

## A Study of the Relativistic Euler Equation

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### 1 Introduction

In this article we study the Cauchy problem to the one-dimensional relativistic Euler equation

$$\begin{aligned}\frac{\partial}{\partial t} \frac{\rho + Pu^2/c^4}{1 - u^2/c^2} + \frac{\partial}{\partial x} \frac{(\rho + P/c^2)u}{1 - u^2/c^2} &= 0, \\ \frac{\partial}{\partial t} \frac{(\rho + P/c^2)u}{1 - u^2/c^2} + \frac{\partial}{\partial x} \frac{P + \rho u^2}{1 - u^2/c^2} &= 0,\end{aligned}\tag{1.1}$$

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x).\tag{1.2}$$

Here  $c$  is a positive constant, the speed of light, and  $P$  is a given function of  $\rho$ . The equation (1.1) governs the one dimensional motion of a perfect gas in the Minkowski space-time. When  $c \rightarrow \infty$ , (1.1) tends to the usual Euler equation of gas dynamics

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (P + \rho u^2)_x &= 0.\end{aligned}\tag{1.3}$$

Many mathematical investigations for this non-relativistic Euler equation were done. But the first mathematical investigation for the relativistic Euler equation (1.1) was done recently by Smoller and Temple [6]. They assume  $P = \sigma^2 \rho$ , where  $\sigma$  is a positive constant  $< c$ . Under this assumption, they showed that if the initial data  $\rho_0(x)$  and  $u_0(x)$  satisfy

$$T.V. \log \rho_0 < \infty, \quad T.V. \log \frac{c + u_0}{c - u_0} < \infty,$$

then there exists a global weak solution to the Cauchy problem (1.1)(1.2). The result was obtained by Glimm's scheme and it is the relativistic version of Nishida's result [5] for the non-relativistic problem.

However we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

$$\begin{aligned} P &= Kc^5 f(y), & \rho &= Kc^3 g(y) \\ f(y) &= \int_0^y \frac{q^4}{\sqrt{1+q^2}} dq, \\ g(y) &= 3 \int_0^y q^2 \sqrt{1+q^2} dq. \end{aligned}$$

For this equation of state, we have  $P \sim \frac{c^2}{3}\rho$  as  $\rho \rightarrow \infty$  but  $P \sim \frac{1}{5}K^{2/3}\rho^{5/3}$  as  $\rho \rightarrow 0$ . So we assume the following properties of the function  $P(\rho)$ :

(A):

$$P(\rho) > 0, \quad 0 < dP/d\rho < c^2, \quad 0 < d^2P/d\rho^2$$

for  $\rho > 0$ , and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}/c^2]_1)$$

as  $\rho \rightarrow 0$ . Here  $A$  and  $\gamma$  are positive constants and

$$\gamma = 1 + \frac{2}{2N+1},$$

$N$  being a positive integer, and  $[X]_1$  denotes a convergent power series of the form  $\sum_{k \geq 1} a_k X^k$ .

The result which we want to generalize to the relativistic problem is those by G.-Q. Chen et al [2]. So we assume that the initial data  $\rho_0(x)$ ,  $u_0(x)$  satisfy

$$0 \leq \rho_0(x) \leq M_0, \quad \left| \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \right| \leq M_0.$$

A weak solution of (1.1)(1.2) is defined as follows.

We write

$$\begin{aligned} E &= \frac{\rho + Pu^2/c^4}{1 - u^2/c^2}, \\ F &= \frac{(\rho + P/c^2)u}{1 - u^2/c^2}, \\ G &= \frac{P + \rho u^2}{1 - u^2/c^2}, \\ U &= (E, F)^T, \quad f(U) = (F, G)^T. \end{aligned}$$

Then (1.1) can be written as

$$U_t + f(U)_x = 0.$$

Let us denote by  $U_0(x)$  the initial data. Then a weak solution  $U(t, x)$  is a bounded measurable function which satisfies

$$\int \int (U \Phi_t + f(U) \Phi_x) dx dt + \int U_0(x) \Phi(0, x) dx = 0$$

for any test function  $\Phi \in C_0^\infty([0, +\infty) \times R)$ .

## 2 Riemann problems

The Riemann problem is the problem to the special initial data of the form

$$U_0(x) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases}$$

In order to solve this we introduce the Riemann invariants

$$w = x + y, \quad z = x - y$$

where

$$x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho.$$

Then (1.1) is diagonalized as

$$w_t + \lambda_2 w_x = 0, \quad z_t + \lambda_1 z_x = 0,$$

where

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}, \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}.$$

the possible states  $U = U_R$  connected to  $U_L$  on the right by rarefaction waves are

$$R_1 : \quad w = w_L, z > z_L$$

and

$$R_2 : \quad w > w_L, z = z_L.$$

The Rankine Hugoniot jump condition

$$\sigma[U] = [f(U)],$$

where  $[U] = U_R - U_L$ ,  $[f(U)] = f(U_R) - f(U_L)$ , gives the shock curve

$$\frac{(u_R - u_L)^2}{(1 - u_R^2/c^2)(1 - u_L^2/c^2)} = \frac{(\rho_R - \rho_L)(P_R - P_L)}{(\rho_L + P_L/c^2)(\rho_R + P_R/c^2)}.$$

Along this curve we have shocks

$$S_1 : \quad \rho_L < \rho_R, u_R < u_L,$$

$$S_2 : \quad \rho_R < \rho_L, u_R < u_L.$$

The Riemann problem can be solved uniquely by using these rarefaction waves and shock waves and vacuum state. The detailed discussion can be found in J. Chen [1].

If we look at a region of the form

$$\Sigma_B = \{(w, z) \mid -B \leq z \leq w \leq B\},$$

we have the following

**Proposition 1** *If the initial data  $U_L, U_R$  belong to  $\Sigma_B$  for some large  $B$ , then the solution of the Riemann problem is confined to  $\Sigma_B$ .*

Moreover if we consider the image of  $\Sigma_B$  in the  $(E, F)$ -space, we have

**Proposition 2** *The region  $\Sigma_B$  is convex in the  $(E, F)$ -plane.*

*Proof.* Let us consider the above hedge  $F = F(E)$  which corresponds to  $w = B, -B < z < B$ . We have to show  $d^2 F/dE^2 < 0$ . Along the hedge  $w = B$ , we have

$$u = c \tanh \frac{1}{c} (B - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2) \frac{\sqrt{P'}}{\rho + P/c^2}.$$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2} = \lambda_1.$$

Differentiating once more we have

$$\frac{d^2 F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^4} \left( \frac{P''}{2\sqrt{P'}} + \left(1 - \frac{P'}{c^2}\right) \frac{\sqrt{P'}}{\rho + P/c^2} \right) < 0.$$

This was to be seen. QED.

From Proposition 2, we have

**Proposition 3** *If  $U(s), s \in [a, b]$ , is confined to a region  $\Sigma_B$ , then the average*

$$\frac{1}{b-a} \int_a^b U(s) ds$$

*belongs to  $\Sigma_B$ .*

Let us look at the shock wave which connects the left state  $U_L$  to the right state  $U_R$  with the shock speed  $\sigma$ .

The right state  $U_R$  and  $\sigma$  are parametrized by  $\rho = \rho_R$ . Then we have the following fact, which will be used in Section 4.

**Proposition 4** *Along  $S_1(\rho_L < \rho)$ , we have  $d\sigma/d\rho < 0$ , and along  $S_2(\rho < \rho_L)$  we have  $d\sigma/d\rho > 0$ .*

Proof. Without loss of generality we can assume  $u_L = 0$ . Then  $u = u_R$  is given by

$$u = -\sqrt{\frac{[\rho][P]}{(\rho_L + P/c^2)(\rho + P_L/c^2)}},$$

where  $[\rho] = \rho - \rho_L$ ,  $[P] = P - P_L$ . We have

$$\sigma = \frac{[F]}{[E]} = \frac{(\rho + P/c^2)u}{\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2)}.$$

By a direct but tedious computations, we have

$$\begin{aligned} \frac{d\sigma}{d\rho} &= \frac{(\rho + P/c^2)(\rho_L + P_L/c^2)[\rho]X}{2(\rho + Pu^2/c^4 - \rho_L(1 - u^2/c^2))^2 u(\rho_L + P/c^2)^2 (\rho + P_L/c^2)^2}, \\ X &= (\rho + P_L/c^2)(\rho + P/c^2)P'[\rho] + \\ &+ (\rho + P_L/c^2)(-(\rho + P_L/c^2) + [P]/c^2)[P] + \\ &- (\rho_L + P/c^2)[P]^2/c^2. \end{aligned}$$

Since  $P'' > 0$  we know  $[P] \leq P'[\rho]$ . Thus

$$\begin{aligned} X &\geq (\rho + P_L/c^2)(\rho + P/c^2)[P] + \\ &+ (\rho + P_L/c^2)(-(\rho_L + P_L/c^2) + [P]/c^2)[P] + \\ &- (\rho_L + P/c^2)[P]^2/c^2 \\ &= [P](\rho + P_L/c^2)([\rho] + [P]/c^2) + ([\rho] - [P]/c^2)[P]/c^2. \end{aligned}$$

But

$$1 > \frac{[\rho] - [P]/c^2}{[\rho]} = 1 - P'(\rho_L + \theta(\rho - \rho_L))/c^2 > 0.$$

Using this, it is easy to see  $X > 0$  both when  $[\rho] > 0$  and when  $[\rho] < 0$ . Since  $u < 0$ , this completes the proof. QED.

### 3 Entropies

A pair of functions  $\eta$  and  $q$  is called an entropy- entropy flux if it satisfies the equation

$$D_U q = D_U \eta \cdot D_U f. \quad (3.1)$$

Using the Riemann invariants, we can write (3.1) as

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

By eliminating  $q$  from the equation, we get the following second order equation:

$$\frac{\partial^2 \eta}{\partial w \partial z} + Q \left( J \frac{\partial \eta}{\partial w} - \frac{1}{J} \frac{\partial \eta}{\partial z} \right) = 0, \quad (3.2)$$

where

$$\begin{aligned} Q &= \frac{1}{4\sqrt{P'}} \left( 1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} P'' \right), \\ J &= \frac{1 - \sqrt{P'} u / c^2}{1 + \sqrt{P'} u / c^2}. \end{aligned}$$

Since this equation tends to the Euler-Poisson-Darboux equation

$$\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w - z} \left( \frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z} \right) = 0 \quad (3.3)$$

as  $c \rightarrow \infty$ , we shall call (3.2) the relativistic Euler-Poisson-Darboux equation.

Among entropies of (3.3) when  $c = \infty$  the kinetic energy

$$\eta = \frac{1}{2} \rho u^2 + \frac{P}{\gamma - 1} \quad (3.4)$$

plays an important role. Therefore we want to find an entropy of (3.2) which tends to (3.4) as  $c \rightarrow \infty$ . Let us look for an entropy-entropy flux of the form

$$\eta = H(\rho, u^2), \quad q = Q(\rho, u^2)u.$$

Inserting this to the equation it is easy to find an entropy-entropy flux

$$\eta^* = -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2 \left( \frac{\rho + P u^2/c^4}{1 - u^2/c^2} \right), \quad (3.5)$$

$$q^* = \left( -\frac{\Psi(\rho)}{(1 - u^2/c^2)^{1/2}} + c^2 \frac{\rho + P/c^2}{1 - u^2/c^2} \right) u, \quad (3.6)$$

$$\Psi = \exp \left( \int_1^\rho \frac{d\rho}{\rho + P/c^2} + K_0 \right), \quad (3.7)$$

where  $K_0$  is determined so that  $\eta^*$  tends to the kinetic energy (3.4) as  $c = \infty$ . We call the entropy  $\eta^*$  defined by (3.5) the relativistic standard entropy. The important fact is

**Proposition 5** *The Hessian  $D_U^2 \eta^*$  is positive definite. For any fixed  $B$  there is a positive constant  $k$  such that*

$$(\xi | D_U^2 \eta^*(U) \xi) \geq k |\xi|^2,$$

for any  $U \in \Sigma_B$  and  $\xi = (\xi_0, \xi_1)$  with  $|\xi|^2 = \xi_0^2 + \xi_1^2$ .

Proof. The proof is due to direct but tedious calculations. We note

$$\begin{aligned}\frac{\partial \rho}{\partial E} &= \frac{1 + u^2/c^2}{1 - P'u^2/c^4}, \\ \frac{\partial u}{\partial E} &= -\frac{(1 + P'/c^2)(1 - u^2/c^2)u}{(\rho + P/c^2)(1 - P'u^2/c^4)}, \\ \frac{\partial \rho}{\partial F} &= -\frac{2u/c^2}{1 - P'u^2/c^4}, \\ \frac{\partial u}{\partial F} &= \frac{(1 - u^2/c^2)(1 + P'u^2/c^4)}{(\rho + P/c^2)(1 - P'u^2/c^4)}.\end{aligned}$$

Using these, we have

$$\begin{aligned}\frac{\partial \eta^*}{\partial E} &= -\frac{\Psi}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2}} + c^2, \\ \frac{\partial \eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho + P/c^2)(1 - u^2/c^2)^{1/2}}, \\ \frac{\partial^2 \eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(P' + 2P'u^2/c^2 + u^2), \\ \frac{\partial^2 \eta^*}{\partial E \partial F} &= \frac{-\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(2P'/c^2 + 1 + P'u^2/c^4)u, \\ \frac{\partial^2 \eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2}(1 + 3P'u^2/c^4).\end{aligned}$$

Therefore we get

$$\begin{aligned}(\xi | D_U^2 \eta^* \xi) &= \eta_{EE}^* \xi_0^2 + 2\eta_{EF}^* \xi_0 \xi_1 + \eta_{FF}^* \xi_1^2 \\ &= \frac{\Psi/c^2}{(1 - P'u^2/c^4)(1 - u^2/c^2)^{1/2}(\rho + P/c^2)^2} Z, \\ Z &= (P' + 2P'u^2/c^2 + u^2)\xi_0^2 - 2(2P'/c^2 + 1 + P'u^2/c^4)u\xi_0\xi_1 + \\ &\quad + (1 + 3P'u^2/c^4)\xi_1^2 \\ &\geq \frac{2P'(1 - u^2/c^2)^2(1 - P'u^2/c^4)}{A + C + \sqrt{(A - C)^2 + 4B^2}}(\xi_0^2 + \xi_1^2), \\ A &= P' + 2P'u^2/c^2 + u^2, \\ B &= (2P'/c^2 + 1 + P'u^2/c^4)u, \\ C &= 1 + 3P'u^2/c^4.\end{aligned}$$

This completes the proof. QED.

## 4 Construction of approximate solutions

Let us construct approximate solutions using the Godunov scheme. The construction is similar if we use the Lax-Friedrichs scheme.

Suppose that the initial data  $U_0(x)$  is confined to an invariant region  $\Sigma_B$ . Put  $\Lambda_0 = \sup\{|\lambda_j(U)| | j = 1, 2, U \in \Sigma_B\}$ . Fixing  $\Lambda_1 > \Lambda_0$ , we take mesh lengths  $\Delta x, \Delta t$  such that  $\Delta x = \Lambda_1 \Delta t$ . We denote  $\Delta = \Delta x$ .

Let us construct the approximate solution  $U^\Delta(t, x)$ . First we put

$$U_0^\Delta(x) = U_0(x) \chi_{[-1/\Delta, 1/\Delta]}.$$

We define

$$U^\Delta(+0, x) = \frac{1}{2\Delta x} \int_{2j\Delta x}^{(2j+2)\Delta x} U_0^\Delta(x) dx$$

for  $2j\Delta x < x \leq (2j+2)\Delta x$ . Solving the Riemann problem on each interval  $[2(j-1)\Delta, 2(j+1)\Delta]$ , we define  $U^\Delta(t, x)$  for  $0 \leq t < \Delta t$ . Since the Courant-Friedrichs-Levi condition is satisfied, the wave from the center  $2j\Delta$  does not intersect. If  $U^\Delta(t, x)$  for  $0 \leq t < n\Delta t$  has been defined, then we define

$$U^\Delta(n\Delta t, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U^\Delta(n\Delta t - 0, x) dx$$

for  $2j\Delta < x \leq (2j+2)\Delta$ . Solving the Riemann problem, we define  $U^\Delta(t, x)$  for  $n\Delta t \leq t < (n+1)\Delta t$ .

By Proposition 1 and 3, it is inductively guaranteed that  $U^\Delta$  remains in  $\Sigma_B$ , say,

**Proposition 6** *The approximate solution  $U^\Delta(t, x)$  satisfies  $U^\Delta(t, x) \in \Sigma_B$ , therefore,*

$$0 \leq \rho^\Delta(t, x) \leq M, \quad \left| \frac{c}{2} \log \frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)} \right| \leq M.$$

Moreover we shall prove

**Proposition 7** *For any test function  $\Phi$  it holds that*

$$\int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta(x) dx = O(\Delta^{1/2}).$$

In order to prove Proposition 7, we prepare

**Proposition 8** *For any shock wave from  $U_L$  to  $U_R$  with the shock speed  $\sigma$  and for any convex entropy  $\eta$ , we have*

$$\sigma[\eta] - [q] \geq 0,$$

where  $[\eta] = \eta(U_R) - \eta(U_L)$ ,  $[q] = q(U_R) - q(U_L)$ .

Proof. The right state of shocks can be parametrized by  $\rho = \rho_R$ . Putting

$$Q(\rho) = \sigma[\eta] - [q],$$

we shall see  $dQ/d\rho \geq 0$  along  $S_1 : [\rho] > 0$  and  $dQ/d\rho \leq 0$  along  $S_2 : [\rho] < 0$ . Using the equation (3.1) and the differentiation of the Rankine-Hugoniot condition, we have

$$\begin{aligned} \frac{dQ}{d\rho} &= \frac{d\sigma}{d\rho}([\eta] - D_U \eta(U) \cdot [U]) \\ &= -\frac{d\sigma}{d\rho} \int_0^1 \theta(U - U_L) D_U^2 \eta(U_L + \theta(U - U_L)) \cdot (U - U_L) d\theta. \end{aligned}$$

We supposed  $D_U^2 \eta \geq 0$ . By Proposition 4, we know  $d\sigma/d\rho < 0$  on  $S_1$  and  $d\sigma/d\rho > 0$  on  $S_2$ . QED.

Proof of Proposition 7.

We fix  $T$  to consider  $U^\Delta$  on  $0 \leq t \leq T$ . First we shall show

$$\sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, (2j+1)\Delta)|^2 dx \leq C. \quad (4.1)$$

Let us consider the standard entropy  $\eta^*$ . Then we have

$$\begin{aligned} 0 &= \int \eta^*(U(T, x)) dx - \int \eta^*(U(0, x)) dx + L + \Sigma, \\ L &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\eta^*(U(n\Delta t - 0, x)) - \eta^*(U(n\Delta t + 0, (2j+1)\Delta))) dx, \\ \Sigma &= \int_0^T \sum_{shocks} (\sigma[\eta^*] - [q^*]) dt. \end{aligned}$$

We write  $U_0 = U(n\Delta t + 0, (2j+1)\Delta)$ ,  $U_1 = U(n\Delta t - 0, x)$ . Since

$$U_0 = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U_1 dx,$$

we see

$$\begin{aligned} L &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \int_0^1 (1-\theta)(U_1 - U_0) D_U^2 \eta^*(U_0 + \theta(U_1 - U_0)) \cdot (U_1 - U_0) d\theta dx \\ &\geq 0. \end{aligned}$$

On the other hand we have  $\Sigma \geq 0$  from Proposition 8. Thus  $L \leq C, \Sigma \leq C$ .

But from Proposition 5, we have  $D_U^2 \eta^* \geq k$ . Therefore

$$C \geq L \geq \frac{k}{2} \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U_1 - U_0|^2 dx.$$

Thus we get (4.1).

Now let us consider a test function  $\Phi$ . Put

$$J = \int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta dx.$$

Since  $U^\Delta$  is a weak solution on each time strip  $n\Delta t < t < (n+1)\Delta t$ , we have

$$\begin{aligned} J &= \sum_n \int \Phi(n\Delta t, x) (U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx \\ &= J_1 + J_2, \\ J_1 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} \Phi(n\Delta t, j\Delta) (U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx, \\ J_2 &= \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} (\Phi(t, x) - \Phi(n\Delta t, j\Delta)) (U(n\Delta t - 0, x) - U(n\Delta t + 0, x)) dx. \end{aligned}$$

Since

$$U(n\Delta t + 0, x) = \frac{1}{2\Delta} \int_{2j\Delta}^{(2j+2)\Delta} U(n\Delta t - 0, x) dx$$

for  $2j\Delta < x < (2j+2)\Delta$ , we see  $J_1 = 0$ . It follows from (4.1) that

$$\begin{aligned} |J_2| &\leq C\Delta^{1/2} \|\Phi\|_{C^1} \left( \sum_{j,n} \int_{2j\Delta}^{(2j+2)\Delta} |U(n\Delta t - 0, x) - U(n\Delta t + 0, x)|^2 dx \right)^{1/2} \\ &\leq C'\Delta^{1/2}. \end{aligned}$$

Here we have used  $T/\Delta t = O(1/\Delta)$ . QED.

Summing up, we have the following theorem.

**Theorem 1** *The approximate solution  $U^\Delta(t, x)$  satisfies*

$$0 \leq \rho^\Delta(t, x) \leq M, \quad \left| \frac{c}{2} \log \frac{c + u^\Delta(t, x)}{c - u^\Delta(t, x)} \right| \leq M$$

and

$$\int \int (\Phi_t U^\Delta + \Phi_x f(U^\Delta)) dx dt + \int \Phi(0, x) U_0^\Delta(x) = O(\Delta^{1/2})$$

for any test function  $\Phi$ .

We expect that  $U^\Delta$  tends to a weak solution everywhere. For the non-relativistic gas dynamics, this was done by DiPerna [3] and G.Q.Chen et al [2]. In their proof the Darboux formula

$$\eta = \int_z^w ((w-s)(s-z))^N \phi(s) ds$$

which gives solutions of the Euler-Poisson-Darboux equation (3.3),  $\phi$  being arbitrary, plays an important role. Section 6 will be devoted to find such an integral formula for the relativistic Euler-Poisson-Darboux equation (3.2).

## 5 Remark

We note that

$$\begin{aligned}\lambda_2 - \lambda_1 &= \frac{\sqrt{P'}(1 - u^2/c^2)}{1 - u^2 P'/c^4} > 0, \\ \frac{\partial \lambda_1}{\partial z} &= \frac{1 - u^2/c^2}{2(1 - \sqrt{P'}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}\right) > 0, \\ \frac{\partial \lambda_2}{\partial w} &= \frac{1 - u^2/c^2}{2(1 + \sqrt{P'}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}\right) > 0\end{aligned}$$

for  $\rho > 0$  and  $|u| < c$ .

This says that the system is strictly hyperbolic and genuinely nonlinear on  $\rho > 0$ . Therefore the Glimm's theory can be applied if

$$\|U_0(x) - U^*\|_{L^\infty} + T.V.U_0$$

is sufficiently small, where  $U^*$  is a constant state such that  $\rho^* > 0, |u^*| < c$ . But the vacuum may not be covered by this application of the general theorem.

## 6 Generalized Darboux formula

In this section we seek an integration formula for solutions of the relativistic Euler-Poisson-Darboux equation. Let us introduce the variables

$$x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} d\rho.$$

Then the relativistic Euler-Poisson-Darboux equation is

$$(EPD) \quad \eta_{xx} - \eta_{yy} + A(x, y)\eta_y + B(x, y)\eta_x = 0,$$

where

$$\begin{aligned}A(x, y) &= \frac{1}{\sqrt{P'}} \left(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} P''\right) \frac{1 + P'u^2/c^4}{1 - P'u^2/c^4}, \\ B(x, y) &= -\frac{2u/c^2}{1 - P'u^2/c^4} \left(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'} P''\right).\end{aligned}$$

The coefficients  $A$  and  $B$  are of the form

$$\begin{aligned}A &= \frac{2N}{y} + a, \quad a = \frac{y}{c^2} (a_0 + [x^2/c^2, y^2/c^2]_1), \\ B &= -\frac{4N}{N+1} \frac{x}{c^2} (1 + [x^2/c^2, y^2/c^2]_1),\end{aligned}$$

where  $[X, Y]_1$  denotes a convergent power series  $\sum_{j+k \geq 1} c_{jk} X^j Y^k$ . In order to remove the singularity in  $A$ , we use the trick of Weinstein [7]. We introduce the sequence of variables  $\eta_j, j = 0, 1, \dots, N$  by

$$\frac{\partial \eta_j}{\partial y} = y \eta_{j+1},$$

or

$$\eta_j(x, y) = I \eta_{j+1}(x, y) = \int_0^y Y \eta_{j+1}(x, Y) dY,$$

where  $\eta_0 = \eta$ . The sequence of formal integro-differential operators  $L_j$  is defined by

$$\begin{aligned} L_j V &= V_{xx} - V_{yy} + \left( \frac{2(N-j)}{y} + a \right) V_y + B V_x + \\ &+ j \tilde{a} V + \sum_{k=1}^j F_{jk} I^k V_x + \sum_{k=1}^j H_{jk} I^k V, \end{aligned}$$

where

$$\tilde{a} = \frac{\partial a}{\partial y} + \frac{a}{y} = \frac{1}{c^2} [x^2/c^2, y^2/c^2]_0.$$

The coefficients  $F_{jk}$  and  $H_{jk}$  are determined inductively by

$$\begin{aligned} F_{j+1,k} &= \begin{cases} F_{j1} + \frac{1}{y} \frac{\partial B}{\partial y} & \text{if } k = 1 \\ F_{jk} + \frac{1}{y} \frac{\partial}{\partial y} F_{j,k-1} & \text{if } k \geq 2 \end{cases} \\ H_{j+1,k} &= \begin{cases} H_{j1} + j \frac{1}{y} \frac{\partial \tilde{a}}{\partial y} & \text{if } k = 1 \\ H_{jk} + \frac{1}{y} \frac{\partial}{\partial y} H_{j,k-1} & \text{if } k \geq 2 \end{cases} \end{aligned}$$

It is easy to see that  $F_{jk}$  are of the form  $\frac{x}{c^2} [x^2/c^2, y^2/c^2]_0$  and  $H_{jk}$  are of the form  $\frac{1}{c^2} [x^2/c^2, y^2/c^2]_0$ . By the definition we have formally

$$\frac{1}{y} \frac{\partial}{\partial y} (L_j \eta_j) = L_{j+1} \eta_{j+1}.$$

Now we consider the equation  $L_N V = 0$  for  $V = \eta_N$  with the initial conditions

$$V = 0, \quad V_y = 2^{N+1} N! \phi(x), \quad \text{at } y = 0.$$

The problem is

$$\begin{aligned} (Q) \quad & V_{yy} - V_{xx} = a V_y + B V_x + N \tilde{a} V + \\ & + \sum_{k=1}^N F_k I^k V_x + \sum_{k=1}^N H_k I^k V, \\ & V = 0, \quad V_y = 2^{N+1} N! \phi(x) \quad \text{at } y = 0. \end{aligned}$$

**Proposition 9** *If  $\phi \in C^1(R)$ , then the problem (Q) admits a unique solution  $V$  in  $C^2(R \times [0, \infty))$ .*

Proof. Let us denote by  $H(x, y, V)$  the right hand side of the equation  $L_N = 0$ . Then (Q) is transformed to the integral equation

$$V(x, y) = 2^N N! \int_{x-y}^{x+y} \phi(\xi) d\xi + \frac{1}{2} \int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V) dX dY.$$

We can solve this integral equation by the iteration

$$\begin{aligned} V_0(x, y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi) d\xi, \\ V^{n+1}(x, y) &= 2^N N! \int_{x-y}^{x+y} \phi(\xi) d\xi + \frac{1}{2} \int_0^y \int_{x-y+Y}^{x+y-Y} H(X, Y, V^n) dX dY. \end{aligned}$$

Fixing  $L$  arbitrarily, we consider  $|x| \leq L$ . Then it is easy to get the estimates

$$|V^{n+1}(x, y) - V^n(x, y)| \leq \frac{M^{n+1} y^{n+1}}{(n+1)!}.$$

Therefore  $V^n$  tends to a limit  $V$  uniformly on  $|x| \leq L, 0 \leq y \leq L$ . The limit is the unique solution of (Q). QED.

Now we put

$$\eta_N = V, \quad \eta_{N-k} = I \eta_{N-k+1}.$$

Since  $\eta_{N-k}$  and its derivatives of order  $\leq 2$  all vanish on  $y = 0$  for  $k \geq 1$ , we see  $\eta = \eta_0$  gives a solution of the relativistic Euler-Poisson-Darboux equation (EPD).

Next we give an integral formula for the solution  $V$  of (Q).

**Proposition 10** *There is a  $C^{N+2}$ -function  $G(x, y, \xi)$  of  $|x| < \infty, y \geq 0, x - y \leq \xi \leq x + y$  such that the solution  $V$  of (Q) satisfies*

$$V(x, y) = \int_{x-y}^{x+y} G(x, y, \xi) \phi(\xi) d\xi. \quad (6.1)$$

Moreover

$$\begin{aligned} G &= 2^N N! + O(y/c^2), \\ \partial_x^{p_1} \partial_\xi^{p_2} \partial_y^{p_3} G &= O(1/c^2) \quad \text{for } 1 \leq p_1 + p_2 + p_3 \leq N + 2 \end{aligned}$$

Proof. We consider the approximate solution  $V^n(x, y)$  which appeared in the iteration of the proof of Proposition 9. By writing  $H$  as

$$H = (aV)_y + (bV)_x + bV + \sum (F_k I^k V)_x + \sum \tilde{H}_k I^k V,$$

where

$$\begin{aligned} b &= N\tilde{a} - a_y - B_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \\ \tilde{H}_k &= H_k - (F_k)_x = \frac{1}{c^2}[x^2/c^2, y^2/c^2]_0, \end{aligned}$$

it is easy to see inductively that there is a kernel  $G^n(x, y, \xi)$  such that

$$V^n(x, y) = \int_{x-y}^{x+y} G^n(x, y, \xi) \phi(\xi) d\xi.$$

In fact  $G^0 = 2$  and  $G^n$  are determined inductively by the formula

$$\begin{aligned} G^{n+1} &= 2 + \frac{1}{2}(G_I^n + G_{II}^n + G_{III}^n + \sum G_{IVk}^n + \sum G_{Vk}^n), \\ G_I &= \int_{(-x+y+\xi)/2}^y a(x-y+Y, Y) G(x-y+Y, Y, \xi) dY + \\ &\quad + \int_{(x+y-\xi)/2}^y a(x+y-Y, Y) G(x+y-Y, Y, \xi) dY, \\ G_{II} &= \int_{(x+y-\xi)/2}^y B(x+y-Y, Y) G(x+y-Y, Y, \xi) dY + \\ &\quad - \int_{(-x+y+\xi)/2}^y B(x-y+Y, Y) G(x-y+Y, Y, \xi) dY, \\ G_{III} &= \int \int_{D(x, y, \xi)} b(X, Y) G(X, Y, \xi) dX dY, \end{aligned}$$

where

$$\begin{aligned} D(x, y, \xi) &= \{(X, Y) | X - Y \leq \xi \leq X + Y, x - y + Y \leq X \leq x + y - Y, 0 \leq Y \leq y\}, \\ G_{IVk} &= \int_{(x-y+\xi)/2}^y F_k(x+y-Y, Y) J^k G(x+y-Y, Y, \xi) dY + \\ &\quad - \int_{(-x+y+\xi)/2}^y F_k(x-y+Y, Y) J^k G(x-y+Y, Y, \xi) dY, \end{aligned}$$

where

$$JG(x, y, \xi) = \int_{|x-\xi|}^y Y G(x, Y, \xi) dY,$$

and

$$G_{Vk} = \int \int_{D(x, y, \xi)} \tilde{H}_k(X, Y) J^k G(X, Y, \xi) dX dY.$$

It is easy to see inductively that

$$|G^{n+1}(x, y, \xi) - G^n(x, y, \xi)| \leq \frac{M^{n+1} y^{n+1}}{(n+1)!}.$$

therefore  $G^n$  converges to a limit  $G$  uniformly and (6.1) holds. Moreover we can differentiate  $G^{n+1}$  supposing that  $G^n$  is differentiable. In fact we have

$$\begin{aligned}
G_{I,x} &= \frac{1}{2}aG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) \\
&\quad - \frac{1}{2}aG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&\quad + \int_{(-x+y+\xi)/2}^y (aG)_x(x-y+Y, Y, \xi)dY \\
&\quad + \int_{(x+y-\xi)/2}^y (aG)_x(x-Y+Y, Y, \xi)dY, \\
G_{I,\xi} &= -\frac{1}{2}aG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
&\quad + \frac{1}{2}aG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&\quad + \int_{(-x+y+\xi)/2}^y aG_\xi(x-y+Y, Y, \xi)dY + \\
&\quad + \int_{(x+y-\xi)/2}^y aG_\xi(x+y-Y, Y, \xi)dY, \\
G_{I,y} &= -\frac{1}{2}aG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
&\quad - \frac{1}{2}aG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&\quad + 2aG(x, y, \xi) + \\
&\quad - \int_{(-x+y+\xi)/2}^y (aG)_x(x-y+Y, Y, \xi)dY + \\
&\quad + \int_{(x+y-\xi)/2}^y (aG)_x(x+y-Y, Y, \xi)dY; \\
G_{II,x} &= -\frac{1}{2}BG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&\quad - \frac{1}{2}BG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
&\quad + \int_{(x+y-\xi)/2}^y (BG)_x(x+y-Y, Y, \xi)dY + \\
&\quad - \int_{(-x+y+\xi)/2}^y (BG)_x(x-y+Y, Y, \xi)dY, \\
G_{II,\xi} &= \frac{1}{2}BG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&\quad + \frac{1}{2}BG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
&\quad + \int_{(x+y-\xi)/2}^y BG_\xi(x+y-Y, Y, \xi)dY + \\
&\quad - \int_{(-x+y+\xi)/2}^y BG_\xi(x-y+Y, Y, \xi)dY,
\end{aligned}$$

$$\begin{aligned}
G_{II,y} &= -\frac{1}{2}BG((x+y+\xi)/2, (x+y-\xi)/2, \xi) + \\
&+ \frac{1}{2}BG((x-y+\xi)/2, (-x+y+\xi)/2, \xi) + \\
&+ \int_{(x+y-\xi)/2}^y (BG)_x(x+y-Y, Y, \xi)dY + \\
&+ \int_{(-x+y+\xi)/2}^y (BG)_x(x-y+Y, Y, \xi)dY; \\
G_{III,x} &= \int_{(x+y-\xi)/2}^y bG(x+y-Y, Y, \xi)dY - \int_{(-x+y+\xi)/2}^y bG(x-y+Y, Y, \xi)dY, \\
G_{III,\xi} &= \int_0^{(x+y-\xi)/2} bG(\xi+Y, Y, \xi)dY + \int_0^{(-x+y+\xi)/2} bG(\xi-Y, Y, \xi)dY + \\
&+ \int \int_{D(x,y,\xi)} bG(X, Y, \xi)dXdY, \\
G_{III,y} &= \int_{(x+y-\xi)/2}^y bG(x+y-Y, Y, \xi)dY + \int_{(-x+y+\xi)/2}^y bG(x-y+Y, Y, \xi)dY;
\end{aligned}$$

and the derivatives of  $G_{IVk}$  are similar to  $G_{II}$  and the derivatives of  $G_{IVk}$  are similar to  $G_{III}$ . Then it is easy to see inductively that

$$|G_x^{n+1} - G_x^n| + |G_\xi^{n+1} - G_\xi^n| + |G_y^{n+1} - G_y^n| \leq \frac{M^n y^n}{n!}.$$

Thus the limit  $G$  is differentiable. In a similar manner we see

$$\begin{aligned}
|G_{xx}^{n+1} - G_{xx}^n| &+ |G_{x\xi}^{n+1} - G_{x\xi}^n| + |G_{xy}^{n+1} - G_{xy}^n| + \\
&+ |G_{\xi\xi}^{n+1} - G_{\xi\xi}^n| + |G_{\xi y}^{n+1} - G_{\xi y}^n| + |G_{yy}^{n+1} - G_{yy}^n| \leq \\
&\leq \frac{M^{n-1} y^{n-1}}{(n-1)!}.
\end{aligned}$$

Thus  $G$  is twice continuously differentiable. In a similar manner we see that  $G$  is  $N+2$ -times continuously differentiable. The rough estimates stated in the propositions is obvious since the coefficients are all of  $O(1/c^2)$ . QED.

The solution  $\eta_{N-k}$  enjoys an integral representation

$$\eta_{N-k} = \int_{x-y}^{x+y} K_{N-k}(x, y, \xi) \phi(\xi) d\xi,$$

where

$$K_{N-k}(x, y, \xi) = JK_{N-k+1}(x, y, \xi) = J^k G(x, y, \xi).$$

So the solution  $\eta$  of the relativistic Euler-Poisson-Darboux equation is given by

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi,$$

where

$$K(x, y, \xi) = J^N G(x, y, \xi).$$

By induction we see

$$J^k G(x, y, \xi) = \frac{2^N N!}{2^k k!} (y^2 - (x - \xi)^2)^k (1 + O(y/c^2)).$$

Thus we have

**Proposition 11** *There is a kernel  $K(x, y, \xi)$  which is of  $C^{N+2}$ -class in  $|x| < \infty, 0 \leq y, x - y \leq \xi \leq x + y$  such that*

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi$$

*gives a solution of the relativistic Euler- Poisson-Darboux equation for any smooth  $\phi$ . Moreover*

$$K(x, y, \xi) = (y^2 - (x - \xi)^2)^N (1 + O(y/c^2)).$$

But in order to apply this integration formula, the generalized Darboux formula, to the study of the relativistic Euler equation, more detailed estimates of the remainder are necessary.

**Proposition 12** *We have*

$$G_y = O(y/c^2).$$

Proof. Since  $a = O(y/c^2)$ , it is clear that  $G_{I,y} = O(y/c^2)$ . Next we see  $G_{II,y} = -B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2) + O(y/c^2)$ .

On the other hand we can write

$$B = \frac{1}{c^2} B_0(x) + O(y^2/c^2)$$

and

$$\frac{x+y+\xi}{2} = x + \frac{y+Z}{2}, \quad \frac{x-y+\xi}{2} = x + \frac{-y+Z}{2}, \quad Z = \xi - x.$$

Therefore we see  $G_{II,y} = O(y/c^2)$ . It is clear that  $G_{III,y} = O(y/c^2)$  and  $G_{IVk,y}, G_{Vky} = O(y^2/c^2)$ . QED.

**Proposition 13** *We have*

$$G = 2^N N! + \frac{1}{c^2} C_0(x, c)(\xi - x) + O(y^2/c^2),$$

where  $C_0(x, c)$  is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2} [x^2/c^2]_0.$$

Proof. It is clear that  $G_I = O(y^2/c^2)$  since  $a = O(y/c^2)$ . Next we see

$$G_{II} = 2^N N! \int_{(x+y-\xi)/2}^y B(x+y-Y, Y) dY - 2^N N! \int_{(-x+y+\xi)/2}^y B(x-y+Y, Y) dY + O(y^2/c^2),$$

since  $G = 2^N N! + O(y/c^2)$ . If we write

$$B = \frac{1}{c^2} B_0(x) + O(y^2/c^2), \quad Z = \xi - x$$

, then we see

$$\begin{aligned} \int_{(x+y-\xi)/2}^y B(x+y-Y, Y) dY - \int_{(-x+y+\xi)/2}^y B(x-y+Y, Y) dY &= \\ &= \frac{1}{c^2} \left( \int_x^{x+\frac{y+Z}{2}} B_0(s) ds - \int_{x+\frac{-y+Z}{2}}^x B_0(s) ds \right) + O(y^2/c^2) \\ &= \frac{1}{c^2} B_0(x) Z + O(y^2/c^2). \end{aligned}$$

Note  $|Z| \leq y$ . It is clear that  $G_{III}, G_{IVk}, G_{Vk} = O(y^2/c^2)$ . QED.

**Proposition 14** *We have*

$$G_x + G_\xi = O(y/c^2).$$

Proof. First we see

$$\begin{aligned} G_{I,x} + G_{I,\xi} &= \int_{(-x+y+\xi)/2}^y ((aG)_x + aG_\xi)(x-y+Y, Y, \xi) dY + \\ &+ \int_{(x+y-\xi)/2}^y ((aG)_x + aG_\xi)(x+y-Y, Y, \xi) dY \\ &= O(y^2/c^2), \end{aligned}$$

since  $a, a_x = O(y/c^2)$ . Next we see

$$\begin{aligned} G_{II,x} + G_{II,\xi} &= \int_{(x+y-\xi)/2}^y ((BG)_x + BG_\xi)(x+y-Y, Y, \xi) dY + \\ &- \int_{(-x+y+\xi)/2}^y ((BG)_x + BG_\xi)(x-y+Y, Y, \xi) dY \\ &= O(y/c^2). \end{aligned}$$

It is clear that  $G_{III,x}, G_{III,\xi}, G_{Vk,x}, G_{Vk,\xi} = O(y/c^2)$ .  $G_{IVk,x} + G_{IVk,\xi}$  is estimated in a similar manner as  $G_{II,x} + G_{II,\xi}$ . QED.

**Proposition 15** *We have*

$$(G_x + G_\xi)_y = O(y/c^2).$$

Proof. First we see

$$\begin{aligned}
(G_{I,x} + G_{I,\xi})_y &= 2((aG)_x + aG_\xi)(x, y, \xi) + \\
&- \frac{1}{2}((aG)_x + aG_\xi)((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\
&- \frac{1}{2}((aG)_x + aG_\xi)((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \\
&- \int_{(-x+y+\xi)/2}^y ((aG)_x + aG_\xi)_x(x - y + Y, Y, \xi) dY + \\
&+ \int_{((x+y-\xi)/2)}^y ((aG)_x + aG_\xi)_x(x + y - Y, Y, \xi) dY \\
&= O(y/c^2),
\end{aligned}$$

since  $a, a_x = O(y/c^2)$ . Next we see

$$\begin{aligned}
(G_{II,x} + G_{II,\xi})_y &= -\frac{1}{2}((BG)_x + BG_\xi)((x + y + \xi)/2, (x + y - \xi)/2, \xi) + \\
&+ \frac{1}{2}((BG)_x + BG_\xi)((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\
&+ \int_{(x+y-\xi)/2}^y ((BG)_x + BG_\xi)_x(x + y - Y, Y, \xi) dY + \\
&+ \int_{(-x+y+\xi)/2}^y ((BG)_x + BG_\xi)_x(x - y + Y, Y, \xi) dY \\
&= 2^{N-1} N! B_x((x - y + \xi)/2, (-x + y + \xi)/2) \\
&- 2^{N-1} N! B_x((x + y + \xi)/2, (x + y - \xi)/2) + \\
&+ O(y/c^2),
\end{aligned}$$

since  $G = 2^N N! + O(y/c^2)$  and  $G_x + G_\xi = O(y/c^2)$ . But

$$B_x = \frac{1}{c^2} B'_0(x) + O(y^2/c^2)$$

and

$$\begin{aligned}
B_x((x - y + \xi)/2, (-x + y + \xi)/2) - B_x((x + y + \xi)/2, (x + y - \xi)/2) &= \\
&= \frac{1}{c^2} B''_0(x)(-y) + O(y^2/c^2) \\
&= O(y/c^2).
\end{aligned}$$

It is clear that

$$\begin{aligned}
(G_{III,x} + G_{III,\xi})_y &= \int_{(x+y-\xi)/2}^y ((bG)_x + bG_\xi)(x + y - Y, Y, \xi) dY + \\
&+ \int_{(-x+y+\xi)/2}^y ((bG)_x + bG_\xi)(x - y + Y, Y, \xi) dY \\
&= O(y/c^2).
\end{aligned}$$

Similarly we can estimate  $(G_{IVk,x} + G_{IVk,\xi})_y, (G_{Vk,x} + G_{Vk,\xi})_y$  bearing in mind that  $(JG)_x + (JG)_\xi = J(G_x + G_\xi)$ . QED.

**Proposition 16** *We have*

$$G_x + G_\xi = \frac{1}{c^2} C_1(x, c)(\xi - x) + O(y^2/c^2),$$

where  $C_1(x, c)$  is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2} [x^2/c^2]_0.$$

Proof. We already observed that  $G_{Ix} + G_{I\xi} = O(y^2/c^2)$ . Next we look at

$$\begin{aligned} G_{II,x} + G_{II,\xi} &= \int_{(x+y-\xi)/2}^y ((BG)_x + BG_\xi)(x+y-Y, Y, \xi) dY + \\ &\quad - \int_{(-x+y+\xi)/2}^y ((BG)_x + BG_\xi)(x-y+Y, Y, \xi) dY \\ &= 2^N N! \int_{(x+y-\xi)/2}^y B_x(x+y-Y, Y) dY - 2^N N! \int_{(-x+y+\xi)/2}^y B_x(x-y+Y, Y) dY + \\ &\quad + O(y^2/c^2), \end{aligned}$$

since  $G = 2 + O(y/c^2)$  and  $G_x + G_\xi = O(y/c^2)$ . Bearing in mind that  $B_y = O(y/c^2)$ , we see

$$\begin{aligned} &\int_{(x+y-\xi)/2}^y B_x(x+y-Y, Y) dY - \int_{(-x+y+\xi)/2}^y B_x(x-y+Y, Y) dY = \\ &= - \int_{(x+y-\xi)/2}^y (-B_x + B_y)(x+y-Y, Y) dY - \int_{(-x+y+\xi)/2}^y (B_x + B_y)(x-y+Y, Y) dY \\ &\quad + O(y^2/c^2) \\ &= -2B(x, y) + B((x+y+\xi)/2, (x+y-\xi)/2) + \\ &\quad + B((x-y+\xi)/2, (-x+y+\xi)/2) + O(y^2/c^2) \\ &= \frac{1}{c^2} (-2B_0(x) + B_0(x + \frac{y+Z}{2}) + B_0(x + \frac{-y+Z}{2})) + O(y^2/c^2) \\ &= \frac{1}{c^2} B'_0(x)Z + O(y^2/c^2). \end{aligned}$$

Next we look at

$$\begin{aligned} G_{III,x} + G_{III,\xi} &= \int_{(x+y-\xi)/2}^y bG(x+y-Y, Y, \xi) dY - \int_{(-x+y+\xi)/2}^y bG(x-y+Y, Y, \xi) dY + \\ &\quad + \int_0^{(x+y-\xi)/2} bG(\xi+Y, Y, \xi) dY - \int_0^{(-x+y+\xi)/2} bG(\xi-Y, Y, \xi) dY + \\ &\quad + \int \int_{D(x,y,\xi)} bG(X, Y, \xi) dX dY. \end{aligned}$$

Putting

$$b(x, y) = \frac{1}{c^2} b_0(x) + O(y^2/c^2),$$

we see

$$\begin{aligned} G_{III,x} + G_{III,\xi} &= 2^N N! \left( \int_x^{x+\frac{y+Z}{2}} b_0(s) ds - \int_{x+\frac{-y+Z}{2}}^x b_0(s) ds + \right. \\ &\quad \left. + \int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s) ds - \int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s) ds \right) + O(y^2/c^2) \\ &= \frac{2^N N!}{c^2} b_0(x) \left( \frac{y+Z}{2} - \frac{y-Z}{2} + \frac{y-Z}{2} - \frac{y+Z}{2} \right) + O(y^2/c^2) \\ &= O(y^2/c^2). \end{aligned}$$

$G_{IVk,x} + G_{IVk,\xi}$  can be estimated in a similar manner as  $G_{II,x} + G_{II,\xi}$ .

Finally  $G_{V k, x}, G_{V k, \xi} = O(y^3/c^2)$  since  $J^k G = O(y^2/c^2)$  for  $k \geq 1$ . QED.

**Proposition 17** *We have*

$$(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2).$$

Proof. First we see

$$\begin{aligned} (G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi &= \\ &= \int_{(-x+y+\xi)/2}^y ((aG)_{xx} + 2(aG_\xi)_x + aG_{\xi\xi})(x-y+Y, Y, \xi) dY + \\ &\quad + \int_{(x+y-\xi)/2}^y ((aG)_{xx} + 2(aG_\xi)_x + aG_{\xi\xi})(x+y-Y, Y, \xi) dY \\ &= O(y^2/c^2), \end{aligned}$$

since  $a, a_x, a_{xx} = O(y/c^2)$ . Next

$$\begin{aligned} (G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi &= \\ &= \int_{(x+y-\xi)/2}^y (((BG)_x + BG_\xi)_x + ((BG)_x + BG_\xi)_\xi)(x+y-Y, Y, \xi) dY + \\ &\quad + \int_{(-x+y+\xi)/2}^y (((BG)_x + BG_\xi)_x + ((BG)_x + BG_\xi)_\xi)(x+y-Y, Y, \xi) dY \\ &= O(y/c^2). \end{aligned}$$

It is easy to see

$$(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi = O(y/c^2).$$

The estimates of  $G_{IVk}$  and  $G_{V k}$  can be seen similarly. QED.

**Proposition 18** *We have*

$$(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = \frac{1}{c^2} C_2(x, c)(\xi - x) + O(y^2/c^2),$$

where  $C_2(x, c)$  is a function of the form

$$[x^2/c^2]_0 + \frac{x}{c^2} [x^2/c^2]_0.$$

*Proof.* We already observed that

$$(G_{I,x} + G_{I,\xi})_x + (G_{I,x} + G_{I,\xi})_\xi = O(y^2/c^2).$$

Next, bearing in mind that  $G_x + G_\xi = O(y/c^2)$  and  $(G_x + G_\xi)_x + (G_x + G_\xi)_\xi = O(y/c^2)$ , we see

$$\begin{aligned} (G_{II,x} + G_{II,\xi})_x + (G_{II,x} + G_{II,\xi})_\xi &= \\ &= \int_{(x+y-\xi)/2}^y (B_{xx}G + 2B_x(G_x + G_\xi) + \\ &+ B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x + y - Y, Y, \xi) dY + \\ &- \int_{(-x+y+\xi)/2}^y (B_{xx}G + 2B_x(G_x + G_\xi) + \\ &+ B((G_x + G_\xi)_x + (G_x + G_\xi)_\xi))(x - y + Y, Y, \xi) dY \\ &= 2^N N! \int_{(x+y-\xi)/2}^y B_{xx}(x + y - Y, Y) dY - 2^N N! \int_{(-x+y+\xi)/2}^y B_{xx}(x - y + Y, Y) dY + \\ &+ O(y^2/c^2). \end{aligned}$$

The same discussion to that of the proof of Proposition 16 can be applied by replacing  $B$  by  $B_x$ . Let us look at  $(G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi$ . Note that

$$\begin{aligned} (bG)_x + bG_\xi &= b_x G + b(G_x + G_\xi) \\ &= 2^N N! b_x + O(y/c^2), \\ bG &= 2^N N! b + O(y/c^2). \end{aligned}$$

Applying the discussion of the proof of Proposition 16 by replacing  $b$  by  $b_x$ , we see

$$\begin{aligned} (G_{III,x} + G_{III,\xi})_x + (G_{III,x} + G_{III,\xi})_\xi &= \\ &= 2^N N! \left( \int_{x+Z}^{x+\frac{y+Z}{2}} b_0(s) ds - \int_{x+\frac{-y+Z}{2}}^{x+Z} b_0(s) ds \right) + \\ &+ O(y^2/c^2) \\ &= -2^N N! b_0(x) Z + O(y^2/c^2). \end{aligned}$$

The estimates of  $G_{IVk}, G_{Vk}$  are parallell. QED.

**Proposition 19** *We have*

$$G_\xi = \frac{1}{c^2} C_3(x, c) + O(y/c^2).$$

*Proof.* It is sufficient to note that

$$\begin{aligned} G_{II,\xi} &= 2^{N-1} N! (B((x+y+\xi)/2, (x+y-\xi)/2) + B((x-y+\xi)/2, (-x+y+\xi)/2)) + \\ &\quad + O(y/c^2) \\ &= \frac{2^{N-1} N!}{c^2} (B_0(x + \frac{y+Z}{2}) + B_0(x + \frac{-y+Z}{2})) + O(y/c^2) \\ &= \frac{2^N N!}{c^2} B_0(x) + O(y/c^2). \end{aligned}$$

QED.

**Proposition 20** *We have*

$$(G_x + G_\xi)_\xi = \frac{1}{c^2} C_4(x, c) + O(y/c^2).$$

*Proof.* We see

$$(G_{I,x} + G_{I,\xi})_\xi = O(y/c^2)$$

by  $a, a_x = O(y/c^2)$ . Next we see

$$\begin{aligned} (G_{II,x} + G_{II,\xi})_\xi &= \\ &= 2^{N-1} N! (B_x((x+y+\xi)/2, (x+y-\xi)/2) + \\ &\quad + B_x((x-y+\xi)/2, (-x+y+\xi)/2)) + O(y/c^2) \\ &= \frac{2^N N!}{c^2} B'_0(x) + O(y/c^2). \end{aligned}$$

And we see

$$\begin{aligned} (G_{III,x} + G_{III,\xi})_\xi &= \\ &= 2^N N! b((x-y+\xi)/2, (-x+y+\xi)/2) + O(y/c^2) \\ &= \frac{2^N N!}{c^2} b_0(x) + O(y/c^2). \end{aligned}$$

Other terms can be estimated similarly. QED.

## 7 Estimates of the derivatives of entropies

Let us consider the entropy  $\eta$  generated by  $\phi$  of  $C^3$ -class, that is,

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi.$$

In this section we will find estimates of the derivatives of  $\eta$  with respect to  $E, F$ . As auxiliary variables we introduce

$$R = y^{2N+1}, \quad M = xy^{2N+1}. \quad (7.1)$$

We are going to prove the following

**Proposition 21** *We have*

$$\frac{\partial \eta}{\partial M} = 2^{2N+1} \int_0^1 (s-s^2)^N D\phi(x + (2s-1)y) ds + O(y^2/c^2), \quad (7.2)$$

$$\begin{aligned} \frac{\partial \eta}{\partial R} &= 2^{2N+1} \int_0^1 (s-s^2)^N \phi ds + \\ &2^{2N+1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1}(2s-1)\right) D\phi ds + \\ &O(y^2/c^2), \end{aligned} \quad (7.3)$$

$$\frac{\partial^2 \eta}{\partial M^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D^2 \phi ds + O(y^{-2N+1}/c^2), \quad (7.4)$$

$$\begin{aligned} \frac{\partial^2 \eta}{\partial R \partial M} &= 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1}(2s-1)\right) D^2 \phi ds + \\ &O(y^{-2N+1}/c^2), \end{aligned} \quad (7.5)$$

$$\begin{aligned} \frac{\partial^2 \eta}{\partial R^2} &= 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(\left(-x + \frac{y}{2N+1}(2s-1)\right)^2 + \right. \\ &\left. \frac{4}{(2N+1)^2} s(1-s)y^2\right) D^2 \phi(x + (2s-1)y) ds + O(y^{-1}/c^2) \end{aligned} \quad (7.6)$$

Proof. We write

$$\eta = 2R^{\frac{1}{2N+1}} \int_0^1 K\left(\frac{M}{R}, R^{\frac{1}{2N+1}}, \frac{M}{R} + (2s-1)R^{\frac{1}{2N+1}}\right) \phi\left(\frac{M}{R} + (2s-1)R^{\frac{1}{2N+1}}\right) ds.$$

Differentiating  $\eta$  with respect to  $M$ , we have

$$\begin{aligned} \frac{\partial \eta}{\partial M} &= (1) + (2), \\ (1) &= 2R^{\frac{1}{2N+1}} \int_0^1 (K_x + K_\xi)(x, y, x + (2s-1)y) \phi(x + (2s-1)y) ds, \\ (2) &= 2R^{\frac{1}{2N+1}} \int_0^1 K(x, y, x + (2s-1)y) D\phi(x + (2s-1)y) ds. \end{aligned}$$

Since  $K(x, y, \xi) = J^N G(x, y, \xi)$ , i.e.

$$K(x, y, \xi) = \int_{|x-\xi|}^y Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1} \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x, Y_1, \xi) dY_1 \cdots Y_N,$$

by Proposition 16 we see

$$\begin{aligned}
& (K_x + K_\xi)(x, y, x + (2s - 1)y) \\
&= \int_{|2s-1|y}^y Y_N \int_{|2s-1|y}^{Y_N} Y_{N-1} \cdots \int_{|2s-1|y}^{Y_2} Y_1 (G_x + G_\xi)(x, Y_1, x + (2s - 1)y) dY_1 \cdots Y_N \\
&= \frac{C_1(x, c)}{2^N N! c^2} y^{2N+1} (2s - 1) (1 - (2s - 1)^2)^N + O(y^{2N+2}/c^2) \\
&= -\frac{2^N C_1(x, c)}{(N + 1)! c^2} y^{2N+1} \frac{d}{ds} (s - s^2)^{N+1} + O(y^{2N+2}/c^2).
\end{aligned}$$

Therefore by integration by part we get

$$\begin{aligned}
(1) &= R^{\frac{-2N}{2N+1}} y^{2N+2} \frac{2^{N+1} C_1(x, c)}{(N + 1)! c^2} \int_0^1 (s - s^2)^{N+1} D\phi ds + O(y^2/c^2) \\
&= O(y^2/c^2).
\end{aligned}$$

By Proposition 13 we see

$$\begin{aligned}
K(x, y, \xi) &= \int_{|x-\xi|}^y Y_N \int_{|x-\xi|}^{Y_N} Y_{N-1} \cdots \int_{|x-\xi|}^{Y_2} Y_1 G(x, Y_1, \xi) dY_1 \cdots Y_N, \\
&= 2^{2N} (s - s^2)^N y^{2N} + \frac{2^N C_0(x, c)}{N! c^2} (2s - 1) (s - s^2)^N y^{2N+1} + O(y^{2N+2}/c^2).
\end{aligned}$$

Therefore by integration by parts we get

$$\begin{aligned}
(2) &= 2^{2N+1} R^{\frac{-2N}{2N+1}} y^{2N} \int_0^1 (s(1 - s))^N D\phi(x + (2s - 1)y) ds \\
&+ R^{\frac{-2N}{2N+1}} O(y^{2N+2}/c^2).
\end{aligned}$$

Thus we have (7.2). Next we show (7.3). We have

$$\begin{aligned}
\frac{\partial \eta}{\partial R} &= (3) + (4) + (5), \\
(3) &= \frac{2}{2N + 1} R^{\frac{-2N}{2N+1}} \int_0^1 K(x, y, x + (2s - 1)y) \phi(x + (2s - 1)y) ds, \\
(4) &= 2R^{\frac{-2N}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{1}{2N + 1} y(K_y + (2s - 1)K_\xi)) \times \\
&\quad \phi(x + (2s - 1)y) ds, \\
(5) &= 2R^{\frac{-2N}{2N+1}} \int_0^1 K(x, y, x + (2s - 1)y) (-x + \frac{y}{2N + 1} (2s - 1)) D\phi(\dots) ds.
\end{aligned}$$

By Proposition 13 we get

$$(3) = \frac{2^{2N+1}}{2N + 1} \int_0^1 (s - s^2)^N \phi(\dots) ds + O(y^2/c^2).$$

As for (4) we use Proposition 16 and

$$K_y + (2s - 1)K_\xi = yJ^{N-1}G - (2s - 1)(\xi - x)G(x, |\xi - x|, \xi)J^{N-1}1 + (2s - 1)J^N G_\xi$$

$$\begin{aligned}
&= 2^{2N+1}N(s-s^2)^N y^{2N-1} + \frac{2^{N-1}C_0(x,c)}{(N-1)!c^2}(2s-1)(s-s^2)^N y^{2N} + \\
&+ \frac{2^N C_3(x,c)}{N!c^2}(2s-1)(s-s^2)^N y^{2N} + O(y^{2N+1}/c^2)
\end{aligned}$$

(See Proposition 19). Then by integration by parts we have

$$(4) = \frac{2^{2N+2}N}{2N+1} \int_0^1 (s-s^2)^N \phi(\dots) ds + O(y^2/c^2).$$

As (2) we get

$$(5) = 2^{2N+1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1}(2s-1)\right) D\phi(\dots) ds + O(y^2/c^2).$$

Thus we get (7.3).

Next we show (7.4). We have

$$\begin{aligned}
\frac{\partial^2 \eta}{\partial M^2} &= (6) + (7) + (8), \\
(6) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 ((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)(x, y, \dots) \times \phi(\dots) ds, \\
(7) &= 4R^{\frac{-4N-1}{2N+1}} \int_0^1 (K_x + K_\xi)(x, y, \dots) D\phi(\dots) ds, \\
(8) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(x, y, \dots) D^2 \phi(\dots) ds.
\end{aligned}$$

By Proposition 18 we have

$$\begin{aligned}
&((K_x + K_\xi)_x + (K_x + K_\xi)_\xi)(x, y, x + (2s-1)y) = \\
&= \frac{2^N C_2(x,c)}{N!c^2} (s-s^2)^N (2s-1) y^{2N+1} + O(y^{2N+2}/c^2).
\end{aligned}$$

Thus by integration by parts we get

$$(6) = O(y^{-2N+1}/c^2).$$

By the same discussion as (1) we see (7) =  $O(y^{-2N+1}/c^2)$ . By the same discussion as (2) we see

$$(8) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D^2 \phi(\dots) ds + O(y^{-2N+1}/c^2).$$

Thus we get (7.4).

Next we show (7.5). We see

$$\frac{\partial^2 \eta}{\partial M \partial R} = (9) + (10) + (11) + (12) + (13) + (14),$$

$$(9) = -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (K_x + K_\xi)(x, y, \dots) \phi(\dots) ds,$$

$$(10) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi)_x + (K_x + K_\xi)_\xi) + \\ + \frac{y}{2N+1} ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi(\dots) ds,$$

$$(11) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (K_x + K_\xi) \left(-x + \frac{y}{2N+1}(2s-1)\right) D\phi ds,$$

$$(12) = -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K D\phi ds,$$

$$(13) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 \left(-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)\right) D\phi ds$$

$$(14) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K \left(-x + \frac{y}{2N+1}(2s-1)\right) D^2\phi ds.$$

We already know that (9) =  $O(y^{-2N+1}/c^2)$ . (Recall (1).) Next we look at (10). The first term is  $O(y^{-2N+1}/c^2)$ . (Recall (6)). By Proposition 16 and 20 we see

$$(K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi = \\ = \frac{2^{N-1}C_1(x, c)}{(N-1)!c^2} y^{2N} (2s-1)(s-s^2)^N + \\ + \frac{2^N C_4(x, c)}{N!c^2} y^{2N} (2s-1)(s-s^2)^N + \\ - \frac{2^{N-1}C_1(x, c)}{(N-1)!c^2} y^{2N} (2s-1)^3 (s-s^2)^{N-1} + O(y^{2N+1}/c^2).$$

By integration by parts we see (10) =  $O(y^{-2N+1}/c^2)$ . We already know (11) =  $O(y^{-2N+1}/c^2)$ . Clearly

$$(12) = -\frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2).$$

We see

$$(13) = O(y^{-2N+1}/c^2) + \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (K_y + (2s-1)K_\xi) D\phi ds.$$

As (4) we have

$$(13) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N D\phi ds + O(y^{-2N+1}/c^2).$$

Finally we see

$$(14) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(-x + (2s-1)\frac{y}{2N+1}\right) D^2\phi ds + O(y^{-2N+1}/c^2).$$

(Recall (5)). Summing up we get (7.5).

Next we show (7.6).

$$\begin{aligned}
\frac{\partial^2 \eta}{\partial R^2} &= \frac{\partial}{\partial R}(3) + \frac{\partial}{\partial R}(4) + \frac{\partial}{\partial R}(5), \\
\frac{\partial}{\partial R}(3) &= (15) + (16) + (17), \\
(15) &= -\frac{4N}{(2N+1)^2} R^{\frac{-4N-1}{2N+1}} \int_0^1 K \phi ds, \\
(16) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \phi ds, \\
(17) &= \frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1)) D\phi ds, \\
\frac{\partial}{\partial R}(4) &= (18) + (19) + (20), \\
(18) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \phi ds, \\
(19) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K'' \phi ds,
\end{aligned}$$

where

$$\begin{aligned}
K'' &= x(K_x + K_\xi) + x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) + \\
&+ \frac{y}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) + \\
&- \frac{xy}{2N+1}((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) + \\
&+ \frac{y^2}{2N+1}((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi) + \\
&+ \frac{y}{(2N+1)^2}(K_y + (2s-1)K_\xi), \\
(20) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \\
&\times (-x + \frac{y}{2N+1}(2s-1)) D\phi ds, \\
\frac{\partial}{\partial R}(5) &= (21) + (22) + (23) + (24), \\
(21) &= -\frac{4N}{2N+1} R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1)) D\phi ds, \\
(22) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 (-x(K_x + K_\xi) + \frac{y}{2N+1}(K_y + (2s-1)K_\xi)) \\
&\times (-x + \frac{y}{2N+1}(2s-1)) D\phi ds, \\
(23) &= 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds,
\end{aligned}$$

$$(24) = 2R^{\frac{-4N-1}{2N+1}} \int_0^1 K(-x + \frac{y}{2N+1}(2s-1))^2 D^2 \phi ds.$$

First we see

$$(15) = -\frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$

$$(16) = \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2),$$

$$(17) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Thus we have

$$\frac{\partial}{\partial R}(3) = \frac{2^{2N+1}}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Since (18) is similar to (16), we have

$$(18) = -\frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).$$

Next let us look at (19). We already know

$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x(K_x + K_\xi) \phi ds = O(y^{-2N+1}/c^2),$$

$$2R^{\frac{-4N-1}{2N+1}} \int_0^1 x^2((K_x + K_\xi)_x + (K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2).$$

Recalling (10), we see

$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} y \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2),$$

$$\frac{2}{2N+1} R^{\frac{-4N-1}{2N+1}} xy \int_0^1 ((K_x + K_\xi)_y + (2s-1)(K_x + K_\xi)_\xi) \phi ds = O(y^{-2N+1}/c^2).$$

When  $N = 1$ , we have

$$\begin{aligned} & (K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi \\ &= 8(s-s^2) + \frac{C_3}{c^2}(2s-1)y - \frac{C_0}{c^2}(2s-1)^3y - \\ & \quad - \frac{2C_3}{c^2}(2s-1)^3y + O(y^2/c^2). \end{aligned}$$

When  $N \geq 2$ , there are bounded functions  $F_j(x, c)$  such that

$$\begin{aligned} & (K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_\xi = \\ &= 2^{2N+1}N(2N-1)(s-s^2)^N y^{2N-2} + \frac{F_1(x, c)}{c^2}(2s-1)(s-s^2)^{N-1} y^{2N-1} + \\ &+ \frac{F_2(x, c)}{c^2}(2s-1)(s-s^2)^{N-2} y^{2N-1} + \frac{F_3(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-2} y^{2N-1} + \\ &+ \frac{F_4(x, c)}{c^2}(2s-1)^3(s-s^2)^{N-1} y^{2N-1} + \frac{F_5(x, c)}{c^2}(2s-1)^5(s-s^2)^{N-2} y^{2N-1} + \\ &+ O(y^{2N}/c^2). \end{aligned}$$

Thus we see

$$\begin{aligned}
& 2R^{\frac{-4N-1}{2N+1}} \frac{y^2}{(2N+1)^2} \int_0^1 ((K_y + (2s-1)K_\xi)_y + (2s-1)(K_y + (2s-1)K_\xi)_y) \phi ds \\
&= \frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds - \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds \\
&+ O(y^{-2N+1}/c^2).
\end{aligned}$$

We have

$$\begin{aligned}
& \frac{2}{(2N+1)^2} R^{\frac{-4N-1}{2N+1}} y \int_0^1 (K_y + (2s-1)K_\xi) \phi ds \\
&= \frac{2^{2N+2}N}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).
\end{aligned}$$

Therefore

$$(19) = \frac{2^{2N+3}N^2}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N \phi ds + O(y^{-2N+1}/c^2).$$

We see

$$(20) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Therefore

$$\frac{\partial}{\partial R}(4) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2).$$

Next we see

$$(21) = -\frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),$$

$$(22) = \frac{2^{2N+2}N}{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),$$

$$(23) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds + O(y^{-2N+1}/c^2),$$

$$(24) = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds + O(y^{-2N+1}/c^2).$$

Therefore we get

$$\begin{aligned}
\frac{\partial}{\partial R}(5) &= 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (x + \frac{y}{(2N+1)^2}(2s-1)) D\phi ds + \\
&+ 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1))^2 D^2\phi ds + O(y^{-2N+1}/c^2).
\end{aligned}$$

Summing up, we have

$$\frac{\partial^2 \eta}{\partial R^2} = 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N (-x + \frac{y}{2N+1}(2s-1)) D\phi ds$$

$$\begin{aligned}
& + 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(x + \frac{y}{(2N+1)^2} (2s-1)\right) D\phi ds \\
& + 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds \\
& = \frac{2^{2N+2}(N+1)}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^N (2s-1) y D\phi ds + \\
& + 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds \\
& = \frac{2^{2N+3}}{(2N+1)^2} y^{-2N-1} \int_0^1 (s-s^2)^{N+1} y^2 D^2\phi ds + \\
& + 2^{2N+1} y^{-2N-1} \int_0^1 (s-s^2)^N \left(-x + \frac{y}{2N+1} (2s-1)\right)^2 D^2\phi ds.
\end{aligned}$$

Thus we get (7.6). QED.

Let us recall the standard entropy  $\eta^*$ . This is generated by

$$\phi^*(x) = A' c^2 \left( \frac{1}{1-u^2/c^2} - \frac{1}{\sqrt{1-u^2/c^2}} \right),$$

where

$$A' = (2N+1)^{-2N} ((2N+1)/(2N+3)A)^{\frac{2N+1}{2}} (2N-1)!!/2^{N+1}N!.$$

We note that

$$D^2\phi^*(x) = A' \left(1 + \frac{u^2/c^2}{1-u^2/c^2}\right) (2 - \sqrt{1-u^2/c^2}) \geq A'.$$

We are going to show that the Hessian  $D_U^2\eta^*$  dominates any  $D_U^2\eta$ .

**Proposition 22** *For each  $\phi$  fixed in  $C^3$  we have on each compact subset of  $\{\rho \geq 0\}$*

$$|(\xi|D_U^2\eta.\xi)| \leq C(\xi|D_U^2\eta^*.\xi),$$

*provided that  $c$  is sufficiently large.*

By the assumption we have

$$\begin{aligned}
R &= y^{2N+1} = K\rho(1 + [\rho^{\frac{2}{2N+1}}/c^2]_1), \\
\frac{dR}{d\rho} &= K + [\rho^{\frac{2}{2N+1}}/c^2]_1, \\
\frac{d^2R}{d\rho^2} &= \frac{\rho^{\frac{1-2N}{2N+1}}}{c^2} [\rho^{\frac{2}{2N+1}}/c^2]_0,
\end{aligned}$$

where  $K = ((2N+3)(2N+1)A)^{\frac{2N+1}{2}}$ . Using these, we have

$$\frac{\partial R}{\partial E} = \frac{dR}{d\rho} \frac{1+u^2/c^2}{1-P'u^2/c^4}$$

$$\begin{aligned}
&= K(1 + u^2/c^2) + O(y^2/c^2), \\
\frac{\partial R}{\partial F} &= -\frac{dR}{d\rho} \frac{2u/c^2}{1 - P'u^2/c^4} \\
&= -K \frac{2u}{c^2} + O(y^2/c^2), \\
\frac{\partial M}{\partial E} &= -\frac{R}{\rho + P/c^2} \frac{1 + P'/c^2}{1 - P'u^2/c^4} u + x \frac{dR}{d\rho} \frac{1 + u^2/c^2}{1 - P'u^2/c^4} \\
&= K(-u + x(1 + u^2/c^2)) + O(y^2/c^2), \\
\frac{\partial M}{\partial F} &= \frac{R}{\rho + P/c^2} \frac{1 + P'u^2/c^4}{1 - P'u^2/c^4} - \frac{dR}{d\rho} \frac{2xu/c^2}{1 - P'u^2/c^4} \\
&= K(1 - 2xu/c^2) + O(y^2/c^2). \tag{7.7}
\end{aligned}$$

Differentiating once more, we see

$$\begin{aligned}
\frac{\partial^2 R}{\partial E^2} &= -\frac{K^2}{y^{2N+1}} 2u^2(1 - u^2/c^2)/c^2 + O(y^{-2N+1}/c^2), \\
\frac{\partial^2 M}{\partial E^2} &= \frac{K^2}{y^{2N+1}} u(-2u^2/c^2 - 2ux(1 - u^2/c^2)/c^2) + O(y^{-2N+1}/c^2), \\
\frac{\partial^2 R}{\partial E \partial F} &= \frac{K^2}{y^{2N+1}} \frac{2u}{c^2} (1 - u^2/c^2) + O(y^{-2N+1}/c^2), \\
\frac{\partial^2 M}{\partial E \partial F} &= \frac{K^2}{y^{2N+1}} (2u^2/c^2 + 2xu(1 - u^2/c^2)/c^2) + O(y^{-2N+1}/c^2), \\
\frac{\partial^2 R}{\partial F^2} &= -\frac{2}{c^2} \frac{K^2}{y^{2N+1}} (1 - u^2/c^2) + O(y^{-2N+1}/c^2), \\
\frac{\partial^2 M}{\partial F^2} &= -\frac{K^2}{y^{2N+1}} 2(u + x(1 - u^2/c^2))/c^2 + O(y^{-2N+1}/c^2). \tag{7.8}
\end{aligned}$$

The chain rule gives

$$\begin{aligned}
\frac{\partial^2 \eta}{\partial E^2} &= \left(\frac{\partial R}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial R^2} + 2 \frac{\partial R}{\partial E} \frac{\partial M}{\partial E} \frac{\partial^2 \eta}{\partial R \partial M} \\
&\quad + \left(\frac{\partial M}{\partial E}\right)^2 \frac{\partial^2 \eta}{\partial M^2} + \frac{\partial^2 R}{\partial E^2} \frac{\partial \eta}{\partial R} + \frac{\partial^2 M}{\partial E^2} \frac{\partial \eta}{\partial M}, \tag{7.9}
\end{aligned}$$

and so on. Inserting (7.7) and (7.8) into (7.9), and using Proposition 21, we have

$$\begin{aligned}
(\xi | D_U^2 \eta, \xi) &= \frac{2^{2N+1} K^2}{y^{2N+1}} \int_0^1 (s - s^2)^N Z[\xi] D^2 \phi ds + \\
&\quad - \frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (1 - u^2/c^2) (u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial R} + \\
&\quad - \frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (u + x(1 - u^2/c^2)) (u\xi_0 - \xi_1)^2 \frac{\partial \eta}{\partial M} + \\
&\quad + O(y^{-2N+1}/c^2),
\end{aligned}$$

where

$$Z[\xi] = Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2,$$

$$\begin{aligned}
Z_{00} &= (1 + u^2/c^2)^2 \left( (-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2} s(1-s)y^2 \right) + \\
&+ 2(1 + u^2/c^2)(-u + x(1 + u^2/c^2))(-x + \frac{y}{2N+1}(2s-1)) \\
&+ (-u + x(1 + u^2/c^2))^2, \\
Z_{01} &= -2(1 + u^2/c^2)u/c^2 \left( (-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2} s(1-s)y^2 \right) + \\
&+ (1 + 3u^2/c^2 - 4x(1 + u^2/c^2)u/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
&+ (-u + x(1 + u^2/c^2))(1 - 2xu/c^2), \\
Z_{11} &= \frac{4u^2}{c^4} \left( (-x + \frac{y}{2N+1}(2s-1))^2 + \frac{4}{(2N+1)^2} s(1-s)y^2 \right) + \\
&- \frac{4u}{c^2} (1 - 2xu/c^2)(-x + \frac{y}{2N+1}(2s-1)) + \\
&+ (1 - 2xu/c^2)^2.
\end{aligned}$$

It can be shown that

$$Z[\xi] \geq \kappa s(1-s)y^2,$$

where  $\kappa$  is a positive constant depending on the compact subset of  $\{\rho \geq 0\}$ .

In fact we see

$$Z_{00}Z_{11} - Z_{01}^2 = (1 - u^2/c^2) \frac{4}{(2N+1)^2} s(1-s)y^2.$$

On the other hand, we can estimate

$$\begin{aligned}
\left| \frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (1 - u^2/c^2) \frac{\partial \eta}{\partial R} \right| &\leq \frac{\epsilon}{y^{2N+1}}, \\
\left| \frac{2K^2}{y^{2N+1}} \frac{1}{c^2} (u + x(1 - u^2/c^2)) \frac{\partial \eta}{\partial M} \right| &\leq \frac{\epsilon}{y^{2N+1}},
\end{aligned}$$

where  $\epsilon = K'/c^2$ . Let us introduce the parameters

$$\zeta_0 = \xi_0, \quad \zeta_1 = \xi_1 - u\xi_0.$$

Then we have

$$Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2,$$

and

$$\begin{aligned}
Q_{00} &= Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x,s)y^2, \\
Q_{01} &= Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x,s)y^2, \\
Q_{11} &= Z_{11} = 1 + O(1/c^2) > 0.
\end{aligned}$$

Therefore if  $|D^2\phi| \leq C$ , we see

$$|(\xi|D_{\bar{v}}^2\eta\xi)| \leq \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] ds$$

$$\begin{aligned}
& + \frac{12\epsilon}{y^{2N+1}} \int_0^1 (s-s^2)^N \zeta^2 ds + O(y^{-2N+1}/c^2) \\
& \leq \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N (Q_{11}(1+\epsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00}\zeta_0^2) ds \\
& + O(y^{-2N+1}/c^2).
\end{aligned}$$

But since  $Q_{00}^{(0)} = Q_{01}^{(0)} = 0$ ,  $\int_0^1 (s-s^2)^N (2s-1) ds = 0$ , we see

$$\int_0^1 (s-s^2)^N (-2\epsilon'Q_{01}\zeta_0\zeta_1 - \epsilon'Q_{00}\zeta_0^2) ds = O(y^{-2N+1}/c^2).$$

Therefore we get

$$|(\xi|D_U^2\eta\xi)| \leq \frac{2^{2N+1}K^2C(1+\epsilon')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] ds + O(y^{-2N+1}/c^2).$$

Similarly, if  $D^2\phi^* \geq \mu$ , we have

$$(\xi|D_U^2\eta^*\xi) \geq \frac{2^{2N+1}K^2\mu(1-\epsilon'')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi] ds + O(y^{-2N+1}/c^2).$$

Thus we get

$$|(\xi|D_U^2\eta\xi)| \leq \frac{C(1+\epsilon')}{\mu(1-\epsilon'')} (\xi|D_U^2\eta^*\xi) + O(y^{-2N+1}/c^2).$$

But we know

$$(\xi|D_U^2\eta^*\xi) \geq \kappa|\xi|^2 y^{-2N+1}.$$

Hence if  $c$  is sufficiently large we get the required estimate. QED.

As for the first derivatives, the following conclusion is now clear.

**Proposition 23** *On each compact subset of  $\{\rho \geq 0\}$ , we have*

$$|\frac{\partial\eta}{\partial E}| + |\frac{\partial\eta}{\partial F}| \leq C.$$

## 8 Usefull entropies

Let us consider an entropy  $\eta$  generated by  $\phi$ , that is,

$$\eta(x, y) = \int_{x-y}^{x+y} K(x, y, \xi) \phi(\xi) d\xi. \quad (8.1)$$

The corresponding entropy flux  $q$  is given by integrating the differential equations

$$\frac{\partial q}{\partial w} = \lambda_2 \frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \lambda_1 \frac{\partial \eta}{\partial z}.$$

We can solve these equations as

$$\begin{aligned} q &= \lambda_2 \eta - \int_z^w \frac{\partial \lambda_2}{\partial w} \eta dw \\ &= \lambda_1 \eta + \int_z^w \frac{\partial \lambda_1}{\partial z} \eta dz. \end{aligned}$$

Thus we get the formula

$$q(x, y) = \int_{x-y}^{x+y} L(x, y, \xi) \phi(\xi) d\xi, \quad (8.2)$$

where

$$\begin{aligned} L(x, y, \xi) &= \lambda_1 K(x, y, \xi) + L_1(x, y, \xi) \\ &= \lambda_2 K(x, y, \xi) + L_2(x, y, \xi), \\ L_1(x, y, \xi) &= 2 \int_{(x+y-\xi)/2}^y \mu_1(x+y-Y, Y) K(x+y-Y, Y, \xi) dY, \\ L_2(x, y, \xi) &= -2 \int_{(-x+y+\xi)/2}^y \mu_2(x-y+Y, Y) K(x-y+Y, Y, \xi) dY, \\ \mu_1(x, y) &= \frac{\partial \lambda_1}{\partial z} \\ &= \frac{1-u^2/c^2}{2(1-\sqrt{P'}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}\right) \\ &= \frac{N}{2N+1} + O(1/c^2), \\ \mu_2(x, y) &= \frac{\partial \lambda_2}{\partial w} \\ &= \frac{1-u^2/c^2}{2(1+\sqrt{P'}u/c^2)} \left(1 - \frac{P'}{c^2} + \frac{(\rho + P/c^2)P''}{2P'}\right) \\ &= \frac{N}{2N+1} + O(1/c^2). \end{aligned}$$

In this section we will construct various kinds of usefull entropies.

1) Let us put

$$\begin{aligned} \eta_k^1(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{k\xi} d\xi, \\ \eta_k^2(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) k^{N+1} e^{-k\xi} d\xi. \end{aligned}$$

**Proposition 24** *If  $1/c^2$  is sufficiently small, we have*

$$\eta_k^1 > 0, \quad \eta_k^2 > 0 \quad \text{for } y > 0, \quad (8.3)$$

$$\begin{aligned} \eta_k^1 &= 2^N N! y^N (1 + O(y/c^2)) e^{k(x+y)} (1 + O(1/k)), \\ \eta_k^2 &= 2^N N! y^N (1 + O(y/c^2)) e^{-k(x-y)} (1 + O(1/k)) \end{aligned} \quad (8.4)$$

uniformly on each compact subset of  $\{y > 0\}$ . Moreover

$$\begin{aligned} q_k^1 &= \eta_k^1(\lambda_2 + O(1/k)), \\ q_k^2 &= \eta_k^2(\lambda_1 + O(1/k)) \end{aligned} \quad (8.5)$$

uniformly on each compact subset of  $\{y \geq 0\}$  and

$$\eta_k^2 q_k^1 - \eta_k^1 q_k^2 = (2^N N!)^2 y^{2(N-1)} \left( \frac{1}{2N+1} + O(1/c^2) \right) e^{2ky} (y + O(1/k))^3. \quad (8.6)$$

Proof. Since  $K = (1 + O(y/c^2))(y^2 - (x - \xi)^2)^N$ , we see

$$\begin{aligned} \eta_k^1 &= (1 + O(y/c^2)) \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N k^{N+1} e^{k\xi} d\xi \\ &= (1 + O(y/c^2)) 2^{2N+1} y^N e^{kx} f(ky) \end{aligned}$$

where

$$\begin{aligned} f(r) &= r^{N+1} e^{-r} \int_0^1 (s(1-s))^N e^{2rs} ds \\ &= e^r \int_0^r \left( \sigma \left( 1 - \frac{\sigma}{r} \right) \right)^N e^{-2\sigma} d\sigma. \end{aligned}$$

It is easy to see

$$e^{-r} f(r) = 2^{-(N+1)} N! + O(1/r)$$

This implies (8.4). We note

$$\begin{aligned} \eta^1 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{k(x+y)} (y + O(1/k)) \\ \eta^2 &= (1 + O(1/c^2)) 2^N N! y^{N-1} e^{-k(x-y)} (y + O(1/k)) \end{aligned}$$

uniformly on  $\{y \geq 0\}$ . Let us consider the flux. We have

$$\begin{aligned} L_2(x, y, \xi) &= -2 \int_{(-x+y+\xi)/2}^y \mu_2(x - y + Y, Y) K(x - y + Y, Y, \xi) dY \\ &= -2 \left( \frac{N}{2N+1} + O(1/c^2) \right) \int_{(-x+y+\xi)/2}^y (Y^2 - (x - y + Y - \xi)^2)^N dY \\ &= - \left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) (y - x + \xi)^N (y + x - \xi)^{N+1}, \\ q^1 - \lambda_2 \eta^1 &= - \left( \frac{N}{(2N+1)(N+1)} + O(1/c^2) \right) \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi. \end{aligned}$$

But

$$\begin{aligned} 0 &\leq \int_{x-y}^{x+y} (y - x + \xi)^N (y + x - \xi)^{N+1} k^{N+1} e^{k\xi} d\xi \\ &= (N+1) k^N \int_{x-y}^{x+y} (y^2 - (x - \xi)^2)^N e^{k\xi} d\xi \end{aligned}$$

$$\begin{aligned}
& - Nk^N \int_{x-y}^{x+y} (y-x+\xi)^{N-1} (y+x-\xi)^{N+1} e^{k\xi} d\xi \\
& \leq (N+1) \frac{1}{k} \int_{x-y}^{x+y} (y^2 - (x-\xi)^2)^N k^{N+1} e^{k\xi} d\xi.
\end{aligned}$$

Thus

$$q^1 - \lambda_2 \eta^1 = O(1/k) \eta^1.$$

Since

$$\lambda_2 - \lambda_1 = \frac{\sqrt{P'}(1 - u^2/c^2)}{1 - P'u^2/c^4} = \left(\frac{1}{2N+1} + O(1/c^2)\right)y,$$

we have

$$\eta^2 q^1 - \eta^1 q^2 = \eta^1 \eta^2 \left( \left( \frac{1}{2N+1} + O(1/c^2) \right) y + O(1/k) \right).$$

This implies (8.6). QED.

2) Let  $\psi$  be a function in  $C_0^\infty(-1, 1)$  such that  $\psi \geq 0$ ,  $\int \psi = 1$ . We put

$$\begin{aligned}
\phi_n^3(x) &= \psi_n(x) = n\psi(n(x-a)), \\
\phi_n^4(x) &= -D\psi_n(x), \\
\eta_n^3(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) \phi_n^3(\xi) d\xi, \\
\eta_n^4(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) \phi_n^4(\xi) d\xi, \\
\eta^3(x, y) &= K(x, y, a)X, \\
\eta^4(x, y) &= K_\xi(x, y, a)X, \\
q^3(x, y) &= L(x, y, a)X, \\
q^4(x, y) &= L_\xi(x, y, a)X, \\
X &= 1 \quad (x-y < a < x+y) \\
&= \frac{1}{2} \quad (|x-a| = y) \\
&= 0 \quad (|x-a| > y).
\end{aligned}$$

**Proposition 25** As  $n \rightarrow \infty$ , we have

$$\eta_n^3 \rightarrow \eta^3, \quad q_n^3 \rightarrow q^3, \quad \eta_n^4 \rightarrow \eta^4, \quad q_n^4 \rightarrow q^4.$$

Moreover

$$|\eta_n^3| \leq My^{2N}, \quad |q_n^3| \leq My^{2N}(|x| + y), \quad (8.7)$$

$$|\eta_n^4| \leq My^{2N-1}, \quad |q_n^4| \leq My^{2N-1}(|x| + y), \quad (8.8)$$

$$\eta^3 q^4 - \eta^4 q^3 = \frac{N}{(2N+1)(N+1)} (1 + O(1/c^2)) (y^2 - (x-a)^2)^{2N} \quad (8.9)$$

Proof. We note

$$\begin{aligned}
K_\xi &= -(\xi - x)G(x, |\xi - x|, \xi) \frac{1}{2^{N-1}(N-1)!} (y^2 - (x - \xi)^2)^{N-1} + J^N G_\xi \\
&= (2N(x - \xi) + O(1/c^2)(\xi - x)^2)(y^2 - (x - \xi)^2)^{N-1} + O(1/c^2)(y^2 - (x - \xi)^2)^N, \\
L_{1,\xi} &= 2 \int_{(x+y-\xi)/2}^y \mu_1(x + y - Y, Y) K_\xi(x + y - Y, Y, \xi) dY.
\end{aligned}$$

The estimates (8.7), (8.8) can be seen easily. Let us consider

$$\eta^3 q^4 - \eta^4 q^3 = (KL_\xi - LK_\xi)(x, y, a).$$

Suppose  $x - a \geq 0$ . Then

$$\begin{aligned}
\frac{1}{2}(KL_\xi - LK_\xi) &= K \int_{(x+y-a)/2}^y \mu_1 K_\xi(x + y - Y, Y, a) dY - \\
&\quad - K_\xi \int_{(x+y-a)/2}^y \mu_1 K(x + y - Y, Y, a) dY.
\end{aligned}$$

We note

$$0 \leq \frac{x + y - a}{2} \leq x - y + Y - a \leq x - a \leq y.$$

Hence we have

$$\begin{aligned}
&\int_{(x+y-a)/2}^y \mu_1 K_\xi(x + y - Y, Y, a) dY \\
&= \left(\frac{N}{2N+1} + O(1/c^2)\right) 2N \int_{(x+y-a)/2}^y (x + y - Y - a)(Y^2 - (x + y - Y - a)^2)^{N-1} dY + \\
&\quad + O(1/c^2) \int_{(x+y-a)/2}^y (Y^2 - (x + y - Y - a)^2)^N dY \\
&= \left(\frac{N^2}{2(2N+1)} + O(1/c^2)\right) (x + y - a)^{N-1} (-x + y + a)^N \frac{1}{N(N+1)} (y + (2N+1)(x - a)) \\
&\quad + O(1/c^2)(y^2 - (x - a)^2)^N.
\end{aligned}$$

Thus

$$\begin{aligned}
&K \int_{(x+y-a)/2}^y \mu_1 K_\xi dY \\
&= \left(\frac{N}{2(2N+1)(N+1)} + O(1/c^2)\right) (y^2 - (x - a)^2)^{2N-1} (-x + y + a)(y + (2N+1)(x - a)) \\
&\quad + O(1/c^2)(y^2 - (x - a)^2)^{2N}.
\end{aligned}$$

Also we have

$$\begin{aligned}
&K_\xi \int_{(x+y-a)/2}^y \mu_1 K dY \\
&= \left(\frac{N^2}{(2N+1)(N+1)} + O(1/c^2)\right) (x - a)(-x + y + a)(y^2 - (x - a)^2)^{2N-1} \\
&\quad + O(1/c^2)(-x + y + a)(y^2 - (x - a)^2)^{2N}.
\end{aligned}$$

Hence

$$\frac{1}{2}(KL_\xi - LK_\xi) = \left(\frac{N}{2(2N+1)(N+1)} + O(1/c^2)\right)(y^2 - (x-a)^2)^{2N}.$$

Here we have used

$$\begin{aligned} 0 &\leq (x-a)(y-(x-a)) \leq y^2 - (x-a)^2, \\ 0 &\leq (y-x+a)(y+(2N+1)(x-a)) \\ &\leq (2N+1)(y^2 - (x-a)^2) \end{aligned}$$

provided that  $0 \leq x-a \leq y$ . When  $x-a \leq 0$ , we can discuss in a similar manner by using  $L_2$ . QED.

3) Let  $\Phi$  be a function in  $C_0^\infty(-1, 1)$  such that  $\int \Phi = 0$  and the support  $\text{supp}\Phi$  is  $[-1+\alpha, 1+\alpha]$ , where  $\alpha$  is a small positive number. We put

$$\begin{aligned} \psi_n(x) &= n\Phi(n(x-a)), \\ \eta_n^5(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) D^{N+1} \psi_n(\xi) d\xi, \\ q_n^5(x, y) &= \int_{x-y}^{x+y} L(x, y, \xi) D^{N+1} \psi_n(\xi) d\xi; \\ \hat{\Phi}(x) &= \frac{d}{dx} \left( x \int_{-1}^x \Phi \right), \\ \hat{\psi}_n(x) &= n\hat{\Phi}(n(x-a)), \\ \eta_n^6(x, y) &= \int_{x-y}^{x+y} K(x, y, \xi) D^{N+1} \hat{\psi}_n(\xi) d\xi, \\ q_n^6(x, y) &= \int_{x-y}^{x+y} L(x, y, \xi) D^{N+1} \hat{\psi}_n(\xi) d\xi; \\ B_n^3 &= \eta_n^3 q_n^5 - \eta_n^5 q_n^3, \\ B_n^4 &= \eta_n^4 q_n^5 - \eta_n^5 q_n^4, \\ B_n &= \eta_n^5 q_n^6 - \eta_n^6 q_n^5. \end{aligned}$$

Let us divide the domain  $\Sigma = \{-B \leq x-y \leq x+y \leq B\}$  into the following 5 parts.

$$\begin{aligned} S_0 &= \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma, \\ S_1 &= \left\{ \frac{1}{n} < x+y-a, x-y-a < -\frac{1}{n} \right\} \cap \Sigma, \\ S_L &= \left\{ -\frac{1}{n} < x+y-a \leq \frac{1}{n}, x-y-a < -\frac{1}{n} \right\} \cap \Sigma, \\ S_R &= \left\{ \frac{1}{n} < x+y-a, -\frac{1}{n} \leq x-y-a < \frac{1}{n} \right\} \cap \Sigma, \\ S &= \Sigma - (S_0 \cup S_1 \cup S_L \cup S_R). \end{aligned}$$

**Proposition 26** *We have*

$$|B_n^3| \leq M/n, \quad |B_n^4| \leq M \quad (8.10)$$

on  $\Sigma$ , and

$$|B_n| \leq M/n \quad (8.11)$$

on  $S_0 \cup S_1 \cup S$ . Moreover, on  $S_L$ , we have

$$B_n = ny^{2N}A_1 + y^NA_2 + A_3, \quad (8.12)$$

where

$$\begin{aligned} A_1 &= \left( \frac{N(2^N N!)^2}{2N+1} + O(1/c^2) \right) \left( \int_{-1}^{n(x+y-a)} \Phi \right)^2, \\ |A_2| &\leq M \left( \left| \int_{-1}^{n(x+y-a)} \Phi \right| + |\Phi(n(x+y-a))| \right), \\ |A_3| &\leq \frac{M}{n}. \end{aligned}$$

On  $S_R$ , we have

$$\begin{aligned} B_n &= ny^{2N}C_1 + y^NC_2 + C_3, \\ C_1 &= \left( \frac{N(2^N N!)^2}{2N+1} + O(1/c^2) \right) \left( \int_{-1}^{n(x-y-a)} \Phi \right)^2, \\ |C_2| &\leq M \left( \left| \int_{-1}^{n(x-y-a)} \Phi \right| + |\Phi(n(x-y-a))| \right), \\ |C_3| &\leq \frac{M}{n}. \end{aligned}$$

Proof. For the simplicity, we write  $\eta_n = \eta_n^5, q_n = q_n^5, \hat{\eta}_n = \eta_n^6, \hat{q}_n = q_n^6$ .

It is easy to see inductively that, for  $G_j = J^j G = K_{N-j}$ , we have

$$\partial_\xi^p G_j = J \partial_\xi^p G_{j-1}$$

for  $j \geq p+1$  and

$$\partial_\xi^p G_p = (-1)^p (\xi - x)^p G(x, |\xi - x|, \xi) + J \partial_\xi^p G_{p-1}.$$

Therefore

$$\partial_\xi^p K = \partial_\xi^p G_N(x, y, \xi) = 0$$

for  $p \leq N-1$  and  $y = |x - \xi|$ . Thus by integration by parts we have

$$\begin{aligned} \eta_n &= (-1)^N \partial_\xi^N K(x, y, x+y) \psi_n(x+y) + \\ &\quad - (-1)^N \partial_\xi^N K(x, y, x-y) \psi_n(x-y) + \\ &\quad + F_n^1(x, y), \\ F_n^1(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi. \end{aligned}$$

We see

$$\partial_\xi^p L_2(x, y, \xi) = -2 \int_{(-x+y+\xi)/2}^y \mu_2 \partial_\xi^p K(x - y + Y, Y, \xi) dY$$

for  $p \leq N - 1$ . Therefore

$$\partial_\xi^p L_2(x, y, x + y) = \partial_\xi^p L_2(x, y, x - y) = 0$$

for  $p \leq N - 1$ . Moreover we see

$$\partial_\xi^N L_2(x, y, x + y) = 0.$$

Therefore by integration by parts we have

$$\begin{aligned} \sigma_n(x, y) &= q_n(x, y) - \lambda_2 \eta_n(x, y) \\ &= -(-1)^N \partial_\xi^N L_2(x, y, x - y) \psi_n(x - y) + \\ &\quad + F_n^2(x, y), \\ F_n^2(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi. \end{aligned}$$

Similarly

$$\begin{aligned} \bar{\sigma}_n(x, y) &= q_n(x, y) - \lambda_1 \eta_n(x, y) = \\ &= (-1)^N \partial_\xi^N L_1(x, y, x + y) \psi_n(x + y) + \\ &\quad + \bar{F}_n^2(x, y), \\ \bar{F}_n^2(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_1(x, y, \xi) \psi_n(\xi) d\xi. \end{aligned}$$

We note

$$\partial_\xi^N K(x, y, \xi) = (-1)^N (\xi - x)^N G(x, |x - \xi|, \xi) + J \partial_\xi^N G_{n-1}.$$

It is easy to see inductively that

$$\begin{aligned} \partial_\xi^{p+1} G_p(x, y, \xi) &= (-1)^p \frac{p(p+1)}{2} (\xi - x)^{p-1} G(x, |x - \xi|, \xi) + \\ &\quad + (\xi - x)^p H_p(x, \xi) + J \partial_\xi^p G_{p-1}, \end{aligned}$$

where  $H_p = O(1/c^2)$ . Therefore

$$\begin{aligned} \partial_\xi^{N+1} K(x, y, \xi) &= (-1)^N \frac{N(N+1)}{2} (\xi - x)^{N-1} G(x, |\xi - x|, \xi) + \\ &\quad + (\xi - x)^N H_N(x, \xi) + J \partial_\xi^N G_{N-1}. \end{aligned}$$

1) Suppose  $(x, y) \in S$ . Then it is clear that  $\eta^3, \eta^4, q^3, q^4, \eta_n, q_n, \hat{\eta}_n, \hat{q}_n, B_n^3, B_n^4, B_n$  all vanish.

2) Suppose  $(x, y) \in S_0$ . Then we see

$$\begin{aligned}
\eta^3 &= K(x, y, a) \\
&= O((y^2 - (x - a)^2)^N) \\
&= O(n^{-2N}), \\
\eta^4 &= K_\xi(x, y, a) \\
&= O(|x - a|(y^2 - (x - a)^2)^{N-1}) + O((y^2 - (x - a)^2)^N) \\
&= O(n^{-2N+1}), \\
\sigma^3 &= L_2(x, y, a) \\
&= -2 \int_{(-x+y+a)/2}^y \mu_2 K(x - y + Y, Y, a) dY \\
&= O(n^{-2N-1}), \\
\sigma^4 &= L_{2,\xi}(x, y, a) \\
&= -2 \int_{(-x+y+a)/2}^y \mu_2 K_\xi(x - y + Y, Y, a) dY \\
&= O(n^{-2N}).
\end{aligned}$$

Since  $y = O(1/n)$  and  $\psi_n = O(n)$ , we see

$$\begin{aligned}
&(-1)^N \partial_\xi^N K(x, y, x + y) \psi_n(x + y) + \\
&- (-1)^N \partial_\xi^N K(x, y, x - y) \psi_n(x - y) = \\
&= O(n^{-N+1}).
\end{aligned}$$

Since  $F_n^1 = O(1)$ , we have  $\eta_n = O(1)$ . We see

$$\partial_\xi^N L_2(x, y, x - y) = -2 \int_0^y \mu_2 \partial_\xi^N K(x - y + Y, Y, x - y) dY = O(n^{-N-1}).$$

Therefore

$$-(-1)^N \partial_\xi^N L_2(x, y, x - y) \psi_n(x - y) = O(n^{-N}).$$

Since

$$\begin{aligned}
\partial_\xi^{N+1} L_2(x, y, \xi) &= \mu_2 \partial_\xi^N K((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\
&- 2 \int_{(-x+y+\xi)/2}^y \partial_\xi^{N+1} K(x - y + Y, Y, \xi) dY \\
&= O((-x + y + \xi)^N) + O(x + y - \xi),
\end{aligned}$$

we see

$$\begin{aligned}
F_n^2(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi \\
&= O(n^{-1}).
\end{aligned}$$

Hence  $\sigma_n = O(n^{-1})$ . Therefore

$$\begin{aligned} B_n^3 &= \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-2N-1}), \\ B_n^4 &= \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{-2N}), \\ B_n &= \eta_n \hat{\sigma}_n - \hat{\eta}_n \sigma_n = O(n^{-1}). \end{aligned}$$

3) Suppose  $(x, y) \in S_1$ , where  $x + y > a + \frac{1}{n}$  and  $x - y < a - \frac{1}{n}$ . Then  $\psi_n(x+y) = \psi_n(x-y) = \hat{\psi}_n(x+y) = \hat{\psi}_n(x-y) = 0$ . So,  $\eta_n = F_n^1, \sigma_n = F_n^2$ , and so on. But

$$\begin{aligned} F_n^1(x, y) &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi \\ &= (-1)^{N+1} \int_{-1}^1 (\partial_\xi^{N+1} K(x, y, a + \frac{s}{n}) - \partial_\xi^{N+1} K(x, y, a)) \Phi(s) ds \\ &= O(1/n) \end{aligned}$$

since  $\int \Phi = 0$  and  $\partial_\xi^{N+1} K$  is Lipschitz continuous. Same estimates hold for  $F_n^2, \hat{F}_n^1, \hat{F}_n^2$ . Thus

$$\begin{aligned} B_n^3 &= \eta^3 F_n^2 - F_n^1 \sigma^3 = O(1/n), \\ B_n^4 &= \eta^4 F_n^2 - F_n^1 \sigma^4 = O(1/n), \\ B_n &= F_n^1 \hat{F}_n^2 - \hat{F}_n^1 F_n^2 = O(1/n^2). \end{aligned}$$

4) Suppose  $(x, y) \in S_L$ , where  $|x + y - a| \leq 1/n$ . It is easy to see  $\eta^3 = O(n^{-N}), \eta^4 = O(n^{-N+1}), \sigma^3 = O(n^{-N-1}), \sigma^4 = O(n^{-N})$ . Since  $n(x-y-a) < -1$ , we have  $\psi_n(x-y) = 0$ . Thus  $\eta_n = O(n), \sigma_n = F_n^2 = O(1)$ . Therefore

$$\begin{aligned} B_n^3 &= \eta^3 \sigma_n - \eta_n \sigma^3 = O(n^{-N}), \\ B_n^4 &= \eta^4 \sigma_n - \eta_n \sigma^4 = O(n^{1-N}). \end{aligned}$$

Let us estimate  $B_n = \eta_n \hat{\sigma}_n - \hat{\eta}_n \sigma_n$ . Since

$$\begin{aligned} \partial_\xi^{N+1} K &= (-1)^N \frac{N(N+1)}{2} (\xi - x)^{N-1} G(x, |x - \xi|, \xi) + \\ &+ (\xi - x)^N H_N(x, \xi) + J \partial_\xi^N G_{N-1}, \end{aligned}$$

we have

$$\begin{aligned} F_n^1 &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} K(x, y, \xi) \psi_n(\xi) d\xi = \\ &= (-1)^{N+1} ((-1)^N \frac{N(N+1)}{2} 2^N N! (a-x)^{N-1} + F'(x, a)) \int_{-1}^{n(x+y-a)} \Phi + \\ &+ O(1/n) = \\ &= -\frac{N(N+1)}{2} 2^N N! y^{N-1} (1 + F''(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + \\ &+ O(1/n), \end{aligned}$$

where  $F' = O(1/c^2)|x - a|^N$ ,  $F'' = O(1/c^2)$ . On the other hand

$$\partial_\xi^N K(x, y, x + y) = (-1)^N y^N G(x, y, x + y).$$

Hence

$$\begin{aligned} \eta_n &= ny^N G(x, y, x + y) \Phi(n(x + y - a)) + \\ &- \frac{N(N+1)}{2} 2^N N! y^{N-1} (1 + F''(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi + \\ &+ O(1/n). \end{aligned}$$

Since

$$\begin{aligned} \partial_\xi^{N+1} L_2(x, y, \xi) &= \mu_2 \partial_\xi^N K((x - y + \xi)/2, (-x + y + \xi)/2, \xi) + \\ &- 2 \int_{(-x+y+\xi)/2}^y \mu_2 \partial_\xi^{N+1} K(x - y + Y, Y, \xi) dY = \\ &= \left( \frac{N}{2N+1} + O(1/c^2) \right) (-1)^N \left( \frac{-x + y + \xi}{2} \right)^N \times \\ &\times G((x + y + \xi)/2, (-x + y + \xi)/2, \xi) + \\ &+ O(x + y - \xi), \end{aligned}$$

we see

$$\begin{aligned} \sigma_n &= F_n^2 = \\ &= (-1)^{N+1} \int_{x-y}^{x+y} \partial_\xi^{N+1} L_2(x, y, \xi) \psi_n(\xi) d\xi = \\ &= -\frac{N}{2N+1} 2^N N! y^N (1 + L'(x, y, a)) \int_{-1}^{n(x+y-a)} \Phi \\ &+ O(1/n), \end{aligned}$$

where  $L' = O(1/c^2)$ . Here we have used

$$\left( \frac{-x + y + a}{2} \right)^N = \left( y - \frac{x + y - a}{2} \right)^N = y^N + O(1/n).$$

Similar estimates hold for  $\hat{\eta}_n, \hat{\sigma}_n$ . Thus

$$B_n = ny^{2N} A_1 + y^N A_2 + A_3,$$

where

$$\begin{aligned} A_1 &= -G \frac{N}{2N+1} 2^N N! (1 + L') \Phi(\beta) \int_{-1}^{\beta} \hat{\Phi} + \\ &+ G \frac{N}{2N+1} 2^N N! (1 + L') \hat{\Phi}(\beta) \int_{-1}^{\beta} \Phi = \\ &= \frac{N}{2N+1} 2^N N! G(1 + L') \left( \int_{-1}^{\beta} \Phi \right)^2, \\ \beta &= n(x + y - a). \end{aligned}$$

The estimates on  $S_R$  can be obtained in a similar manner considering  $\bar{\sigma}^3, \bar{\sigma}^4, \bar{\sigma}_n$ . QED.

If we put

$$\begin{aligned}\hat{B}_n^3 &= \eta^3 \eta_n^6 - \eta_n^6 q^3, \\ \hat{B}_n^4 &= \eta^4 q_n^6 - \eta_n^6 q^4,\end{aligned}$$

then the same estimates hold.

## 9 Compactness of $\eta_t + q_x$

Let us consider an entropy  $\eta$  generated by  $\phi$  through the generalized Darboux formula and its flux  $q$ . In this section we will prove

**Lemma 1** *Let  $U^\Delta$  be the approximate solutions constructed in Section 4. Then  $\eta(U^\Delta)_t + q(U^\Delta)_x$  lies in a compact subset of  $H_{loc}^{-1}(\Omega)$ ,  $\Omega$  being a bounded open subset of  $\{t \geq 0\}$ .*

Proof. Let  $\Phi$  be a test function and we consider

$$\begin{aligned}J &= \int \int (\eta(U^\Delta) \Phi_t + q(U^\Delta) \Phi_x) dx dt \\ &= N + L + \Sigma, \\ N &= - \int \eta(U^\Delta(+0, x)) \Phi(0, x) dx, \\ L &= \sum_n \int [\eta(U^\Delta(t, x))]_{t=n\Delta t+0}^{t=n\Delta t-\Delta t} \Phi(n\Delta t, x) dx, \\ \Sigma &= \int \sum_{shock} (\sigma[\eta] - [q]) \Phi dt.\end{aligned}$$

Since  $U^\Delta$  is bounded, we see

$$|N| \leq M \|\Phi\|_C.$$

Let us look at  $L$ . We see

$$\begin{aligned}L &= L_1 + L_2, \\ L_1 &= \sum_{j,n} \Phi(n\Delta t, (2j+1)\Delta x) \int_{2j\Delta x}^{(2j+2)\Delta x} [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-\Delta t} dx, \\ L_2 &= \sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} (\Phi(n\Delta t, x) - \Phi(n\Delta t, (2j+1)\Delta x) \times \\ &\quad \times [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-\Delta t} dx.\end{aligned}$$

We note

$$\begin{aligned} [\eta(U^\Delta)]_{t=n\Delta t+0}^{t=n\Delta t-0} &= D_U \eta(U^\Delta(n\Delta t + 0, x)) [U^\Delta] \\ &+ \int_0^1 (1-\theta) ([U^\Delta] |D_U^2(U^\Delta(n\Delta t + 0) + \theta[U^\Delta])| [U^\Delta]) d\theta. \end{aligned}$$

and

$$\int_{2j\Delta x}^{(2j+2)\Delta x} [U^\Delta] dx = 0$$

by the scheme. Therefore

$$|L_1| \leq M \|\Phi\|_C \sum_{j,n} \int \int_0^1 (1-\theta) |F(\theta, \eta)| d\theta dx,$$

where

$$F(\theta, \eta) = ([U^\Delta] |D_U^2 \eta(U^\Delta(n\Delta t + 0) + \theta[U^\Delta])| [U^\Delta]).$$

By Proposition 22 we know  $|F(\theta, \eta)| \leq M F(\theta, \eta^*)$ . But in the proof of Proposition 7 we know

$$\sum_{j,n} \int \int_0^1 (1-\theta) F(\theta, \eta^*) d\theta dx \leq C.$$

Thus we know

$$|L_1| \leq M \|\Phi\|_C.$$

In the proof of Proposition 7 we know

$$\sum_{j,n} \int_{2j\Delta x}^{(2j+2)\Delta x} |[U^\Delta]|^2 dx \leq C.$$

Therefore

$$\begin{aligned} |L_2| &\leq 2^\alpha \|\Phi\|_{C^\alpha} \sum_n \int (\Delta x)^\alpha |[\eta(U^\Delta)]| dx \\ &\leq 2^{\alpha-1} \|\Phi\|_{C^\alpha} \sum_n \int ((\Delta x)^{\alpha+\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} |[\eta(U^\Delta)]|^2) dx \\ &\leq M \|\Phi\|_{C^\alpha} ((\Delta x)^{\alpha-\frac{1}{2}} + (\Delta x)^{\alpha-\frac{1}{2}} \sum \int |[U^\Delta]|^2 dx) \\ &\leq M' (\Delta x)^{\alpha-\frac{1}{2}} \|\Phi\|_{C^\alpha}, \end{aligned}$$

where we use the boundedness of  $D_U \eta$  and  $n = O(1/(\Delta x))$ . Next we look at  $\Sigma$ . Along the shock we have

$$\begin{aligned} \sigma[\eta(U)] &- [q(U)] \\ &= \int_{\rho_L}^{\rho_R} \left( -\frac{d\sigma}{d\rho} \int_0^1 \theta (U - U_L |D_U^2 \eta(U_L + \theta(U - U_L))| (U - U_L)) d\theta \right) d\rho. \end{aligned}$$

This implies

$$|\sigma[\eta] - [q]| \leq M(\sigma[\eta^*] - [q^*]).$$

But we know

$$\int \sum_{shock} (\sigma[\eta^*] - [q^*]) dt \leq C$$

in the proof of Proposition 7. Therefore

$$|\Sigma| \leq M \|\Phi\|_C.$$

Summing up, we know the compactness. QED.

## 10 Convergence of approximate solutions

We consider the approximate solutions  $U^\Delta$  constructed in Section 4. Since  $U^\Delta$  is bounded, there is a sequence  $U^{\Delta_n}$  and a family of Young measures  $\nu_{t,x}$  such that  $\text{supp } \nu_{t,x} \subset \Sigma = \Sigma_B$  and for any continuous function  $f$

$$f(U^{\Delta_n}(t, x)) \rightarrow \bar{f} = \langle \nu_{t,x}, f \rangle$$

in  $L^\infty$  weak star topology. By Lemma 1, we can apply the compensated compactness theory, and we can assume

$$(\eta q' - \eta' q)(U^{\Delta_n}) \rightarrow \langle \nu, q \rangle \langle \nu, q' \rangle - \langle \nu, \eta' \rangle \langle \nu, q \rangle$$

in  $L^\infty$  weak star. Here  $\eta, q; \eta', q'$  are arbitrary Darboux entropy pairs. Thus we have

**Lemma 2** *For any pairs  $(\eta, q), (\eta', q')$  of Darboux entropies-entropy flux, the identity*

$$\langle \nu, \eta q' - \eta' q \rangle = \langle \nu, \eta \rangle \langle \nu, q' \rangle - \langle \nu, \eta' \rangle \langle \nu, q \rangle$$

*holds a.e.-(t, x), where  $\nu = \nu_{t,x}$ .*

Since entropies we will use are countably many, we can assume that the above identity holds outside a null set which is common to all  $\eta$ . We fix  $(t, x)$  at which the identity holds, and we write  $\nu = \nu_{t,x}$ . Of course  $\text{supp } \nu \subset \Sigma$ . Suppose that  $\text{supp } \nu \cap \{\rho > 0\} \neq \emptyset$ . Let  $\Sigma_0$  be the smallest triangle  $\{z_0 \leq z \leq w \leq w_0\}$  such that  $\text{supp } \nu \cap \{\rho > 0\} \subset \Sigma_0$ . Let us denote by  $P_0$  the state  $(w_0, z_0)$ . It will be verified that  $\nu = \delta_{P_0}$ . (the Dirac measure). First we show

**Proposition 27**

$$P_0 \in \text{supp } \nu.$$

Proof. Suppose  $P_0 \notin \text{supp.}\nu$ . Since  $\Sigma_0$  is the smallest triangle containing  $\text{supp.}\nu \cap \{\rho > 0\}$ ,  $w = w_0$  and  $z = z_0$  intersect with  $\text{supp.}\nu \cap \{\rho > 0\}$ . On neighborhoods of these intersection points we have

$$\begin{aligned}\eta^1 &\geq \frac{1}{M}e^{k(w_0-\epsilon)}, \\ \eta^2 &\geq \frac{1}{M}e^{-k(z_0+\epsilon)}.\end{aligned}$$

(See Proposition 24). Since  $\nu, \eta^1, \eta^2$  are nonnegative, we see

$$\begin{aligned}\langle \nu, \eta^1 \rangle &\geq \frac{1}{M}e^{k(w_0-\epsilon)}, \\ \langle \nu, \eta^2 \rangle &\geq \frac{1}{M}e^{-k(z_0+\epsilon)}.\end{aligned}$$

Since  $P_0 \notin \text{supp.}\nu$ , we have

$$\langle \nu, \eta^2 q^1 - \eta^1 q^2 \rangle \leq M e^{k(w_0 - z_0 - \delta)}.$$

Taking  $2\epsilon < \delta$ , we have

$$\begin{aligned}\left| \frac{\langle \nu, q^1 \rangle}{\langle \nu, \eta^1 \rangle} - \frac{\langle \nu, q^2 \rangle}{\langle \nu, \eta^2 \rangle} \right| &= \left| \frac{\langle \nu, \eta^2 q^1 - \eta^1 q^2 \rangle}{\langle \nu, \eta^1 \rangle \langle \nu, \eta^2 \rangle} \right| \\ &\leq M e^{-k(\delta - 2\epsilon)} \\ &\rightarrow 0\end{aligned}$$

as  $k \rightarrow \infty$ . Let  $\beta$  be a sufficiently small positive number, and we put

$$\begin{aligned}\Sigma_2 &= \{z_0 \leq z \leq w < w_0 - \beta\} \\ \Sigma_3 &= \{z_0 \leq z \leq w \leq w_0, w_0 - \beta \leq w\}.\end{aligned}$$

Then

$$\eta^1 e^{-kw} = (1 + O(1/c^2)) 2^N N! y^{N-1} (y + O(1/k))$$

is bounded on  $\Sigma_0$  and we have

$$\langle \nu|_{\Sigma_2}, \eta^1 \rangle \leq M e^{k(w_0 - \beta)}.$$

Taking  $\epsilon = \beta/2$ , we know

$$\frac{\langle \nu|_{\Sigma_2}, \eta^1 \rangle}{\langle \nu, \eta^1 \rangle} \leq M e^{-\beta k/2} \rightarrow 0.$$

Since  $\partial \lambda_2 / \partial w > 0$ , we know

$$\lambda_2(w, z) \geq \lambda_2(w_0 - \beta, z_0)$$

on  $\Sigma_3$ . Therefore we have

$$\begin{aligned} \frac{\langle \nu, q^1 \rangle}{\langle \nu, \eta^1 \rangle} &= \frac{\langle \nu|_{\Sigma_2}, \eta^1 \lambda_2 \rangle}{\langle \nu, \eta^1 \rangle} + \frac{\langle \nu|_{\Sigma_3}, \eta^1 \lambda_2 \rangle}{\langle \nu, \eta^1 \rangle} + \\ &+ O(1/k) \\ &\geq o(1) + \lambda_2(w_0 - \beta, z_0) \end{aligned}$$

Similarly we see

$$\frac{\langle \nu, q^2 \rangle}{\langle \nu, \eta^2 \rangle} \leq o(1) + \lambda_1(w_0, z_0 + \beta).$$

Therefore we have

$$\lambda_2(w_0 - \beta, z_0) - \lambda_1(w_0, z_0 + \beta) \leq 0 + o(1).$$

Passing to the limit, we know

$$\lambda_2(w_0, z_0) \leq \lambda_1(w_0, z_0).$$

But this means  $P_0 \in \{\rho = 0\}$ , a contradiction. QED.

Let us fix  $a$  such that  $z_0 < a < w_0$ . We have

$$\begin{aligned} \langle \nu, B_n^3 \rangle &= \langle \nu, \eta^3 \rangle \langle \nu, q_n^5 \rangle - \langle \nu, \eta_n^5 \rangle \langle \nu, q^3 \rangle, \\ \langle \nu, B_n^4 \rangle &= \langle \nu, \eta^4 \rangle \langle \nu, q_n^5 \rangle - \langle \nu, \eta_n^5 \rangle \langle \nu, q^4 \rangle, \\ \langle \nu, \eta^3 q^4 - \eta^4 q^3 \rangle &= \langle \nu, \eta^3 \rangle \langle \nu, q^4 \rangle - \langle \nu, \eta^4 \rangle \langle \nu, q^3 \rangle, \\ \langle \nu, B_n \rangle &= \langle \nu, \eta_n^5 \rangle \langle \nu, q_n^6 \rangle - \langle \nu, \eta_n^6 \rangle \langle \nu, q_n^5 \rangle. \end{aligned}$$

From (8.8) we know

$$\langle \nu, \eta^3 q^4 - \eta^4 q^3 \rangle > 0$$

and from (8.10) we know

$$\langle \nu, B_n^3 \rangle \rightarrow 0$$

Using these we can prove the following propositions. Proofs can be found in Chen et al [2].

**Proposition 28** As  $n \rightarrow \infty$ ,  $\langle \nu, \eta_n^5 \rangle, \langle \nu, q_n^5 \rangle, \langle \nu, q_n^6 \rangle, \langle \nu, q_n^6 \rangle$  are bounded.

**Proposition 29** As  $n \rightarrow \infty$ , we have  $\langle \nu, B_n \rangle \rightarrow 0$ .

Now, taking

$$\Phi_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

we put

$$\Phi(x) = \frac{1}{\beta}(\Phi_0(\frac{x+\beta}{\beta}) - \Phi_0(\frac{x-\beta}{\beta}))$$

for the generating function of  $\eta_n^5$ . Here  $\beta = (1 - \alpha)/2$ . We put

$$\begin{aligned} S_+ &= \{z \leq w, |w - a| \leq \frac{1 - 3\alpha}{n}\}, \\ S_- &= \{z \leq w, |z - a| \leq \frac{1 - 3\alpha}{n}\}. \end{aligned}$$

**Proposition 30** *As  $n \rightarrow \infty$ , we have*

$$\langle \nu|_{S_+}, ny^{2N} \rangle + \langle \nu|_{S_-}, ny^{2N} \rangle \rightarrow 0.$$

Proof. Put  $S'_L = S_+ \cap S_L$ ,  $S'_R = S_- \cap S_R$ . It is sufficient to prove that

$$\langle \nu|_{S'_L}, ny^{2N} \rangle + \langle \nu|_{S'_R}, ny^{2N} \rangle \rightarrow 0.$$

From (8.11) we have

$$\langle \nu|_{S_L}, ny^{2N} A_1 + y^N A_2 \rangle + \langle \nu|_{S_R}, ny^{2N} C_1 + y^N C_2 \rangle \rightarrow 0.$$

Note

$$A_1 = (\frac{N(2^N N!)^2}{2N+1} + O(1/c^2)) (\int_{-1}^{n(x+y-a)} \Phi)^2 \geq \frac{1}{M_0} > 0$$

on  $S'_L$ . Put

$$E_n = \{0 \leq y \leq (\frac{1}{n})^\mu\},$$

where  $\mu$  is a positive parameter. Then  $|y^N A_2| \leq M(1/n)^{\mu N} = o(1)$  on  $S_L \cap E_n$  and  $|y^N A_2| \leq Mny^{2N}(1/n)^{1-\mu N}$  on  $S_L - E_n$ . Choose  $d_n \searrow 0$  such that

$$\int_{-1+\alpha}^{1-\alpha-d_n} \Phi = - \int_{1-\alpha-d_n}^{1-\alpha} \Phi \geq (1/n)^{\mu_0}.$$

Then

$$(\int_{-1}^H \Phi)^2 \geq (1/n)^{2\mu_0}$$

for  $|H| \leq 1 - \alpha - d_n$ , and

$$|\Phi(H)| + |\int_{-1}^H \Phi| = o(1)$$

for  $1 - \alpha - d_n \leq |H| \leq 1$ . Put

$$S_+^n = S_L \cap \{|w - a| \leq \frac{1 - \alpha - d_n}{n}\}.$$

Then  $S'_L \subset S_+^n \subset S_L$  and

$$|y^N A_2| = o(1)$$

on  $S_L - S_+^n$  and

$$\begin{aligned} ny^{2N}A_1 + y^NA_2 &\geq ny^{2N}\left(\frac{1}{M}(1/n)^{2\mu_0} - M(1/n)^{1-\mu N}\right) \\ &\geq 0 \end{aligned}$$

on  $S_+^n - E_n$ . Here we take  $0 < 2\mu_0 < 1 - \mu N$ . Then

$$\begin{aligned} \langle \nu|_{S_L}, ny^{2N}A_1 + y^NA_2 \rangle &= \langle \nu|_{S_L \cap E_n}, ny^{2N}A_1 \rangle + \\ &+ \langle \nu|_{S_L - E_n}, ny^{2N}A_1 + y^NA_2 \rangle + \\ &+ o(1) \\ &\geq \frac{1}{M_0} \langle \nu|_{S'_L \cap E_n}, ny^{2N} \rangle + \\ &+ \langle \nu|_{S_L - S_+^n \cap E_n}, ny^{2N}A_1 \rangle + \\ &+ \langle \nu|_{S'_L - E_n}, ny^{2N}A_1 + y^NA_2 \rangle + \\ &+ \langle \nu|_{S_+^n - S'_L - E_n}, ny^{2N}A_1 + y^NA_2 \rangle + \\ &+ o(1) \\ &\geq \frac{1}{M_0} \langle \nu|_{S'_L \cap E_n}, ny^{2N} \rangle + \\ &+ \langle \nu|_{S'_L - E_n}, ny^{2N}\left(\frac{1}{M_0} - M(1/n)^{1-\mu N}\right) \rangle + \\ &+ o(1) \\ &\geq \frac{1}{2M_0} \langle \nu|_{S'_L}, ny^{2N} \rangle + \\ &+ o(1). \end{aligned}$$

Similarly we know

$$\langle \nu|_{S_R}, ny^{2N}C_1 + y^NC_2 \rangle \geq \frac{1}{2M_0} \langle \nu|_{S'_R}, ny^{2N} \rangle + o(1)$$

Thus we see

$$\langle \nu|_{S'_L}, ny^{2N} \rangle + \langle \nu|_{S'_R}, ny^{2N} \rangle \rightarrow 0.$$

QED.

**Proposition 31** *We have*

$$\nu|_{\{\rho > 0\}} = \delta_{P_0}.$$

Proof. Proposition 30 says that the projections  $P_w \tilde{\nu}, P_z \tilde{\nu}$  of the measure  $\tilde{\nu} = y^{2N} \nu$  admits the Lebesgue lower derivatives which vanish at any  $a$ . Therefore we can claim that

$$\text{supp.} \nu \cap \{\rho > 0\} = \{P_0\}.$$

Since  $\nu$  is a probability measure, we have

$$\nu|_{\{\rho>0\}} = C\delta_{P_0}.$$

But

$$C(\eta^3 q^4 - \eta^4 q^3) = C^2(\eta^3 q^4 - \eta^4 q^3)$$

at  $P_0$ . Hence  $C = 1$ . QED.

Summing up we get the final

**Theorem 2** *For any  $M_0$  there is a positive number  $\epsilon_0$  such that if the initial data satisfy*

$$0 \leq \rho_0(x) \leq M_0, \quad \left| \frac{c}{2} \log \frac{c + u_0(x)}{c - u_0(x)} \right| \leq M_0.$$

*and if  $1/c^2 \leq \epsilon_0$ , then a subsequence of the approximate solutions  $U^\Delta$  converges a.e. to a limit  $U$  which is a weak solution of the relativistic Euler equation.*

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