

The scaling limit of eigenfunctions for 1d random Schrödinger operator

Fumihiko Nakano *

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Abstract

We report our results on the scaling limit of the eigenvalues and the corresponding eigenfunctions for the 1-d random Schrödinger operator with random decaying potential. The formulation of the problem is based on the paper by Rifkind-Virag [9].

1 Introduction

In this note we consider the following one-dimensional Schrödinger operator with random decaying potential :

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t)$$

where $a \in C^\infty(\mathbf{R})$, $a(-t) = a(t)$, $a(t)$ is monotone decreasing for $t > 0$ and

$$a(t) = t^{-\alpha}(1 + o(1)), \quad t \rightarrow \infty$$

for some $\alpha > 0$. $F \in C^\infty(M)$ is a smooth function on a torus M such that

$$\langle F \rangle := \int_M F(x)dx = 0$$

*Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima-ku, Tokyo, 171-8588, Japan. e-mail : fumihiko@math.gakushuin.ac.jp

and $\{X_t\}_{t \in \mathbf{R}}$ is the Brownian motion on M . Since $a(t)F(X_t)$ is a compact perturbation with respect to $(-\Delta)$, the spectrum $\sigma(H) \cap (-\infty, 0)$ on the negative real axis is discrete. The spectrum $\sigma(H) \cap [0, \infty)$ on the positive real axis is [4] :

$$\sigma(H) \cap [0, \infty) \text{ is } \begin{cases} \text{a.c.} & (\alpha > 1/2) \\ \text{p.p. on } [0, E_c] \text{ and s.c. on } [E_c, \infty) & (\alpha = 1/2) \\ \text{p.p.} & (\alpha < 1/2) \end{cases}$$

where E_c is a deterministic constant. For the level statistics problem, we consider the point process ξ_{L, E_0} composed of the rescaling eigenvalues $\{L(\sqrt{E_j(L)} - \sqrt{E_0})\}_j$ of the finite box Dirichlet Hamiltonian $H_L := H|_{[0, L]}$ whose behavior as $L \rightarrow \infty$ is given by [3, 6, 8]

$$\xi_{L, E_0} \xrightarrow{d} \begin{cases} Clock(\theta(E_0)) & (\alpha > 1/2) \\ Sine(\beta(E_0)) & (\alpha = 1/2) \\ Poisson(d\lambda/\pi) & (\alpha < 1/2) \end{cases}$$

where $Clock(\theta) := \sum_{n \in \mathbf{Z}} \delta_{n\pi + \theta}$, is the clock process for some random variable θ on $[0, \pi)$, and $Sine(\beta)$ is the Sine $_{\beta}$ -process which is the bulk scaling limit of the Gaussian beta emsemble [10]. For $\alpha = 1/2$, $\beta(E_0) = \tau(E_0)^{-1}$ is equal to the reciprocal of the Lyapunov exponent $\tau(E_0)$ such that the solution to the Schrödinger equation $H\varphi = E\varphi$ has the power-law decay : $\varphi(x) \simeq |x|^{-\tau(E)}$, $|x| \rightarrow \infty$. Since $\lim_{E_0 \downarrow 0} \beta(E_0) = 0$ and $\lim_{E_0 \uparrow \infty} \beta(E_0) = \infty$, small (resp. large) E_0 corresponds to small (resp. large) repulsion of eigenvalues, which is consistent to the following fact [1, 7] :

$$Sine(\beta) \xrightarrow{d} \begin{cases} Poisson(d\lambda/\pi) & (\beta \downarrow 0) \\ Clock(unif[0, \pi)) & (\beta \uparrow \infty) \end{cases}$$

In this note, we consider the scaling limit of the measure corresponding to the eigenfunction of H_L under the formulation studied by Rifkind-Virag [9]. To formulate the problem, we need some notations. Let $\{E_j(L)\}_j$ be the positive eigenvalues of H_L , and $\{\psi_{E_j(L)}^{(L)}\}$ be the corresponding eigenfunctions. We consider the associated random probability measure $\mu_{E_j(L)}^{(L)}$ on $[0, 1]$.

$$\mu_{E_j(L)}^{(L)}(dt) := C \left(|\psi_{E_j(L)}^{(L)}(Lt)|^2 + \frac{1}{E_j(L)} \left| \frac{d}{dt} \psi_{E_j(L)}^{(L)}(Lt) \right|^2 \right) dt.$$

Let $J := [a, b] (\subset (0, \infty))$ be an interval, $\mathcal{E}_J^{(L)} := \{E_j(L)\}_j \cap J$ be the eigenvalues of H_L on J , and $E_J^{(L)}$ be the uniform distribution on $\mathcal{E}_J^{(L)}$. Our aim is to consider the large L limit of the eigenvalue-eigenvector pairs :

$$\mathbf{Q} : \left(E_J^{(L)}, \mu_{E_J^{(L)}}^{(L)} \right) \xrightarrow{d} ?$$

For d -dimensional discrete random Schrödinger operator, if J is in the localized region, we have [2, 5]

$$\left(E_J^{(L)}, \mu_{E_J^{(L)}}^{(L)} \right) \xrightarrow{d} (E_J, \delta_{\text{unif}[0,1]^d})$$

where E_J is the random variable obeying $\frac{1_J(E)}{N(J)} dN(E)$, where dN is the density of states measure. Rifkind-Virag studied the 1-d discrete Schrödinger operator with critical decaying coupling constant, and obtained that the limit of $\mu_{E_J^{(L)}}^{(L)}$ is given by an exponential Brownian motion [9] :

$$\left(E_J^{(L)}, \mu_{E_J^{(L)}}^{(L)} \right) \xrightarrow{d} \left(E_J, \frac{\exp\left(2\mathcal{Z}_{\tau(E_J)(t-U)} - 2\tau(E_J)|t-U|\right) dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_J)(s-U)} - 2\tau(E_J)|s-U|\right) ds} \right).$$

To state our result, we need notations further. Let $N(E) := \pi^{-1}\sqrt{E}$ be the integrated density of states, $N(J) := N(b) - N(a)$, and

$$\tau(E) := \frac{1}{8E} \int_M |\nabla(L + 2i\sqrt{E})^{-1}F|^2 dx.$$

where L is the generator of (X_t) . Moreover, let E_J be the random variable whose distribution is equal to $N(J)^{-1}1_J(E)dN(E)$, let U be the uniform distribution on $[0, 1]$, and let $\mathcal{Z}$ be the 2-sided Brownian motion, where E_J , U , and $\mathcal{Z}$ are independent.

Theorem 1.1

$$\begin{aligned} & \left(E_J^{(L)}, \mu_{E_J^{(L)}}^{(L)} \right) \\ & \xrightarrow{d} \begin{cases} (E_J, 1_{[0,1]}(t)dt) & (\alpha > 1/2) \\ \left(E_J, \frac{\exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{t}{U}} - 2\tau(E_J)\left|\log\frac{t}{U}\right|\right) dt}{\int_0^1 \exp\left(2\mathcal{Z}_{\tau(E_J)\log\frac{s}{U}} - 2\tau(E_J)\left|\log\frac{s}{U}\right|\right) ds} \right) & (\alpha = 1/2) \\ (E_J, \delta_{\text{unif}[0,1]}(dt)) & (\alpha < 1/2) \end{cases} \end{aligned}$$

When $\alpha < 1/2$, this result is the same as that in [2, 5], while for $\alpha > 1/2$ this result is natural. For $\alpha = 1/2$, this result implies that, the localization center U of the eigenfunction ψ is uniformly distributed and ψ has the power law decay around U with Brownian fluctuation. Since $\lim_{E \downarrow 0} \tau(E) = \infty$ and $\lim_{E \uparrow \infty} \tau(E) = 0$, ψ is localized (resp. delocalized) for $E \downarrow 0$ (resp. $E \uparrow \infty$) which is consistent with the previous picture.

2 Sketch of Proof

For the proof, we mostly follow the strategy in [2, 5, 9], except for some technical points.

2.1 Step 1 : renormalize the radial coordinate

In what follows, we describe the solution x_t to the equation $Hx_t = \kappa^2 x_t$ in terms of the Prüfer variables :

$$\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = r_t(\kappa) \begin{pmatrix} \sin \theta_t(\kappa) \\ \cos \theta_t(\kappa) \end{pmatrix}$$

Introducing $\rho_t(\kappa)$ defined by $r_t(\kappa) := \exp(\rho_t(\kappa))$, we have

$$\rho_t(\kappa) = \frac{1}{2\kappa} \operatorname{Im} \int_0^t e^{2i\theta_s(\kappa)} a(s) F(X_s) ds$$

Let $\kappa_\lambda := \kappa_0 + \frac{\lambda}{n}$, $\kappa_0 := \sqrt{E_0}$, $\tilde{\rho}_t^{(n)}(\kappa) := \rho_{nt}(\kappa) - \langle Fg_\kappa \rangle \int_0^n a(s)^2 ds$, $g_\kappa := (L + 2i\kappa)^{-1}F$, $t \in [0, 1]$. We then have

Lemma 2.1 *If $\alpha = 1/2$, then*

$$\begin{aligned} \tilde{\rho}_t^{(n)}(\kappa_\lambda) &\xrightarrow{d} \tilde{\rho}_t(\lambda), \quad t \in [0, 1], \text{ locally uniformly} \\ d\tilde{\rho}_t^{(n)}(\kappa_\lambda) &= \frac{\tau(\kappa_0^2)}{t} dt + \sqrt{\frac{\tau(\kappa_0^2)}{t}} dB_t^\lambda, \quad t > 0 \end{aligned}$$

where $\{B_t^\lambda\}$ is a family of Brownian motion.

2.2 Step 2 : limit of the local version

Let $\Xi^{(n)}$ be the local version of our problem :

$$\Xi^{(n)} := \sum_j \delta \left(n \left(\sqrt{E_j(n)} - \sqrt{E_0} \right), \mu_{E_j(n)}^{(n)} \right)$$

It then follows that

Lemma 2.2 $\Xi^{(n)} \xrightarrow{d} \Xi$, where

$$\Xi = \begin{cases} \sum_{j \in \mathbf{Z}} \delta_{j\pi + \theta} \otimes \delta_{1_{[0,1]}(t)dt} & (\alpha > 1/2) \\ \sum_{\lambda: \text{Sine}_\beta} \delta_\lambda \otimes \delta \left(\frac{\exp(2\tilde{\rho}_t(\lambda))dt}{\int_0^1 \exp(2\tilde{\rho}_s(\lambda))ds} \right) & (\alpha = 1/2) \\ \sum_{j \in \mathbf{Z}} \delta_{P_j} \otimes \delta_{\tilde{P}_j}, & (\alpha < 1/2) \end{cases}$$

where $\{P_j\} : \text{Poisson}(d\lambda/\pi)$, $\{\tilde{P}_j\} : \text{Poisson}(1_{[0,1]}(t)dt)$. The intensity measure of Ξ is given by

$$\begin{aligned} & \mathbf{E} [G(\lambda, \nu) d\Xi(\lambda, \nu)] \\ &= \frac{1}{\pi} \begin{cases} \int d\lambda \mathbf{E} [G(\lambda, 1_{[0,1]}(t)dt)] & (\alpha > 1/2) \\ \int d\lambda \mathbf{E} \left[G \left(\lambda, \frac{\exp \left(2\mathcal{Z}_{\tau(E_0)} \log \frac{t}{\tilde{t}} - 2\tau(E_0) \log \left| \frac{t}{\tilde{t}} \right| \right) dt}{\int_0^1 \exp \left(2\mathcal{Z}_{\tau(E_0)} \log \frac{s}{\tilde{s}} - 2\tau(E_0) \log \left| \frac{s}{\tilde{s}} \right| \right) ds} \right) \right] & (\alpha = 1/2) \\ \int d\lambda \mathbf{E} [G(\lambda, \delta_U)] & (\alpha < 1/2) \end{cases} \end{aligned}$$

where $U := \text{unif}[0, 1]$.

2.3 Step 3 : averaging over the reference energy

Following [9], we introduce

$$\begin{aligned} g_1(x) &:= (1 - |x|)1(|x| \leq 1) \\ G_L(E) &:= \sum_{E_j(L) \in J} g_1 \left(L \left(\sqrt{E_j(L)} - \sqrt{E_0} \right) \right) \cdot g_2 \left(E_j(L), \mu_{E_j(L)}^{(L)} \right) \end{aligned}$$

where $g_2 \in C_b(\mathbf{R} \times \mathcal{P}(0, 1))$. We compute $\int \frac{dN(E)}{N(J)} \mathbf{E}[G_L(E)]$ by the following two ways, and then equate them by the Fubini theorem, which leads to the

conclusion.

(1) Since $\int \frac{dN(E)}{N(J)} g_1 = 1/(L\pi)$, we have

$$\begin{aligned} & \mathbf{E} \left[\int \frac{dN(E)}{N(J)} G_L(E) \right] \\ &= \frac{1}{N(J)} \frac{1}{\pi L} \mathbf{E} \left[\sum_{E_j(L) \in J} g_2 \left(E_j(L), \mu_{E_j(L)}^{(L)} \right) \right] \\ &= \mathbf{E} \left[\frac{1}{\#\{\text{eigenvalues of } H_L \text{ on } J\} (1 + o(1))} \cdot \frac{1}{\pi} \cdot \sum_{E_j(L) \in J} g_2 \left(E_j(L), \mu_{E_j(L)}^{(L)} \right) \right] \end{aligned}$$

(2)

$$\begin{aligned} & \int \frac{dN(E)}{N(J)} \mathbf{E}[G_L(E)] \\ &= \int \frac{dN(E)}{N(J)} \mathbf{E} \left[\sum_{E_j(L) \in J} g_1 \left(L \left(\sqrt{E_j(L)} - \sqrt{E_0} \right) \right) \cdot g_2 \left(E_j(L), \mu_{E_j(L)}^{(L)} \right) \right] \\ &\sim \int \frac{dN(E)}{N(J)} \mathbf{E} \left[\int g_1(\lambda) g_2(E, \mu) d\Xi^{(L)}(\lambda, \mu) \right] \\ &\rightarrow \int \frac{dN(E)}{N(J)} \mathbf{E} \left[\int g_1(\lambda) g_2(E, \mu) d\Xi(\lambda, \mu) \right] \\ &= \int \frac{dN(E)}{N(J)} \frac{1}{\pi} \left\{ \begin{array}{ll} \int d\lambda \mathbf{E} [g_2(E, 1_{[0,1]}(t) dt)] & (\alpha > 1/2) \\ \int d\lambda \mathbf{E} \left[g_2 \left(E, \frac{\exp \left(2\mathcal{Z}_{\tau(E_0)} \log \frac{t}{T} - 2\tau(E_0) \log \left| \frac{t}{T} \right| \right) dt}{\int_0^1 \exp \left(2\mathcal{Z}_{\tau(E_0)} \log \frac{s}{T} - 2\tau(E_0) \log \left| \frac{s}{T} \right| \right) ds} \right) \right] & (\alpha = 1/2) \\ \int d\lambda \mathbf{E} [g_2(E, \delta_U)] & (\alpha > 1/2) \end{array} \right. \end{aligned}$$

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References

- [1] Allez, R., Dumaz, L., : From sine kernel to Poisson statistics, *Elec. J. Prob.* **19**(2014), 1-25.
- [2] Killip, R., Nakano, F., : Eigenfunction statistics in the localized Anderson model, *Annales Henri Poincaré.* **8**, no.1 (2007), 27-36.
- [3] Kotani, S., Nakano, F., : Level statistics for the one-dimensional Schrödinger operator with random decaying potentials, *Interdisciplinary Mathematical Sciences*, Vol. 17(2014), 343-373.
- [4] Kotani, S. Ushiroya, N. : One-dimensional Schrödinger operators with random decaying potentials, *Commun. Math. Phys.* **115**(1988), 247-266.
- [5] Nakano, F., : Distribution of localization centers in some discrete random systems, *Rev. Math. Phys.* **19**(2007), 941-965.
- [6] Nakano, F., : Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble, *J. Stat. Phys.* **156**(2014), 66-93.
- [7] Nakano, F., : Limit of Sine_β and Sch_τ processes, *RIMS Kokyuroku*, **1970**(2015), 83 - 89.
- [8] Kotani, S., Nakano, F., : Poisson statistics for the one-dimensional Schrödinger operator with random decaying potentials, *Elec. J. of Prob.* **22**(2017), no.69, 1-31.
- [9] Rifkind, B., Virág, B. : Eigenvectors of the 1-dimensional critical random Schrödinger operator, *Geom. Funct. Anal.* **28** (2018), 1394-1419.
- [10] Valkó, B., Virág, V. : Continuum limits of random matrices and the Brownian carousel, *Invent. Math.* **177**(2009), 463-508.