

# The commutator subgroup of the general $\Lambda$ -quadratic group $\mathbf{GQ}(A, \Lambda)$

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## 1. INTRODUCTION

Let  $A$  be a ring with the unity,  $- : A \rightarrow A$  an involution,  $\lambda \in \text{center}(A)$  a symmetry, and  $\Lambda$  a form parameter on  $A$  in the sense of [1, Section 1]. We refer to the tuple  $(A, (-, \lambda), \Lambda)$  as a *form ring*. The *general  $\Lambda$ -quadratic group*  $\mathbf{GQ}_{2n}(A, \Lambda)$  is defined to be the matrix group corresponding to the automorphism group on the  $\Lambda$ -hyperbolic module  $\Lambda - H(A^n)$ . The *elementary  $\Lambda$ -quadratic group*  $\mathbf{EQ}_{2n}(A, \Lambda)$  is the subgroup of  $\mathbf{GQ}_{2n}(A, \Lambda)$  generated by all elementary  $\Lambda$ -quadratic  $2n \times 2n$ -matrices. If  $\mathfrak{q}$  is an involution invariant ideal of  $A$ , then the *relative congruence subgroup*  $\mathbf{GQ}_{2n}(A, \Lambda, \mathfrak{q})$  is defined to be

$$\ker[\mathbf{GQ}_{2n}(A, \Lambda) \rightarrow \mathbf{GQ}_{2n}(A/\mathfrak{q}, \Lambda/\mathfrak{q})],$$

where

$$\Lambda/\mathfrak{q} = \text{image}[\Lambda \rightarrow A/\mathfrak{q}],$$

and the *relative elementary subgroup*  $\mathbf{EQ}_{2n}(A, \Lambda, \mathfrak{q})$  is defined to be the normal subgroup of  $\mathbf{EQ}_{2n}(A, \Lambda)$  generated by all elementary  $\Lambda$ -quadratic matrices belonging to  $\mathbf{GQ}_{2n}(A, \Lambda, \mathfrak{q})$ . The groups  $\mathbf{GQ}(A, \Lambda, \mathfrak{q})$  and  $\mathbf{EQ}(A, \Lambda, \mathfrak{q})$  are defined to be the inductive limits of  $\mathbf{GQ}_{2n}(A, \Lambda, \mathfrak{q})$  and  $\mathbf{EQ}_{2n}(A, \Lambda, \mathfrak{q})$ , respectively, as  $n \rightarrow \infty$ . It is evident that  $\mathbf{EQ}_{2n}(A, \Lambda, \mathfrak{q})$  is canonically embedden in  $\mathbf{GQ}_{2n}(A, \Lambda, \mathfrak{q})$  as a subgroup. The next result is given as [1, Corollary 3.9].

**Theorem 1.1.** *The commutator subgroup  $[\mathbf{GQ}_{2n}(A, \Lambda, \mathfrak{q}), \mathbf{GQ}_{2n}(A, \Lambda)]$  is equal to  $\mathbf{EQ}_{2n}(A, \Lambda, \mathfrak{q})$ .*

The proof of the theorem in [1] uses the lemma:

**Lemma 1.2.** *Let  $G_{2n} = \text{GQ}_{2n}((A, \Lambda) \ltimes q)$  and  $E_{2n} = \text{EQ}_{2n}((A, \Lambda) \ltimes q)$ . Then*

$$[G_{2n}, G_{2n}] \cong [\text{GQ}_{2n}(A, \Lambda), \text{GQ}_{2n}(A, \Lambda)] \ltimes [\text{GQ}_{2n}(A, \Lambda), \text{GQ}_{2n}(A, \Lambda, q)]$$

and

$$E_{2n} \cong \text{EQ}_{2n}(A, \Lambda) \ltimes \text{EQ}_{2n}(A, \Lambda, q).$$

In [1], isomorphisms above are obtained by implicitly identifying  $\text{GQ}_{2n}(A, \Lambda, q)$  with

$$\text{GQ}_{2n}(A, \Lambda, q)' = \ker[\text{GQ}_{2n}((A, \Lambda) \ltimes q) \longrightarrow \text{GQ}_{2n}(A, \Lambda)]$$

and  $\text{EQ}_{2n}(A, \Lambda, q)$  with

$$\text{EQ}_{2n}(A, \Lambda, q)' = \ker[\text{EQ}_{2n}((A, \Lambda) \ltimes q) \longrightarrow \text{EQ}_{2n}(A, \Lambda)],$$

respectively.

The purpose of this paper is to prove the lemma in a precise formulation (Lemma 1.3 below) without employing the groups  $\text{GQ}_{2n}(A, \Lambda, q)'$  and  $\text{EQ}_{2n}(A, \Lambda, q)'$  so that we can clarify the proof of Theorem 1.1.

**Lemma 1.3.** *Let  $G_{2n} = \text{GQ}_{2n}((A, \Lambda) \ltimes q)$  and  $E_{2n} = \text{EQ}_{2n}((A, \Lambda) \ltimes q)$ . Then there is a canonical map*

$$\psi : G_{2n} \longrightarrow \text{GQ}_{2n}(A, \Lambda) \ltimes \text{GQ}_{2n}(A, \Lambda, q)$$

such that the restrictions

$$\psi_{G_{2n}} : [G_{2n}, G_{2n}] \longrightarrow [\text{GQ}_{2n}(A, \Lambda), \text{GQ}_{2n}(A, \Lambda)] \ltimes [\text{GQ}_{2n}(A, \Lambda), \text{GQ}_{2n}(A, \Lambda, q)]$$

and

$$\psi_{E_{2n}} : E_{2n} \longrightarrow \text{EQ}_{2n}(A, \Lambda) \ltimes \text{EQ}_{2n}(A, \Lambda, q)$$

are well-defined and isomorphisms.

## 2. SMASH PRODUCTS OF GROUPS AND OF RINGS, $\Lambda$ -QUADRATIC ELEMENTARY MATRICES

In this section we define the *smash product* of groups, one of rings and elementary matrices. They will be used in the proof Lemma 1.3.

**Definition 2.1.** Let  $\Gamma$  be a group and  $H$  a subgroup of  $\Gamma$ . If  $G$  is a subgroup of  $N_\Gamma(H)$ , we define the *smash product*  $G \ltimes H$  by

$$G \ltimes H = \{(\sigma, \rho) \mid \sigma \in G, \rho \in H\}$$

with multiplication

$$(2.1) \quad (\sigma', \rho') \cdot (\sigma, \rho) = (\sigma' \sigma, (\sigma^{-1} \rho' \sigma) \rho).$$

Let  $(A, (-, \lambda), \Lambda)$  be a form ring and  $\mathfrak{q}$  an involution invariant ideal of  $A$ . A *form ideal* of level  $\mathfrak{q}$  of  $(A, \Lambda)$  is a pair  $(\mathfrak{q}, \Lambda_{\mathfrak{q}})$  where  $\Lambda_{\mathfrak{q}}$  is an additive subgroup of  $A$  such that

- (1)  $\{q - \lambda \bar{q} \mid q \in \mathfrak{q}\} + \left\{ \sum_i q_i \Lambda \bar{q}_i \mid q_i \in \mathfrak{q} \right\} \subset \Lambda_{\mathfrak{q}} \subset \mathfrak{q} \cap \Lambda$  and
- (2)  $a \Lambda_{\mathfrak{q}} \bar{a} \subset \Lambda_{\mathfrak{q}} \ (a \in A)$ .

**Definition 2.2.** (a) Let  $A$  be a ring. If  $\mathfrak{q}$  is a both sides ideal of  $A$ , we define the *smash product ring*  $A \ltimes \mathfrak{q}$  by

$$A \ltimes \mathfrak{q} = \{(a, q) \mid a \in A, q \in \mathfrak{q}\}$$

with addition :  $(a, q) + (a', q') = (a + a', q + q')$  and multiplication :  $(a, q)(a', q') = (aa', qa' + aq' + qq')$ .

(b) If  $(\mathfrak{q}, \Lambda_{\mathfrak{q}})$  is a form ideal of  $(A, \Lambda)$ , we define the *smash product form ring*

$$(A, \Lambda) \ltimes (\mathfrak{q}, \Lambda_{\mathfrak{q}}) = (A \ltimes \mathfrak{q}, \Lambda \ltimes \Lambda_{\mathfrak{q}})$$

where the involution on  $A \ltimes \mathfrak{q}$  is defined by  $(a, q) \mapsto (\bar{a}, \bar{q})$ , and  $\Lambda \ltimes \Lambda_{\mathfrak{q}} = \{(a, q) \mid a \in \Lambda, q \in \Lambda_{\mathfrak{q}}\}$ . If  $\Lambda_{\mathfrak{q}} = \mathfrak{q} \cap \Lambda$ , then we shall write  $(A, \Lambda) \ltimes \mathfrak{q}$  instead of  $(A, \Lambda) \ltimes (\mathfrak{q}, \mathfrak{q} \cap \Lambda)$ .

We have the ring homomorphism

$$f : A \ltimes \mathfrak{q} \longrightarrow A; (a, q) \longmapsto a,$$

its splitting

$$i : A \longrightarrow (A \ltimes \mathfrak{q}); a \longmapsto (a, 0),$$

the form ring homomorphism

$$g : (A, \Lambda) \ltimes (\mathfrak{q}, \Lambda_{\mathfrak{q}}) \longrightarrow (A, \Lambda)$$

induced by  $f$ , and its splitting

$$j : (A, \Lambda) \longrightarrow (A, \Lambda) \ltimes (\mathfrak{q}, \Lambda_{\mathfrak{q}})$$

induced by  $i$ .

Let  $M_{n,n}(A)$  denote the set of all  $n \times n$ -matrices with entries in  $A$ , and  $M_{n,n}(\mathfrak{q})$  the set of all  $n \times n$ -matrices with entries in  $\mathfrak{q}$ . If  $P = (p_{ij}) \in M_{n,n}(A)$  and  $Q = (q_{ij}) \in M_{n,n}(\mathfrak{q})$ , then we have the  $n \times n$ -matrix  $(r_{ij})$  with entries  $r_{ij} := (p_{ij}, q_{ij}) \in A \ltimes \mathfrak{q}$ . The correspondence  $M_{n,n}(A) \times M_{n,n}(\mathfrak{q}) \longrightarrow M_{n,n}(A \ltimes \mathfrak{q}); ((p_{ij}), (q_{ij})) \longmapsto (r_{ij})$ , is clearly a bijection. Thus we abuse the notation  $(P, Q)$  for the assigned matrix  $(r_{ij})$  in  $M_{n,n}(A \ltimes \mathfrak{q})$ . By definition, the formula of multiplication

$$(2.2) \quad (P, Q)(P', Q') = (PP', PQ' + QP' + QQ')$$

holds for  $(P, Q)$  and  $(P', Q') \in M_{n,n}(A \ltimes \mathfrak{q})$ .

**Definition 2.3.** A matrix having one form among the following  $2n \times 2n$ -matrices is called a  $\Lambda$ -quadratic elementary matrix.

$$\begin{aligned} \mathbf{H}(\varepsilon_{i,j}(a)) \ (i \neq j, a \in A) : & \begin{cases} (k, k)\text{-entry} = 1 & (k = 1, \dots, 2n), \\ (i, j)\text{-entry} = a, \\ (n+j, n+i)\text{-entry} = -\bar{a}, \\ \text{all other entries} = 0. \end{cases} \\ \varepsilon_{n+i,j}(a) \ (i \neq j, a \in A) : & \begin{cases} (k, k)\text{-entry} = 1 & (k = 1, \dots, 2n), \\ (i, n+j)\text{-entry} = a, \\ (j, n+i)\text{-entry} = -\bar{\lambda}\bar{a}, \\ \text{all other entries} = 0. \end{cases} \end{aligned}$$

$$\begin{aligned}
\varepsilon_{i,n+j}(a) \ (i \neq j, a \in A) : & \begin{cases} (k, k)\text{-entry} = 1 & (k = 1, \dots, 2n), \\ (n+i, j)\text{-entry} = a, \\ (n+j, i)\text{-entry} = -\lambda \bar{a}, \\ \text{all other entries} = 0. \end{cases} \\
\varepsilon_{n+i,i}(a) \ (a \in \bar{\Lambda}) : & \begin{cases} (k, k)\text{-entry} = 1 & (k = 1, \dots, 2n), \\ (i, n+i)\text{-entry} = a, \\ \text{all other entries} = 0. \end{cases} \\
\varepsilon_{i,n+i}(a) \ (a \in \Lambda) : & \begin{cases} (k, k)\text{-entry} = 1 & (k = 1, \dots, 2n), \\ (n+i, i)\text{-entry} = a, \\ \text{all other entries} = 0. \end{cases}
\end{aligned}$$

**Lemma 2.4** (A. Bak [1, Lemma 3.1]). Let  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_{2n}(A)$  with  $\alpha, \beta, \gamma, \delta \in M_{n,n}(A)$ . Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GQ}_{2n}(A, \Lambda) \iff \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} & \lambda \bar{\beta} \\ \bar{\lambda} \bar{\gamma} & \bar{\alpha} \end{pmatrix}.$$

### 3. PROOF OF LEMMA 1.3

Throughout this section, let  $P \in M_{2n,2n}(A)$  and  $Q \in M_{2n,2n}(q)$  and therefore  $(P, Q) \in M_{2n,2n}(A \ltimes q)$ . Let  $I_{2n}$  (or  $I$  if the context is clear) denote the identity matrix in  $M_{2n,2n}(A)$  and  $O_{2n}$  (or  $O$  if the context is clear) the null matrix in  $M_{2n,2n}(A)$ .

**Lemma 3.1.** *The following (1) and (2) hold.*

- (1)  $P \in \text{GL}_{2n}(A)$  if and only if  $(P, O_{2n}) \in \text{GL}_{2n}(A \ltimes q)$ .
- (2) If  $P \in \text{GL}_{2n}(A)$  and  $(P, Q) \in \text{GL}_{2n}(A \ltimes q)$  then  $(I_{2n}, P^{-1}Q) \in \text{GL}_{2n}(A \ltimes q)$ .

*Proof.* Claim (1) is obvious. Suppose  $P$  and  $(P, Q)$  are as in (2). Then, since  $(P, O)^{-1} = (P^{-1}, O)$ ,

$$(3.1) \quad (P, O)^{-1}(P, Q) = (P^{-1}, O)(P, Q) = (I, P^{-1}Q).$$

By  $(P^{-1}, O)$  and  $(P, Q) \in \text{GL}_{2n}(A \ltimes q)$ ,  $(I, P^{-1}Q) \in \text{GL}_{2n}(A \ltimes q)$ . □

**Lemma 3.2.** *The following (1) and (2) hold.*

- (1)  $P \in \text{GQ}_{2n}(A, \Lambda)$  if and only if  $(P, O_{2n}) \in \text{GQ}_{2n}((A, \Lambda) \ltimes q)$ .

(2) If  $P \in \text{GQ}_{2n}(A, \Lambda)$  and  $(P, Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$  then  $(I_{2n}, P^{-1}Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ .

*Proof.* We check  $P \in \text{GQ}_{2n}(A, \Lambda) \Rightarrow (P, O) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ . If  $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  with  $\alpha, \beta, \gamma, \delta \in M_{n,n}(A)$ , then  $P^{-1} = \begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \bar{\lambda}\bar{\gamma} & \bar{\alpha} \end{pmatrix}$  by Lemma 2.4. The equality

$$\begin{pmatrix} (\alpha, O) & (\beta, O) \\ (\gamma, O) & (\delta, O) \end{pmatrix} \begin{pmatrix} \overline{(\delta, O)} & \lambda\overline{(\beta, O)} \\ \overline{\lambda(\gamma, O)} & \overline{(\alpha, O)} \end{pmatrix} = \begin{pmatrix} (I, O) & (O, O) \\ (O, O) & (I, O) \end{pmatrix}$$

clearly holds. Thus  $(P, O) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ . The implication “ $(P, O) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q}) \Rightarrow P \in \text{GQ}_{2n}(A, \Lambda)$ ” is similarly checked. Suppose  $P$  and  $(P, Q)$  are as in (2). Then  $(I, P^{-1}Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$  follows from (3.1) and  $(P^{-1}, O), (P, Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ .  $\square$

**Lemma 3.3.** If  $(I_{2n}, Q) \in \text{GL}_{2n}(A \ltimes \mathfrak{q})$  then  $I_{2n} + Q \in \text{GL}_{2n}(A)$ .

*Proof.* For the inverse matrix  $(I, Q')$  of  $(I, Q)$ ,

$$(I, Q)(I, Q') = (I, Q + Q' + QQ') = (I, O).$$

Thus,

$$Q + Q' + QQ' = O.$$

This implies

$$(I + Q)(I + Q') = I$$

and hence

$$I + Q \in \text{GL}_{2n}(A).$$

$\square$

**Lemma 3.4.** If  $(I_{2n}, Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$  then  $I_{2n} + Q \in \text{GQ}_{2n}(A, \Lambda)$ .

*Proof.* Suppose  $(I, Q) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ . Writing  $Q = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$ , with  $x, y, u, v \in M_{n,n}(\mathfrak{q})$ , we have the equality

$$\begin{pmatrix} (I, x) & (O, y) \\ (O, u) & (I, v) \end{pmatrix} \begin{pmatrix} \overline{(I, v)} & \lambda\overline{(O, y)} \\ \overline{\lambda(O, u)} & \overline{(I, x)} \end{pmatrix} = \begin{pmatrix} (I, O) & (O, O) \\ (O, O) & (I, O) \end{pmatrix}.$$

This provides the equality

$$(3.2) \quad \begin{pmatrix} (I, \bar{v} + x + x\bar{v} + \bar{\lambda}y\bar{u}) & (O, \lambda\bar{y} + \lambda x\bar{y} + y + y\bar{x}) \\ (O, u + u\bar{v} + \bar{\lambda}\bar{u} + \bar{\lambda}v\bar{u}) & (I, \lambda u\bar{y} + \bar{x} + v + v\bar{x}) \end{pmatrix} \\ = \begin{pmatrix} (I, O) & (O, O) \\ (O, O) & (I, O) \end{pmatrix}.$$

On the other hand, we have

$$\begin{pmatrix} I+x & y \\ u & I+v \end{pmatrix} \begin{pmatrix} \overline{I+v} & \lambda\bar{y} \\ \bar{\lambda}\bar{u} & \overline{I+x} \end{pmatrix} \\ = \begin{pmatrix} I + \bar{v} + x + x\bar{v} + \bar{\lambda}y\bar{u} & \lambda\bar{y} + \lambda x\bar{y} + y + y\bar{x} \\ u + u\bar{v} + \bar{\lambda}\bar{u} + \bar{\lambda}v\bar{u} & I + \lambda u\bar{y} + \bar{x} + v + v\bar{x} \end{pmatrix} \\ = \begin{pmatrix} I & O \\ O & I \end{pmatrix} \text{ by (3.2).}$$

By Lemma 2.4,  $I + Q = \begin{pmatrix} I+x & y \\ u & I+v \end{pmatrix}$  lies in  $GQ_{2n}(A, \Lambda)$ . □

**Lemma 3.5.** *If  $(P, Q) \in GQ_{2n}((A, \Lambda) \ltimes q)$ , then  $(P, I_{2n} + P^{-1}Q) \in GQ_{2n}(A, \Lambda) \ltimes GQ_{2n}(A, \Lambda, q)$ .*

*Proof.* If  $(P, Q) \in GQ_{2n}((A, \Lambda) \ltimes q)$ , then  $P \in GQ_{2n}(A, \Lambda)$  clearly. By Lemma 3.2 (2), we obtain  $(I, P^{-1}Q) \in GQ_{2n}((A, \Lambda) \ltimes q)$ . Then by Lemma 3.4,  $I + P^{-1}Q \in GQ_{2n}(A, \Lambda, q)$ . □

We define the map

$$\psi : GQ_{2n}((A, \Lambda) \ltimes q) \longrightarrow GQ_{2n}(A, \Lambda) \ltimes GQ_{2n}(A, \Lambda, q);$$

$$(P, Q) \longmapsto (P, I + P^{-1}Q).$$

The well-definedness follows from Lemma 3.5.

**Lemma 3.6.** *The map  $\psi$  is a homomorphism.*

*Proof.* If  $(P, Q)$  and  $(P', Q')$  belong to  $GQ_{2n}((A, \Lambda) \ltimes q)$ , then by (2.2),

$$\psi((P, Q)(P', Q')) = (PP', I + P'^{-1}Q + P'^{-1}P^{-1}QP' + P'^{-1}P^{-1}QQ').$$

On the other hand,

$$\begin{aligned}
 \psi(P, Q)\psi(P', Q') &= (P, I + P^{-1}Q) \cdot (P', I + P'^{-1}Q') \\
 &= (PP', P'^{-1}(I + P^{-1}Q)P'(I + P'^{-1}Q')) \text{ by (2.1)} \\
 &= (PP', I + P'^{-1}Q + P'^{-1}P^{-1}QP' + P'^{-1}P^{-1}QQ').
 \end{aligned}$$

Thus  $\psi((P, Q)(P', Q')) = \psi(P, Q)\psi(P', Q')$ .  $\square$

**Lemma 3.7.** *If  $A \in \text{GQ}_{2n}(A, \Lambda)$  and  $B \in \text{GQ}_{2n}(A, \Lambda, \mathfrak{q})$ , then  $(A, AB - A) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ .*

*Proof.* If  $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  with  $\alpha, \beta, \gamma, \delta \in M_{n,n}(\mathfrak{q})$ , then the equality

$$\begin{aligned}
 &\begin{pmatrix} (I, \alpha - I) & (O, \beta) \\ (O, \gamma) & (I, \delta - I) \end{pmatrix} \begin{pmatrix} \overline{(I, \delta - I)} & \overline{\lambda(O, \beta)} \\ \overline{\lambda(O, \gamma)} & \overline{(I, \alpha - I)} \end{pmatrix} \\
 &= \begin{pmatrix} (I, \alpha\bar{\delta} + \bar{\lambda}\beta\bar{\gamma} - I) & (O, \lambda\alpha\bar{\beta} + \beta\bar{\alpha}) \\ (O, \gamma\bar{\delta} + \bar{\lambda}\delta\bar{\gamma}) & (I, \lambda\gamma\bar{\beta} + \delta\bar{\alpha} - I) \end{pmatrix} = \begin{pmatrix} (I, O) & (O, O) \\ (O, O) & (I, O) \end{pmatrix}.
 \end{aligned}$$

holds. By Lemma 2.4,  $(I, B - I) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ .

Next, if  $A = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$  with  $x, y, u, v \in M_{n,n}(A)$ , then  $(A, O)$  clearly belong to  $\text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$ . Thus by (2.2),

$$(A, AB - A) = (A, O)(I, B - I) \in \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q}).$$

$\square$

We define the map

$$\phi : \text{GQ}_{2n}(A, \Lambda) \ltimes \text{GQ}_{2n}(A, \Lambda, \mathfrak{q}) \longrightarrow \text{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q});$$

$$(A, B) \longmapsto (A, AB - A).$$

The well-definedness of the map follows from Lemma 3.7.

**Lemma 3.8.** *The map  $\phi$  is a homomorphism.*

*Proof.* If  $(A, B)$  and  $(A', B')$  belong to  $\text{GQ}_{2n}(A, \Lambda) \ltimes \text{GQ}_{2n}(A, \Lambda, \mathfrak{q})$ , then by (2.1),

$$\phi((A, B) \cdot (A', B')) = \phi(AA', A'^{-1}BA'B') = (AA', ABA'B' - AA').$$

On the other hand,

$$\begin{aligned}
& \phi(A, B)\phi(A', B') \\
&= (A, AB - A)(A', A'B' - A') \\
&= (AA', A(A'B' - A') + (AB - A)A' + (AB - A)(A'B' - A')) \\
&= (AA', AA'B' - AA' + ABA' - AA' + ABA'B' - ABA' - AA'B' + AA') \\
&= (AA', ABA'B' - AA').
\end{aligned}$$

Thus  $\phi((A, B)(A', B')) = \phi(A, B)\phi(A', B')$ .  $\square$

**Lemma 3.9.** *The compositions  $\psi \circ \phi$  and  $\phi \circ \psi$  of the maps  $\psi$  and  $\phi$  are the identity maps.*

*Proof.* By definition,  $(\psi \circ \phi)(A, B) = \psi(A, AB - A) = (A, B)$  and  $(\phi \circ \psi)(A, Q) = \phi(A, I - A^{-1}Q) = (A, Q)$ .  $\square$

We have shown

$$\mathrm{GQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q}) \cong \mathrm{GQ}_{2n}(A, \Lambda) \ltimes \mathrm{GQ}_{2n}(A, \Lambda, \mathfrak{q}).$$

We define the map

$$\psi_E : \mathrm{EQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q}) \longrightarrow \mathrm{EQ}_{2n}(A, \Lambda) \ltimes \mathrm{EQ}_{2n}(A, \Lambda, \mathfrak{q})$$

to be the restriction of  $\psi$ , and

$$\phi_E : \mathrm{EQ}_{2n}(A, \Lambda) \ltimes \mathrm{EQ}_{2n}(A, \Lambda, \mathfrak{q}) \longrightarrow \mathrm{EQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$$

to be restriction of  $\phi$ .

The well-definedness of  $\psi_E$  is checked: for example in the case of

$$H(\varepsilon_{ij}(x)) = \left( \begin{array}{c|ccc} 1 & & x_{ij} & \\ & \ddots & & \\ & & 1 & \\ \hline & & & 1 \\ & & & -\bar{x}_{ji} & \ddots \\ & & & & 1 \end{array} \right) \in \mathrm{EQ}_{2n}((A, \Lambda) \ltimes \mathfrak{q})$$

with  $(x_{ij} = (a_{ij}, q_{ij}))$ ,

$$\begin{aligned}
& \psi_E(H(\varepsilon_{ij}(x))) \\
&= \left( \left( \begin{array}{ccc|ccc} 1 & & a_{ij} & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & -\bar{a}_{ji} & & 1 \end{array} \right), \left( \begin{array}{ccc|ccc} 1 & & q_{ij} & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & -\bar{q}_{ji} & & 1 \end{array} \right) \right) \\
& \in \text{EQ}_{2n}(A, \Lambda) \ltimes \text{EQ}_{2n}(A, \Lambda, \mathbf{q}).
\end{aligned}$$

The well-definedness of  $\phi_E$  is checked as follows. For example in the case of

$$(H(\varepsilon_{ij}(a)), \varepsilon_{i,n+j}(q)) \in \text{EQ}_{2n}(A, \Lambda) \ltimes \text{EQ}_{2n}(A, \Lambda, \mathbf{q}),$$

by (2.2), the equality

$$\begin{aligned}
& \phi_E(H(\varepsilon_{ij}(a)), \varepsilon_{i,n+j}(q)) \\
&= \left( \left( \begin{pmatrix} I_n + (a_{ij}) & 0 \\ 0 & I_n + (-\bar{a}_{ji}) \end{pmatrix}, \begin{pmatrix} 0 & (q_{ij}) + (\lambda \bar{q}_{ji}) + (-\lambda a \bar{q}_{ii}) \\ 0 & 0 \end{pmatrix} \right) \right) \\
&= \left( \left( \begin{pmatrix} I_n + (a_{ij}) & 0 \\ 0 & I_n + (-\bar{a}_{ji}) \end{pmatrix}, O \right), I, \begin{pmatrix} 0 & (q_{ij}) + (-\lambda \bar{q}_{ji}) \\ 0 & 0 \end{pmatrix} \right)
\end{aligned}$$

holds. Since these two matrices belong to  $\text{EQ}_{2n}((A, \Lambda) \ltimes \mathbf{q})$ ,

$$\phi_E(H(\varepsilon_{ij}(a)), \varepsilon_{i,n+j}(q)) \in \text{EQ}_{2n}((A, \Lambda) \ltimes \mathbf{q}).$$

By Lemma 3.9, the compositions  $\psi_E \circ \phi_E$  and  $\phi_E \circ \psi_E$  of the maps  $\psi_E$  and  $\phi_E$  are clearly the identity maps. Thus, we have shown

$$\text{EQ}_{2n}((A, \Lambda) \ltimes \mathbf{q}) \cong \text{EQ}_{2n}(A, \Lambda) \ltimes \text{EQ}_{2n}(A, \Lambda, \mathbf{q}).$$

#### REFERENCES

- [1] A. Bak, *K-Theory of Forms*, Princeton University Press, Princeton, 1981.