

## Parabolic Variational Inequality for the Cahn-Hilliard Equation with Constraint

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### 1. Introduction

In this paper we study the Cahn-Hilliard equation with constraint by means of subdifferential operator techniques. Such a state constraint problem was recently proposed by Blowey-Elliott [1] as a model of diffusive phase separation. The questions of the existence, uniqueness and asymptotic behaviour of solutions, treated in [1] for the special case of the deep quench limit, are considered in our paper without such a restriction.

The standard Cahn-Hilliard equation is a model of diffusive phase separation in isothermal binary systems, and in terms of the concentration  $u$  of one of the components it has the form

$$u_t + \nu \Delta^2 u - \Delta f(u) = 0 \quad \text{in } Q_T = (0, T) \times \Omega. \quad (1.1)$$

Here  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$ , with a smooth boundary  $\Gamma = \partial\Omega$ ,  $\nu$  is a positive constant related to the surface tension,  $f(u)$  corresponds to the volumetric part of the chemical potential difference between components and is given by

$$f(u) = F'(u), \quad (1.2)$$

where  $F(u)$  is a homogeneous (volumetric) free energy parametrized by temperature  $\theta$ , with the characteristic double-well form for  $\theta$  below the critical temperature  $\theta_c$ . Usually the free energy is approximated by polynomials  $F : \mathbb{R} \rightarrow \mathbb{R}$ , e.g. in the simplest case by quartic polynomial

$$F(u) = F_o(\theta) + \alpha_2(\theta - \theta_c)u^2 + \alpha_4 u^4 \quad (1.3)$$

with constants  $\alpha_2, \alpha_4 > 0$  and a given function  $F_o(\theta)$  of temperature. To preserve an explicit physical sense, the state variable  $u$  often is subject to some constraints, e.g. in the case of concentration natural limitation is

$$0 \leq u \leq 1. \quad (1.4)$$

Then the free energy  $F(u)$  can be assumed in the form of the so-called regular solution model

$$F(u) = F_o(\theta) + \alpha_o \theta [u \log u + (1-u) \log(1-u)] + \alpha_1 (\theta - \theta_c) u(u-1) \quad (1.5)$$

with a function  $F_o(\theta)$  and positive constants  $\alpha_o, \alpha_1$ . The corresponding form of the chemical potential  $f(u)$  is shown in Fig. 1. Moreover, as the deep quench limit of (1.5), i.e. as the

(b)

$$tX(t, v(t)) + \int_0^t \tau |v'(\tau)|_{V^*}^2 d\tau \leq \int_0^t \{\tau |\alpha'(\tau)| + X(\tau, v(\tau))\} d\tau \cdot \exp\left(\int_0^t |\alpha'(\tau)| d\tau\right)$$

for all  $t > 0$ ,

and

$$X(t, v(t)) + \int_s^t |v'|_{V^*}^2 d\tau \leq \{X(s, v(s)) + \int_s^t |\alpha'(\tau)| d\tau\} \cdot \exp\left(\int_s^t |\alpha'(\tau)| d\tau\right) \quad (2.1)$$

for all  $0 < s < t$ .

In particular, if  $v_0 \in D$ , then (2.1) holds for  $0 = s < t$ , too.

The third theorem is concerned with the large time behaviour of the solution  $v(t)$  of (VI).

**Theorem 2.3.** In addition to the assumptions  $(\varphi 1) - (\varphi 3)$  and  $(p)$  suppose that  $\alpha' \in L^1(\mathbb{R}_+)$ , and

$(\varphi 4)$   $\varphi^t$  converges to a proper l.s.c. convex function  $\varphi^\infty$  on  $H$  in the sense of Mosco [11] as  $t \rightarrow \infty$ , i.e.

(M1) for any  $z \in D(\varphi^\infty)$  there exists a function  $w : \mathbb{R}_+ \rightarrow H$  such that  $w(t) \rightarrow z$  in  $H$  and  $\varphi^t(w(t)) \rightarrow \varphi^\infty(z)$  as  $t \rightarrow \infty$ ;

(M2) if  $w : \mathbb{R}_+ \rightarrow H$  and  $w(t) \rightarrow z$  weakly in  $H$  as  $t \rightarrow \infty$ , then  $\liminf_{t \rightarrow \infty} \varphi^t(w(t)) \geq \varphi^\infty(z)$ .

Let  $v$  be the solution of (VI) on  $\mathbb{R}_+$  associated with initial datum  $v_0 \in D_*$ , and denote by  $\omega(v_0)$  the  $\omega$ -limit set of  $v(t)$  in  $H$  as  $t \rightarrow \infty$ , i.e.  $\omega(v_0) := \{z \in H; v(t_n) \rightarrow z \text{ in } H \text{ for some } t_n \text{ with } t_n \rightarrow \infty\}$ . Then  $\omega(v_0) \neq \emptyset$  and

$$\partial\varphi^\infty(v_\infty) + p(v_\infty) \ni 0 \quad \text{for all } v_\infty \in \omega(v_0).$$

Finally we give a result on the continuous dependence of solutions of (VI) upon the data  $v_0, \{\varphi^t\}$  and  $p(\cdot)$ .

**Theorem 2.4.** Let  $\{\varphi_n^t\}$  be a sequence of families of proper l.s.c. convex functions on  $H$  such that conditions  $(\varphi 1) - (\varphi 3)$  are satisfied for common positive constants  $C_0, C_1$  and a common function  $\alpha \in W_{loc}^{1,1}(\mathbb{R}_+)$ . Also, let  $p_n$  be a sequence of Lipschitz continuous operators in  $H$  such that condition  $(p)$  is satisfied for a common Lipschitz constant  $L_0 > 0$  and a non-negative  $C^1$ -function  $P_n$  on  $H$ . Suppose that for each  $t \leq 0$ ,  $\varphi_n^t$  converges to  $\varphi^t$  on  $H$  in the sense of Mosco as  $n \rightarrow \infty$ , i.e.

(m1) for any  $z \in D$ , there exists  $\{z_n\} \subset H$  such that  $z_n \in D_n (= D(\varphi_n^t))$ ,  $z_n \rightarrow z$  in  $H$  and  $\varphi_n^t(z_n) \rightarrow \varphi^t(z)$  as  $n \rightarrow \infty$ ;

(m2) if  $z_n \in H$  and  $z_n \rightarrow z$  weakly in  $H$  as  $n \rightarrow \infty$ , then  $\liminf_{n \rightarrow \infty} \varphi_n^t(z_n) \geq \varphi^t(z)$ .

Furthermore suppose that for each  $z \in H$ ,

$$p_n(z) \rightarrow p(z) \quad \text{in } H, \quad P_n(z) \rightarrow P(z) \quad \text{as } n \rightarrow \infty.$$

The cases (1.3), (1.5) and (1.6) of free energies can be written in the form (1.7) with appropriate functions  $\hat{\beta}$  and  $\hat{g}$ , and these special cases have been studied by Blowey-Elliott [1] and Elliott-Luckhaus [5].

## 2. Abstract results

We shall study evolution system (1.8)-(1.10) in an abstract framework.

Let  $H$  and  $V$  be (real) Hilbert spaces such that  $V$  is densely and compactly embedded in  $H$ .  $V^*$  will be the dual of  $V$ . Then, identifying  $H$  with its dual, we have

$$V \subset H \subset V^*$$

with dense and compact injections. Further, let  $J^*$  be the duality mapping from  $V^*$  onto  $V$ , and for  $t \in \mathbb{R}_+ = [0, \infty)$ , let  $\varphi^t(\cdot)$  be a proper, l.s.c., non-negative and convex function on  $H$ . We shall consider the following problem (VI):

$$\begin{cases} J^*(v'(t)) + \partial\varphi^t(v(t)) + p(v(t)) \ni 0 & \text{in } H, t > 0, \\ v(0) = v_0, \end{cases}$$

where  $v' = (\frac{d}{dt})v$ ,  $\partial\varphi^t$  is the subdifferential of  $\varphi^t$  in  $H$ ;  $p(\cdot) : H \rightarrow H$  is a Lipschitz continuous operator and  $v_0$  a given initial datum.

When it is necessary to indicate the data  $\varphi^t, p$  and  $v_0$  explicitly, (VI) is denoted by  $(VI; \varphi^t, p, v_0)$ .

Throughout this paper we use the following notations:

$(\cdot, \cdot)$ : the inner product in  $H$ ;

$\langle \cdot, \cdot \rangle$ : the duality pairing between  $V^*$  and  $V$ ;

$|\cdot|_W$ : the norm in  $W$  for any normed space  $W$ ;

$J$ : the duality mapping from  $V$  onto  $V^*$ , hence  $J^* = J^{-1}$ .

We use some basic notions and results about monotone operators and subdifferentials of convex functions; for details we refer to Brézis [2] and Lions [10].

We shall discuss  $(VI) = (VI; \varphi^t, p, v_0)$  under the following additional hypotheses:

( $\varphi 1$ ) The effective domain  $D(\varphi^t) (= \{z \in H; \varphi^t(z) < \infty\})$  of  $\varphi^t$  is independent of  $t \in \mathbb{R}_+$ ,  $D := D(\varphi^t) \subset V$  and

$$\varphi^t(z) \geq C_0 |z|_V^2 \quad \text{for all } z \in V \text{ and all } t \in \mathbb{R}_+,$$

where  $C_0$  is a positive constant.

( $\varphi 2$ )  $(z_1^* - z_2^*, z_1 - z_2) \geq C_1 |z_1 - z_2|_V^2$  for all  $z_i \in D$ ,  $z_i^* \in \partial\varphi^t(z_i)$ ,  $i = 1, 2$ , and all  $t \in \mathbb{R}_+$ , where  $C_1$  is a positive constant.

( $\varphi 3$ ) There is a function  $\alpha \in W_{loc}^{1,1}(\mathbb{R}_+)$  such that

$$\varphi^t(z) - \varphi^s(z) \leq |\alpha(t) - \alpha(s)|(1 + \varphi^s(z))$$

for all  $z \in D$  and  $s, t \in \mathbb{R}_+$  with  $s \leq t$ .

- (p)  $p$  is a Lipschitz continuous operator in  $H$  and there is a non-negative  $C^1$ -function  $P : H \rightarrow \mathbb{R}$  whose gradient coincides with  $p$ , i.e.  $p = \nabla P$ ; hence

$$\frac{d}{dt}P(w(t)) = (p(w(t)), w'(t)) \quad \text{for a.e. } t \in \mathbb{R}, \text{ if } w \in W_{loc}^{1,2}(\mathbb{R}_+; H).$$

We now introduce a notion of the solution in a weak sense to problem (VI).

**Definition 2.1.** (i) Let  $0 < T < \infty$ . Then a function  $v : [0, T] \rightarrow H$  is called a solution of (VI) on  $[0, T]$ , if  $v \in L^2(0, T; V) \cap C([0, T]; V^*)$ ,  $v' \in L_{loc}^2((0, T]; V^*)$ ,  $v(0) = v_0$ ,  $\varphi^{(\cdot)}(v) \in L^1(0, T)$  and

$$-J^*(v'(t)) - p(v(t)) \in \partial\varphi^t(v(t)) \quad \text{for a.e. } t \in [0, T].$$

(ii) A function  $v : \mathbb{R}_+ \rightarrow H$  is called a solution of (VI) on  $\mathbb{R}_+$ , if the restriction of  $v$  to  $[0, T]$  is a solution of (VI) on  $[0, T]$  for every finite  $T > 0$ .

Our results for (VI) are given as follows.

**Theorem 2.1.** Assume that  $(\varphi 1) - (\varphi 3)$  and (p) are satisfied. Let  $T$  be any positive number. Then the following two statements (a) and (b) hold:

(a) If  $v_0$  is given in the closure  $D_*$  of  $D$  in  $V^*$ , then (VI) has one and only one solution  $v$  on  $[0, T]$  such that

$$t^{\frac{1}{2}}v' \in L^2(0, T; V^*), \quad \sup_{0 < t \leq T} t\varphi^t(v(t)) < \infty.$$

(b) If  $v_0 \in D$ , then the solution  $v$  of (VI) on  $[0, T]$  satisfies that

$$v' \in L^2(0, T; V^*), \quad \sup_{0 \leq t \leq T} \varphi^t(v(t)) < \infty;$$

hence  $v \in C([0, T]; H)$ .

The second theorem is concerned with the energy inequality for (VI).

**Theorem 2.2.** Assume that  $(\varphi 1) - (\varphi 3)$  and (p) hold. Let  $v$  be the solution of (VI) on  $\mathbb{R}_+$  associated with initial datum  $v_0 \in D_*$ . Define

$$X(t, z) = \varphi^t(z) + P(z) \quad \text{for } z \in D \text{ and } t \in \mathbb{R}_+.$$

Then: (a)

$$\sup_{0 \leq \tau \leq t} |v(\tau)|_{V^*}^2 + \int_0^t \varphi^\tau(v(\tau)) d\tau \leq M_0 \{|v_0|_{V^*}^2 + \int_0^t \varphi^\tau(z) d\tau + (|z|_H^2 + 1)\} e^{M_0 t}$$

for all  $z \in D$  and  $t > 0$ ,

where  $M_0$  is a positive constant dependent only on  $C_0$  in  $(\varphi 1)$ , the Lipschitz constant  $L_p$  of  $p(\cdot)$  and the value  $|p(0)|_H$ .

limit of (1.5) as  $\theta \rightarrow 0$ , the non-smooth free energy

$$F(u) = \begin{cases} F_0(\theta) + \alpha_1 \theta_c u(1-u) & \text{if } 0 \leq u \leq 1, \\ \infty & \text{otherwise} \end{cases} \quad (1.6)$$

is obtained (see Fig. 2); the constraint (1.4) is included in formula (1.6). This type of free energy (1.6) was introduced by Oono-Puri [12], and the corresponding Cahn-Hilliard equation was numerically studied by them; subsequently this model was analyzed theoretically, too, by Blowey-Elliott [1].

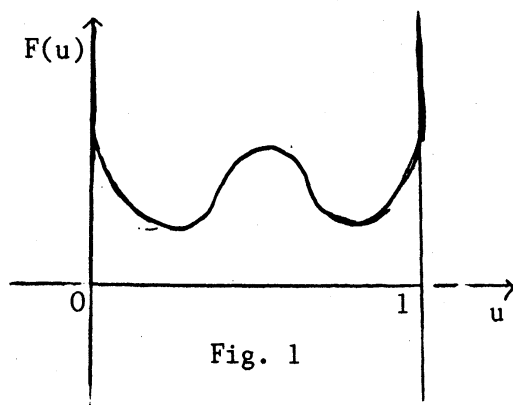


Fig. 1

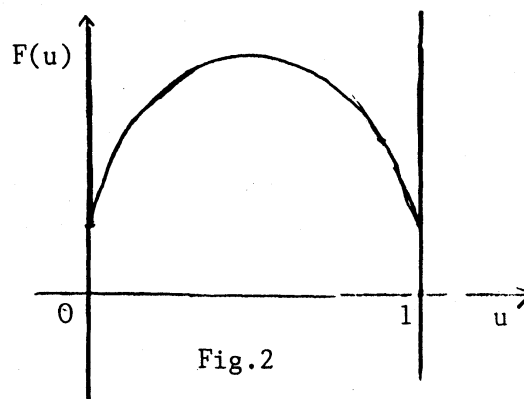


Fig. 2

For generality we propose in this paper the representation of (possibly non-smooth) free energy in the form

$$F(u) = \hat{\beta}(u) + \hat{g}(u), \quad (1.7)$$

where  $\hat{\beta}$  is a proper, l.s.c. and convex function on  $\mathbb{R}$  and  $\hat{g}$  is a non-negative function of  $C^1$ -class on  $\mathbb{R}$  with Lipschitz continuous derivative  $g = \hat{g}'$  on  $\mathbb{R}$ . In such a non-smooth case of free energy functionals, the formula (1.2), giving the volumetric part  $f(u)$  of the chemical potential difference, does not make sense any longer. Therefore, following the idea in [1], we introduce a generalized notion of chemical potential which is represented in terms of the multivalued function

$$F(u) = \{\xi + g(u); \xi \in \beta(u)\},$$

where  $\beta$  is the subdifferential of  $\hat{\beta}$  in  $\mathbb{R}$ . Then the Cahn-Hilliard equation (1.1) is extended to the general form

$$u_t + \nu \Delta^2 u - \Delta(\xi + g(u)) = 0, \quad \xi \in \beta(u) \quad \text{in } Q_T. \quad (1.8)$$

Equation (1.8) is to be satisfied together with boundary conditions

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial}{\partial n}(\nu \Delta u + \xi + g(u)) = 0 \quad \text{on } \Sigma_T := (0, T) \times \gamma \quad (1.9)$$

and initial condition

$$u(0, \cdot) = u_0 \quad \text{in } \Omega, \quad (1.10)$$

where  $u_0$  is a given initial datum, and  $\frac{\partial}{\partial n}$  denotes the outward normal derivative on  $\Gamma$ .

Let  $\{v_{on}\}$  be a sequence in  $V^*$  such that  $v_{on} \in D_{n*}$  (=the closure of  $D_n$  in  $V^*$ ),  $v_o \in D_*$  and  $v_{on} \rightarrow v_o$  in  $V^*$  as  $n \rightarrow \infty$ . Then the solution  $v_n$  of  $(VI)_n := (VI; \varphi_n^t, p_n, v_{on})$  converges to the solution  $v$  of  $(VI) := (VI; \varphi^t, p, v_o)$  as  $n \rightarrow \infty$  in the following sense: for every finite  $T > 0$  and every  $0 < \delta < T$ ,

$$\begin{aligned} v_n &\rightarrow v && \text{in } C([0, T]; V^*), \\ t^{\frac{1}{2}} v'_n &\rightarrow t^{\frac{1}{2}} v' && \text{weakly in } L^2(0, T; V^*), \\ v_n &\rightarrow v && \text{in } C([\delta, T]; H) \text{ and weakly}^* \text{ in } L^\infty(\delta, T; V), \end{aligned}$$

as  $n \rightarrow \infty$ .

### 3. Sketch of the proofs

We sketch the proofs of the main theorems.

(1) (Uniqueness) Let  $v_i$ ,  $i = 1, 2$ , be two solutions of (VI) on  $[0, T]$  and put  $v := v_1 - v_2$ . Multiply the difference of two equations, which  $v_1$  and  $v_2$  satisfy, by  $v$ , and then use the inequality

$$|z|_H^2 \leq \varepsilon |z|_V^2 + C(\varepsilon) |z|_{V^*}^2 \quad \text{for all } z \in V,$$

where  $\varepsilon$  is an arbitrary positive number and  $C(\varepsilon)$  is a suitable positive constant dependent only on  $\varepsilon$ . Then we have an inequality of the form

$$\frac{1}{2} \frac{d}{dt} |v(t)|_{V^*}^2 + k_1 |v(t)|_V^2 \leq k_2 |v(t)|_{V^*}^2 \quad \text{for a.e. } t \in [0, T],$$

where  $k_1$  and  $k_2$  are some positive constants. Therefore, Gronwall's lemma implies that  $v = 0$ .

(2) (Approximate problems) Let  $v_o \in D$  and  $\mu$  be any parameter in  $(0, 1]$ . Consider the following approximate problem  $(VI)_\mu$  for (VI):

$$\begin{cases} (J^* + \mu I)(v'_\mu(t)) + \partial \varphi^t(v_\mu(t)) + p(v_\mu(t)) \ni 0 & \text{in } H, \quad 0 < t < T, \\ v_\mu(0) = v_o. \end{cases}$$

By making use of the results in [9] this problem  $(VI)_\mu$  has one only one solution  $v_\mu \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$ . Also, multiplying the equation of  $(VI)_\mu$  by  $v_\mu$ ,  $v'_\mu$  and  $tv'_\mu$ , we have similar estimates as those in Theorem 2.2.

(3) (Existence and estimates for (VI)) In the case when  $v_o \in D$ , by the standard monotonicity and compactness methods we can prove that the solution  $v_\mu$  tends to the solution  $v$  of (VI) as  $\mu \rightarrow 0$  in the sense that

$$\begin{aligned} v_\mu &\rightarrow v && \text{in } C([0, T]; H) \text{ and weakly}^* \text{ in } L^\infty(0, T; V), \\ v'_\mu &\rightarrow v' && \text{weakly in } L^2(0, T; V^*), \\ \mu v'_\mu &\rightarrow 0 && \text{in } L^2(0, T; H). \end{aligned}$$

Moreover we have the estimates in Theorem 2.2 for  $v$ . In the case when  $v_o \in D_*$ , it is enough to approximate  $v_o$  by a sequence  $\{v_{on}\} \subset D$  and to see the convergence of the solution  $v_n$  associated with initial datum  $v_{on}$ .

(4) (Proof of Theorem 2.3) From the energy estimates which were obtained in Theorem 2.2, it follows that  $v' \in L^2(1, \infty; V^*)$  and  $v \in L^\infty(1, \infty; V)$ ; hence Theorem 2.3 holds.

(5) (Proof of Theorem 2.4) Under the assumptions of Theorem 2.4, we see from the energy estimates for  $v_n$  that  $\{v_n\}$  is bounded in  $C([0, T]; H) \cap L^2(0, T; V) \cap L_{loc}^\infty((0, T]; V) \cap W_{loc}^{1,2}((0, T]; V^*)$ . Hence by the usual monotonicity and compactness argument we have the assertions of Theorem 2.4.

#### 4. Application to the Cahn-Hilliard equation with constraint

We denote by (CHC) the Cahn-Hilliard equation with constraint (1.8)-(1.10). Here we suppose that

- (A1)  $g : \mathbf{R} \rightarrow \mathbf{R}$  is a Lipschitz continuous function with a non-negative primitive  $\hat{g}$  on  $\mathbf{R}$ .
- (A2)  $\beta$  is a maximal monotone graph in  $\mathbf{R} \times \mathbf{R}$  such that  $0 \in R(\beta)$  and  $\text{int}.D(\beta) \neq \emptyset$ ; we may assume that there is a non-negative proper l.s.c. convex function on  $\mathbf{R}$  such that its subdifferential  $\partial\hat{\beta}$  coincides with  $\beta$  in  $\mathbf{R}$ .
- (A3)  $u_o \in L^2(\Omega)$ ,  $u_o(x) \in \overline{D(\beta)}$  for a.e.  $x \in \Omega$ .

**Definition 4.1.** Let  $0 < T < \infty$ . Then  $u : [0, T] \rightarrow H$  is called a (weak) solution of (CHC) on  $[0, T]$ , if  $u$  satisfies the following properties (w1)-(w3):

- (w1)  $u \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; (H^1(\Omega))^*) \cap L_{loc}^2((0, T]; H^2(\Omega)) \cap L_{loc}^\infty((0, T]; H^1(\Omega)) \cap W_{loc}^{1,2}((0, T]; (H^1(\Omega))^*)$  and  $\beta(u) \in L^1(Q_T)$ ;
- (w2)  $u(0, \cdot) = u_o$  a.e. in  $\Sigma_T$ ;
- (w3) there is a function  $\xi : [0, T] \rightarrow L^2(\Omega)$  such that

$$\xi \in L_{loc}^2((0, T]; L^2(\Omega)), \quad \xi \in \beta(u) \quad \text{a.e. in } Q_T$$

and

$$\frac{d}{dt}(u(t), \eta) + \nu(\Delta u(t), \Delta \eta) - (\xi(t) + g(u(t)), \Delta \eta) = 0$$

for all  $\eta \in H^2(\Omega)$  with  $\frac{\partial \eta}{\partial n}$  a.e. on  $\Gamma$ , and for a.e.  $t \in [0, T]$ .

Applying Theorems 2.1-2.4 to (CHC) we have:

**Theorem 4.1.** Assume that (A1)-(A3) hold and

$$m := \frac{1}{|\Omega|} \int_{\Omega} u_o dx \in \text{int}.D(\beta).$$

Then for every finite  $T > 0$  problem (CHC) has one and only one solution  $u$  on  $[0, T]$ , and the following statements (a) and (b) hold:

(a)  $u \in L^\infty(\delta, \infty; H^1(\Omega))$ ,  $u'(\delta, \infty; (H^1(\Omega))^*)$  for every  $\delta > 0$ , and hence the  $\omega$ -limit set  $\omega(u_o) := \{z \in L^2(\Omega); u(t_n) \rightarrow z \text{ in } L^2(\Omega) \text{ for some } t_n \text{ with } t_n \rightarrow \infty\}$  is non-empty;

(b)  $\omega(u_o) \subset H^2(\Omega)$ , and any  $u_\infty \in \omega(u_o)$  with some  $\mu_\infty \in \mathbb{R}$  and  $\xi_\infty \in L^2(\Omega)$  solves the following stationary problem

$$-\nu \Delta u_\infty + \xi_\infty + g(u_\infty) = \mu_\infty \quad \text{in } \Omega, \quad \xi_\infty \in \beta(u_\infty) \quad \text{a.e. in } \Omega,$$

$$\frac{\partial u_\infty}{\partial n} = 0 \quad \text{a.e. on } \Gamma, \quad \frac{1}{|\Omega|} \int_\Omega u_\infty dx = m.$$

Now, let us reformulate (CHC) as an evolution problem of the form (VI) in the space

$$H := \{z \in L^2(\Omega); \int_\Omega z dx = 0\} \quad \text{with } |z|_H = |z|_{L^2(\Omega)};$$

put also

$$V := H \cap H^1(\Omega) \quad \text{with } |z|_V = |\nabla z|_{L^2(\Omega)}.$$

For this purpose we consider the data  $\varphi^t = \varphi$ ,  $p(\cdot)$  and  $v_o$  as follows:

$$\varphi(z) := \begin{cases} \frac{\nu}{2} |\nabla z|_{L^2(\Omega)}^2 + \int_\Omega \hat{\beta}(z+m) dx & \text{if } z \in V, \\ \infty & \text{otherwise,} \end{cases}$$

where  $m = \frac{1}{|\Omega|} \int_\Omega u_o dx$ ;

$$p(z) := \pi(g(z+m)), \quad P(z) := \int_\Omega \hat{g}(z+m) dx, \quad z \in H;$$

$$v_o := u_o - m.$$

By virtue of the following lemma, problems (CHC) and (VI) associated with the data defined above are equivalent.

**Lemma 4.1.** *Let  $\ell \in L^2(\Omega)$ . Then  $\pi(\ell) \in \partial\varphi(z)$  if and only if  $z_m = z + m$  satisfies that there are  $\mu_m \in \mathbb{R}$  and  $\xi_m \in L^2(\Omega)$  such that*

$$-\nu \Delta z_m + \xi_m = \ell + \mu_m \quad \text{in } L^2(\Omega), \quad \xi_m \in \beta(z_m) \quad \text{a.e. in } \Omega,$$

$$\frac{\partial z_m}{\partial n} = 0 \quad \text{a.e. on } \Gamma, \quad \frac{1}{|\Omega|} \int_\Omega z_m dx = m;$$

hence  $z_m \in H^2(\Omega)$ . Moreover,  $\mu_m$  can be chosen so that

$$|\mu_m| \leq M(1 + |\ell|_{L^2(\Omega)}),$$

where  $M > 0$  is a certain constant dependent only upon  $\beta$  and  $m$ , and  $z_m$  satisfies that

$$\nu |\Delta z_m|_{L^2(\Omega)} \leq |\ell|_{L^2(\Omega)} + |\mu_m| |\Omega|^{\frac{1}{2}}.$$

By Theorem 2.1 problem (VI) has one and only one solution  $v$ . Moreover we see from the above lemma that the function  $u := v + m$  is the unique solution of (CHC), and from Theorems 2.2 and 2.3 that (a) and (b) hold.

When the state constraint  $\xi \in \beta(u)$  is not imposed, the system (1.8)-(1.10) becomes the standard Cahn-Hilliard problem. For such a problem various existence, uniqueness and asymptotic results have been established; see e.g. Elliott [3], Elliott-Zheng [6] and Zheng [15]. For related results in abstract setting we refer to Temam [13] and von Wahl [14]. For the Cahn-Hilliard models with non-smooth free energy functionals we refer to Elliott-Mikelic [4]. The structure of stationary solutions corresponding to the Cahn-Hilliard equation was studied by Gurtin-Matano [7]; their analysis covers also some cases of free energy  $F(u)$  with infinite walls.

Finally we give examples of  $\beta$  and the corresponding Cahn-Hilliard equations.

**Example 4.1.** (i) (Logarithmic form) For constants  $\alpha_o > 0$  and  $\theta > 0$ ,  $\theta$  being a parameter,

$$\beta(u) := \beta^\theta(u) = \begin{cases} \{\alpha_o \theta \log \frac{u}{1-u}\} & \text{for } 0 < u < 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Given any Lipschitz continuous function  $\bar{g}$  on  $[0, 1]$ , we extend it to a Lipschitz continuous function  $g$ , with support in  $[-1, 2]$ , on the whole line  $\mathbf{R}$ .

(ii) (The limit of  $\beta^\theta$  as  $\theta \rightarrow 0$ )

$$\beta(u) := \beta^0(u) = \begin{cases} [0, \infty) & \text{if } u = 1, \\ \{0\} & \text{if } 0 < u < 1, \\ (-\infty, 0] & \text{if } u = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

and  $g$  is the same as in (i).

**Example 4.2.** Denote by  $(\text{CHC})_\theta$  and  $(\text{CHC})_0$  the Cahn-Hilliard equations (CHC) associated with  $\beta = \beta^\theta$  and  $\beta = \beta^0$ , respectively. Then, by the theorems proved above,  $(\text{CHC})_\theta$  and  $(\text{CHC})_0$  have the unique solutions  $u^\theta$  and  $u^0$ , respectively, and moreover  $u^\theta \rightarrow u^0$  as  $\theta \rightarrow 0$  in the similar sense as Theorem 2.4.

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