}(\lambda)|\mathcal{F}_n]\}, \quad n \in N$$

and define a stopping time

$$\sigma_n^i(\lambda) = \inf\{k \geq n | \tilde{X}_k(\lambda) \geq E[\tilde{X}_{k+1}(\lambda)|\mathcal{F}_k], \tilde{X}_k(\lambda) = X_i(k; \lambda)\}, \quad n \in N,$$

that is,

$$\sigma_n^i(\lambda)(\omega) = \inf\{k \geq n | \omega \in A_n(\lambda), \tilde{X}_k(\lambda) = X_i(k; \lambda)\}, \quad \omega \in \Omega, n \in N$$

where $\inf \phi = +\infty$. $\sigma_n^i(\lambda)$ is called one-step-look-ahead (OLA) or myopic rule.

For each λ in S we introduce the following condition :

CONDITION $M(\lambda)$. For every $n \in N$, $A_n(\lambda) \subset A_{n+1}(\lambda)$ and $\lim_{n \rightarrow \infty} P(A_n(\lambda)) = 1$.

When the condition $M(\lambda)$ is satisfied for a given $\lambda \in S$, the scalarized stopping problem corresponding λ is in a well known monotone case.

THEOREM 4.1. *Suppose that Condition $M(\lambda)$ is satisfied for a given λ in S . Then for each $n \in N$ $\sigma_n^i(\lambda)$ is a. s. equal to $\tau_i^0(n)$ and $\min_i \sigma_n^i(\lambda)$ is a.s. finite, and hence $(\sigma_n^1(\lambda), \sigma_n^2(\lambda), \dots, \sigma_n^p(\lambda))$ is 0-Pareto optimal at n .*

PROOF The first and second part : $\sigma_n^i(\lambda) = \tau_i^0(n)$ and $\min_i \sigma_n^i(\lambda) < \infty$ a.s. are proved similarly to Chow et al. [2]. Hence Theorem 3.3 implies that $(\sigma_n^1(\lambda), \sigma_n^2(\lambda), \dots, \sigma_n^p(\lambda))$ is 0-Pareto optimal at n . $\square$

Next we consider applications for monotone case. First in the special model discussed in section 2, where $Y_k^i(n) = Y_k(n)$, $n \in N$, $1, k = 1, 2, \dots, p$, let

$$Y_k(n) = \max_{0 \leq m \leq n} W_m^k - c_n, \quad n \in N,$$

where $(W_n^k)_{n=0}^\infty$ be a sequence of independent and identically distributed random variables with finite mean for each k , and $(c_n)_{n=0}^\infty$ is any strictly increasing sequence of positive constants. Then we have

$$\tilde{X}_{n+1} - \tilde{X}_n = \max_k Y_k(n+1) - \max_k Y_k(n) = (\max_k W_{n+1}^k - m_n)^+ - b_n,$$

where

$$m_n = \max_{k, 0 \leq m \leq n} W_m^k,$$

$$b_n = c_{n+1} - c_n.$$

By the way analogous as in Chow et al. [2, p.56], it follows that if $b_{n+1} \geq b_n$ for all $n \in N$, that is, (c_n) is convex with regard to n , then $A_n \subset A_{n+1}$ for any $n \in N$ and

$$\lim_{n \rightarrow \infty} P(A_n) = P(\sigma < \infty) = 1,$$

where

$$A_n = \{\tilde{X}_n \geq E[\tilde{X}_{n+1} | \mathcal{F}_n]\},$$

$$\sigma = \inf\{n \geq 0 | \tilde{X}_n \geq E[\tilde{X}_{n+1} | \mathcal{F}_n]\} = \inf\{n \geq 0 | m_n \geq \gamma_n\}$$

and γ_n is the unique solution of the equation

$$E[(\max_k W_n^k - \gamma_n)^+] = b_n, \quad n \in N.$$

Hence condition $M(\lambda)$ is satisfied, since $\tilde{X}_n = \tilde{X}_n(\lambda)$ for all $\lambda \in S$. We define stopping times by

$$\begin{aligned} \sigma_n^i &= \inf\{k \geq n | \tilde{X}_k \geq E[\tilde{X}_{k+1} | \mathcal{F}_k], \tilde{X}_k = Y_i(k)\} \\ &= \inf\{k \geq n | m_k \geq \gamma_k, \tilde{X}_k = Y_i(k)\}. \end{aligned}$$

Then from Theorem 4.1 an OLA rule $(\sigma_n^1, \sigma_n^2, \dots, \sigma_n^p)$ is 0-Pareto optimal at n .

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