

Donsker のデルタ関数と SDE の離散化による熱核の近似

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Kloeden と Platen の本 [K1-P192] に SDE の解の strong Ito-Taylor scheme による近似が論じられている。ここでは主に L^2 -近似が考察されているが、本報告における主要な結果は γ が任意の $Kp < \infty$ に対し L^p -近似の意味で、さらに SDE の係数が十分滑らかなときは Malliavin 解析における $\mathbb{D}_\alpha^p$ -近似の意味でもなりたつことを示した点である。このことを用いて Malliavin 解析の方法を適用すれば、固定された時刻 $t > 0$ において解が Malliavin の意味で非退化のとき、対応する熱核（すなわち解の分布密度）に対する approximation scheme を具体的に与えることが出来る。この近似の精度は order γ の Ito-Taylor approximation に対しては $O(|\pi|^\gamma)$ ($|\pi|$ は時間分割の step) となり L^2 -近似の精度と変わらない。

以上が大体 研究集会で報告した内容の要旨であるが、

これは 中国, 武漢科学院の Yaozhong Hu 氏との共同研究であり, 共同論文

Yaozhong Hu and Shinzo Watanabe : Donsker's delta functions and approximation of heat kernels by the time discretization method

として 発表予定であるので, ここではこれ以上詳しく述べない。たゞこの論文中の1つの Lemma は [K1-P192] において L^2 場合に論じられている Lemma 5.7.3 および Lemma 10.8.1 を自然に拡張 及至 精密化したもので L^p -近似を証明する際の key となる。この Lemma はそれ自身, Kloeden-Platen の本を読まれる際の参考として 興味あるものと思うので 以下に この Lemma の部分だけ 詳しく 論じることにした。

Let $W = W_0^r$ be the classical r -dimensional Wiener space:

$$W_0^r = \{w \in C([0, \infty) \rightarrow \mathbf{R}^r) | w(0) = 0\}$$

and P be the standard Wiener measure on W_0^r . Then $w(t) = (w^1(t), \dots, w^r(t))$ for $w \in W$ is a realization of r -dimensional Brownian motion on W . Also we write w_i^j for $w^j(t)$.

For a given finite $T > 0$ and $l = 1, 2, \dots$, let

$$\Delta = \{(s_1, s_2, \dots, s_l) \in \mathbf{R}^l \mid 0 \leq s_1 < s_2 < \dots < s_l \leq T\}$$

and $L^2(\Delta)$ be the usual L^2 -space of real square-integrable functions on Δ . Let $f(s_1, \dots, s_l)$ be an $L^2(\Delta)$ -valued Wiener functional on W such that $f(s_1, \dots, s_l)$ is $\mathcal{B}_{s_1}$ -measurable for each fixed $s_1 < \dots < s_l$. Define for a multi-index $\alpha = (\alpha_1, \dots, \alpha_l)$ with $\alpha_i \in \{0, 1, \dots, r\}$ and $0 \leq u \leq v \leq T$,

$$I_\alpha(f)_{u,v} = \int_{u < s_1 < \dots < s_l < v} f(s_1, \dots, s_l) dw_{s_1}^{\alpha_1} \dots dw_{s_l}^{\alpha_l}$$

by iterated Itô's stochastic integrals. Set, for $0 \leq u < s \leq T$,

$$\begin{aligned} \|f\|_u(s) &= |f(s)| & \text{if } l = 1 \\ &= \sup_{u < s_1 < \dots < s_{l-1} < s} |f(s_1, \dots, s_{l-1}, s)| & \text{if } l > 1. \end{aligned}$$

Also we introduce the following notations for multi-indices; for a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ with $\alpha_i \in \{0, 1, \dots, r\}$, we write $l = l(\alpha)$ and set $n(\alpha) = \#\{k; \alpha_k = 0\}$. If $l(\alpha) \geq 2$, we set $-\alpha = (\alpha_2, \dots, \alpha_l)$ and $\alpha- = (\alpha_1, \dots, \alpha_{l-1})$.

Lemma 0.1. (1) For $p \geq 1$ and $0 \leq u < v \leq T$,

$$(0.1) \quad E \left[\sup_{u \leq t \leq v} |I_\alpha(f)_{u,t}|^{2p} \right] \leq C(u-v)^{p[l(\alpha)+n(\alpha)]-1} \int_u^v E[\|f\|_u(t)^{2p}] dt$$

(2) Let $\pi : 0 = t_0 < t_1 < \dots < t_n = T$ be a partition of $[0, T]$. Set $|\pi| = \sup_i (t_{i+1} - t_i)$ and $m(s) = m$ if $t_m \leq s < t_{m+1}$. Consider the following expectation for each $0 \leq t \leq T$:

$$F_t^\alpha = E \left(\sup_{0 \leq s \leq t} \left| \sum_{m=0}^{m(s)-1} I_\alpha(f)_{t_m, t_{m+1}} + I_\alpha(f)_{t_{m(s)}, s} \right|^{2p} \right).$$

Then, for $p \geq 1$ and $0 \leq t \leq T$,

$$\begin{aligned} (0.2) \quad F_t^\alpha &\leq C|\pi|^{2p[l(\alpha)-1]} \int_0^t E(\|f\|_{t_{m(s)}}(s)^{2p}) ds & \text{if } l(\alpha) = n(\alpha), \\ &\leq C|\pi|^{p[l(\alpha)+n(\alpha)]-1} \int_0^t E(\|f\|_{t_{m(s)}}(s)^{2p}) ds & \text{if } l(\alpha) \neq n(\alpha). \end{aligned}$$

Here, C are positive constants depending on T, p and α which may vary from lines to lines.

Proof: (1) can be proved by induction on the length $l(\alpha)$ of α : If $l(\alpha) > 1$,

$$I_\alpha(f)_{u,t} = \int_u^t I_{\alpha-}(f^s)_{u,s} dw_s^{\alpha_l}$$

where $f^s(s_1, \dots, s_{l-1}) = f(s_1, \dots, s_{l-1}, s)$. If $\alpha_l = 0$, then

$$\begin{aligned} E \left[\sup_{u \leq t \leq v} |I_\alpha(f)_{u,t}|^{2p} \right] &\leq E \left\{ \left[\int_u^v |I_{\alpha-}(f^s)_{u,s}| ds \right]^{2p} \right\} \\ &\leq (v-u)^{2p-1} \int_u^v E[|I_{\alpha-}(f^s)_{u,s}|^{2p}] ds \end{aligned}$$

by the Hölder inequality. If $\alpha_l \neq 0$, then

$$\begin{aligned} E \left[\sup_{u \leq t \leq v} |I_\alpha(f)_{u,t}|^{2p} \right] &\leq C \cdot E \left\{ \left[\int_u^v |I_{\alpha-}(f^s)_{u,s}|^2 ds \right]^p \right\} \\ &\leq C(v-u)^{p-1} \int_u^v E[|I_{\alpha-}(f^s)_{u,s}|^{2p}] ds \end{aligned}$$

by a standard martingale inequality of the Burkholder-Davis-Gundy type for stochastic integrals (cf. [IW89], p.110) and the Hölder inequality. The inequality (0.1) for the case $l(\alpha) = 1$ can be obtained by the same estimates. Then we can conclude the proof by induction if we note the following facts: $l(\alpha-) = l(\alpha) - 1$, $n(\alpha-) = n(\alpha)$ or $n(\alpha) - 1$ according as $\alpha_l \neq 0$ or $\alpha_l = 0$ and $\|f^s\|_u(t) \leq \|f\|_u(s)$ if $t \leq s$.

Next, we prove (2). We note that

$$\Xi(t) := \sum_{m=0}^{m(t)-1} I_\alpha(f)_{t_m, t_{m+1}} + I_\alpha(f)_{t_{m(t)}, t} = \int_0^t I_{\alpha-}(f^s)_{t_{m(s)}, s} dw_s^{\alpha_l}.$$

If $n(\alpha) = l(\alpha)$, then

$$\begin{aligned} E\left[\sup_{0 \leq s \leq t} |\Xi(s)|^{2p}\right] &\leq E\left\{\left[\int_0^t |I_{\alpha-}(f^s)_{t_{m(s)}, s}| ds\right]^{2p}\right\} \\ &\leq C \cdot \int_0^t E\left\{|I_{\alpha-}(f^s)_{t_{m(s)}, s}|^{2p}\right\} ds \\ &\leq C \cdot \left\{\sum_{m=0}^{m(t)} \int_{t_m}^{t_{m+1} \wedge t} E\left\{|I_{\alpha-}(f^s)_{t_m, s}|^{2p}\right\} ds\right\}. \end{aligned}$$

By the estimate (0.1), this is dominated by

$$C|\pi|^{2pl(\alpha-)-1} \cdot \sum_{m=0}^{m(t)} \int_{t_m}^{t_{m+1} \wedge t} ds \int_{t_m}^s E\left[\|f^s\|_{t_m}(\tau)^{2p}\right] d\tau \leq C|\pi|^{2pl(\alpha-)} \int_0^t E\left[\|f\|_{t_{m(s)}}(s)^{2p}\right] ds.$$

Since $l(\alpha-) = l(\alpha) - 1$, (0.2) is obtained in this case.

If $n(\alpha) \neq l(\alpha)$, and $\alpha_l \neq 0$, then by the Burkholder-Davis-Gundy inequality applied to stochastic integral $\Xi(s)$, we have

$$\begin{aligned} E\left[\sup_{0 \leq s \leq t} |\Xi(s)|^{2p}\right] &\leq C \cdot E\left\{\left[\int_0^t |I_{\alpha-}(f^s)_{t_{m(s)}, s}|^2 ds\right]^p\right\} \\ &\leq C \cdot \int_0^t E\left\{|I_{\alpha-}(f^s)_{t_{m(s)}, s}|^{2p}\right\} ds. \end{aligned}$$

By the same estimate as above using (0.1), this is further dominated by

$$C|\pi|^{p[l(\alpha-)+n(\alpha-)]} \int_0^t E\left[\|f\|_{t_{m(s)}}(s)^{2p}\right] ds.$$

Since $\alpha_l \neq 0$, we have $l(\alpha-) + n(\alpha-) = l(\alpha) + n(\alpha) - 1$ and hence (0.2) is obtained.

Finally we consider the case $n(\alpha) \neq l(\alpha)$ and $\alpha_l = 0$. We have

$$\begin{aligned} F_t^\alpha &\leq C \cdot E\left\{\sup_{0 \leq s \leq t} \left|\sum_{m=0}^{m(s)-1} I_\alpha(f)_{t_m, t_{m+1}}\right|^{2p}\right\} + C \cdot E\left\{\sup_{0 \leq s \leq t} |I_\alpha(f)_{t_{m(s)}, s}|^{2p}\right\} \\ &:= I_1 + I_2 \end{aligned}$$

and estimate these two terms separately. We first note that

$$I_1 = C \cdot E\left\{\sup_{0 \leq k \leq m(t)-1} \left|\sum_{m=0}^k I_\alpha(f)_{t_m, t_{m+1}}\right|^{2p}\right\}$$

and $S_k = \sum_{m=0}^k I_\alpha(f)_{t_m, t_{m+1}}$ forms a discrete time martingale. Then we can apply the Burkholder-Davis-Gundy inequality for the discrete time martingale (cf. [Ga73]) to obtain that

$$\begin{aligned} I_1 &\leq C \cdot E\left\{\left[\sum_{m=0}^{m(t)-1} |I_\alpha(f)_{t_m, t_{m+1}}|^2\right]^p\right\} \\ &= C \cdot E\left\{\left[\sum_{m=0}^{m(t)-1} \left|\int_{t_m}^{t_{m+1}} I_{\alpha-}(f^s)_{t_m, s} ds\right|^2\right]^p\right\}. \end{aligned}$$

Since

$$\left| \int_{t_m}^{t_{m+1}} I_{\alpha-}(f^s)_{t_m,s} ds \right|^2 \leq (t_{m+1} - t_m) \cdot \int_{t_m}^{t_{m+1}} |I_{\alpha-}(f^s)_{t_m,s}|^2 ds,$$

this is further dominated by

$$\begin{aligned} & C|\pi|^p \cdot E \left\{ \left[\sum_{m=0}^{m(t)-1} \int_{t_m}^{t_{m+1}} |I_{\alpha-}(f^s)_{t_m,s}|^2 ds \right]^p \right\} \\ & \leq C|\pi|^p \cdot E \left\{ \left[\int_0^t |I_{\alpha-}(f^s)_{t_m(s),s}|^2 ds \right]^p \right\} \\ & \leq C|\pi|^p \cdot \int_0^t E \left\{ |I_{\alpha-}(f^s)_{t_m(s),s}|^{2p} \right\} ds. \end{aligned}$$

Then by the same estimate as above using (0.1), we deduce that the above is dominated by

$$C|\pi|^{p[l(\alpha-)+n(\alpha-)+1]} \int_0^t E(\|f\|_{t_m(s)}(s)^{2p}) ds.$$

Since $l(\alpha-) + n(\alpha-) + 1 = l(\alpha) + n(\alpha) - 1$, I_1 is now dominated as desired.

As for I_2 , we have,

$$\begin{aligned} E \left[\sup_{0 \leq s \leq t} |I_{\alpha}(f)_{t_m(s),s}|^{2p} \right] &= E \left[\sup_{0 \leq s \leq t} \left| \int_{m(s)}^s I_{\alpha-}(f^{\tau})_{t_m(s),\tau} d\tau \right|^{2p} \right] \\ &\leq E \left\{ \sup_{0 \leq s \leq t} \left[(s - t_m(s))^{2p-1} \int_{m(s)}^s |I_{\alpha-}(f^{\tau})_{t_m(s),\tau}|^{2p} d\tau \right] \right\} \\ &\leq |\pi|^{2p-1} E \left[\sum_{m=0}^{m(t)} \int_{t_m}^{t_{m+1} \wedge t} |I_{\alpha-}(f^s)_{t_m,s}|^{2p} ds \right]. \end{aligned}$$

By using (0.1), this can be dominated by

$$\begin{aligned} & C|\pi|^{2p-1} \cdot |\pi|^{p[l(\alpha-)+n(\alpha-)]} \int_0^t E(\|f\|_{t_m(s)}(s)^{2p}) ds \\ &= C|\pi|^{p-1} \cdot |\pi|^{p[l(\alpha)+n(\alpha)-1]} \int_0^t E(\|f\|_{t_m(s)}(s)^{2p}) ds. \end{aligned}$$

Since $p \geq 1$, we obtained the desired estimate for I_2 and the proof of (0.2) is now complete. $\square$

References

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- [Kl-Pl92] P. E. Kloeden and E. Platen, Numerical solutions of stochastic differential equations, Springer, 1992.