

On products of countable tightness

Yoshio Tanaka (Tokyo Gakugei Univ.)

Introduction and definitions: The results in this paper are due to a joint paper with Gary Gruenhage (Auburn Univ., U.S.A.), " Products of k -spaces and spaces of countable tightness ", to appear in Trans. Amer. Math. Soc.

If a space Y has countable tightness, not much can be said about the tightness of Y^2 . We consider what must be true if Y^2 has countable tightness and Y is a closed image of some " nice " space X under a map f . We prove some fairly general theorems concerning the behavior of the map f , and then we apply these results to more special cases.

All our spaces are assumed to be regular and T_1 . We recall some basic definitions.

A space X has the weak topology with respect to a cover $\mathcal{C}$ of (not necessarily closed) subsets, if a subset A of X is closed in X whenever $A \cap C$ is closed in C for each $C \in \mathcal{C}$.

A space X is a k -space (resp. sequential space), if X has the weak topology with respect to its compact (resp. compact metric) subsets. Thus every sequential space is k .

The tightness [1], $t(X)$, of a space X is the least cardinal α such that whenever $A \subset X$ and $x \in \overline{A}$, there is a subset $B \subset A$ with $|B| \leq \alpha$ and $x \in \overline{B}$. If $t(X) \leq \omega$, then

X is said to have countable tightness. It is easy to show that $t(X) \leq \omega$ if and only if X has the weak topology with respect to its countable subsets. Thus every sequential space has countable tightness.

A space X is strongly collectionwise Hausdorff if whenever $\{x_\alpha; \alpha \in A\}$ is a closed discrete subset of X , there exists a discrete collection $\{U_\alpha; \alpha \in A\}$ of open subsets such that $x_\alpha \in U_\alpha$ for each $\alpha \in A$. Note that every collectionwise normal space is strongly collectionwise Hausdorff.

1. General results. Let c denote the cardinality of the continuum. A space X is called c -compact, if every subset of A with cardinality c has an accumulation point in X .

Theorem 1.1. Suppose that $f: X \rightarrow Y$ is closed, with X strongly collectionwise Hausdorff. If $t(Y^2) \leq \omega$, then the boundary, $\partial f^{-1}(y)$, of $f^{-1}(y)$ is c -compact.

We remark that, if Y^2 is a k -space with $t(Y) \leq \omega$, then $t(Y^2) \leq \omega$. Thus, assuming the continuum hypothesis (CH), we have

Corollary 1.2. (CH). Suppose that $f: X \rightarrow Y$ is closed, with X paracompact. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k -space with $t(Y) \leq \omega$, or $t(Y^2) \leq \omega$.

We don't know whether the CH assumption in Corollary 1.2 can be omitted or not. However, in case where Y is sequential,

we can omit (CH) as will be seen in Corollary 1.4.

Theorem 1.3. Suppose that $f: X \rightarrow Y$ is closed with X strongly collectionwise Hausdorff and Y is sequential. If $t(Y^2) \leq \omega$, then each $\bigcap f^{-1}(y)$ is ω_1 -compact.

Corollary 1.4. Suppose that $f: X \rightarrow Y$ is closed with X paracompact, Y sequential. If $t(Y^2) \leq \omega$, then each $\bigcap f^{-1}(y)$ is Lindelöf.

The following example shows that the assumption " Y^2 is a k -space" is not sufficient to obtain " $\bigcap f^{-1}(y)$ is Lindelöf" in Corollary 1.2.

Example 1.5. There exists $f: X \rightarrow Y$ closed with X locally compact and paracompact, such that Y^2 is a k -space, but $\bigcap f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

Indeed, for each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $[0, \omega_1]$. Let X be the free union of $\{S(\alpha); \alpha < \omega_1\}$. Let Y be the space obtained from X by identifying the point ω_1 in each copy to a single point ∞ . Let $f: X \rightarrow Y$ be the quotient map. Then X is paracompact and locally compact, f is closed, and $f^{-1}(\infty)$ is not Lindelöf. We can prove that Y^2 is a k -space.

2. Applications. A collection $\mathcal{N}$ of (not necessarily open) subsets of a space X is a k -network for X if, whenever $C \subseteq U$ with C compact and U open, then $C \subseteq \bigcup \mathcal{F} \subseteq U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{N}$. An $\mathcal{H}_o$ -space is a space with a countable k -network, and an $\mathcal{H}$ -space is a space with a σ -locally finite k -network. The concept of $\mathcal{H}_o$ -spaces;

$\mathcal{K}$ -spaces is introduced by E. Michael (5); P. O'Meara (8).

We say that X is a locally $\mathcal{K}_0$ -space if each point of X has a neighborhood which is an $\mathcal{K}_0$ -space.

Theorem 2.1. (CH). Let $f: X \rightarrow Y$ be a closed map.

Let X be a paracompact, locally $\mathcal{K}_0$ -space. Then the following are equivalent:

- (a) $t(Y^2) \leq \omega$.
- (b) each $\partial f^{-1}(y)$ is Lindelöf.
- (c) Y is locally $\mathcal{K}_0$.
- (d) Y is locally separable.

Corollary 2.2. Let $f: X \rightarrow Y$ be a closed map with X

locally separable, metric. Then the following are equivalent:

- (a) $t(Y^2) \leq \omega$.
- (b) each $\partial f^{-1}(y)$ is Lindelöf.
- (c) Y is locally separable.
- (d) Y is locally Lindelöf.
- (e) Y is an $\mathcal{K}$ -space.

A decreasing sequence (A_n) in a space X is a k-sequence (7), if $K = \bigcap_{n=1}^{\infty} A_n$ is compact and every neighborhood of K contains some A_n . A space Y is a bi-k-space (7) if, whenever a filter base $\mathcal{F}$ accumulating at $y \in Y$, then there exists a k-sequence (A_n) in Y such that $y \in \overline{F \cap A_n}$ for all $n \in \mathbb{N}$ and all $F \in \mathcal{F}$. It is shown that (7) Y is a bi-k-space if and only if Y is a bi-quotient image of a paracompact M-space X . Then, by (13), spaces of pointwise countable type are bi-k.

Recall that a space X is a k_ω -space (6), if it has the weak topology with respect to a countable covering of compact subsets of X . For a space Y , we shall say that Y is a locally k_ω -space, if each point of Y has a neighborhood whose closure is a k_ω -space.

Theorem 2.3. (CH). Let $f: X \rightarrow Y$ be a closed map with X paracompact bi- k . If $t(Y) \leq \omega$, then the following are equivalent. When Y is sequential, the CH assumption can be omitted.

- (a) Y^2 is a k -space.
- (b) Y is locally k_ω , or each $\bigcap f^{-1}(y)$ is compact.
- (c) Y is locally k_ω , or bi- k .

Corollary 2.4. Let $f: X \rightarrow Y$ be a closed map with X or Y sequential. Let X be a paracompact space of pointwise countable type. Then Y^2 is sequential if and only if Y is locally k_ω , or bi- k .

Theorem 2.5. Let $f: X \rightarrow Y$ be a closed map with X a paracompact $\mathcal{K}$ -space. Then Y^2 is a k -space if and only if Y is metrizable, or Y is an $\mathcal{K}$ -space which is locally k_ω .

References

1. A. V. Arhangel'skii, The frequency spectrum of a topological space and the classification of spaces, Soviet Math. Dokl. 13 (1972), 1185-1189.
2. G. Gruenhage, k -spaces and products of closed images of metric spaces, Proc. Amer. Math. Soc. 80(1980), 478-482.
3. V. I. Malyhin, On the tightness and suslin number in $\exp X$ and in a product of spaces, Soviet Math. Dokl. 13(1972), 496-499.
4. E. Michael, A note on closed maps and compact sets, Israel J. Math. 2(1964), 173-176.
5. ———, $\mathcal{K}_0$ -spaces, J. Math. Mech. 15(1966), 983-1002.
6. ———, Bi-quotient maps and cartesian products of quotient maps, Ann. Inst. Fourier (Grenoble), 18(2)(1968), 287-302.
7. ———, A quintuple quotient quest, General Topology and Appl. 2(1972), 91-138.
8. P. O'Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc. 29(1971), 183-189.
9. Y. Tanaka, A characterization for the products of k - and $\mathcal{K}_0$ -spaces and related results, Proc. Amer. Math. Soc. 59(1976), 149-154.
10. ———, Some necessary conditions for products of k -spaces, Bull. of Tokyo Gakugei Univ. Ser. IV, 30(1978), 1-16.

11. ———, Closed maps on metric spaces, *Topology and its Applications*, 11(1980), 87-92.
12. ———, Point-countable k -systems and products of k -spaces, to appear in *Pacific J. Math.*
13. H. Wicke, On the Hausdorff open continuous images of Hausdorff paracompact p -spaces, *Proc. Amer. Math. Soc.* 22(1969), 136-140.