

保型形式の周期について

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F を n 次の総実代数体、 B を F 上の quaternion algebra T

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{n-r}, \quad r > 0$$

をみたすものとする。 $G = \text{Res}_{F/\mathbb{Q}}(B^\times)$ とおく。 B の split する F の archimedean places の集合を δ_1 とする。

($|\delta_1| = r$.) F' は $\sum_{\tau \in \delta_1} \tau(\lambda)$, $\lambda \in F$ で生成される代数体とする。 K を G_{A_f} の open compact subgroup, K_∞ を $G_{\infty, f}$ の maximal compact subgroup とする。このとき、 F' の上に定義される Shimura variety (non-connected canonical model) S'_K が定まり、 $S'_K(\mathbb{C}) = G_{\mathbb{C}} \backslash G_A / K K_\infty$ となる。 S'_K の F 上の zeta 函数 $Z(s, S'_K/F')$ は Langlands [8] により決定され、 $\prod_{\pi} L(s, \pi, r_1)$ の形 (一般には若干の修正項がつく) である。ここに π は K により決る G_A の有限個の automorphic representation を動く。 r_1 は G の L -group の $2^r[F':\mathbb{Q}]$ 次元表現である。

この様な π (Hilbert modular form) には、 $\text{Gal}(\overline{\mathbb{Q}}/F)$ の λ -adic 表現 σ_λ が L 函数を保つ形で打ち立てている。(cf. Taylor [18], 及びその引用文献。) 他方 $Z(s, S'_K/F')$ は、 S'_K の cohomology 群における $\text{Gal}(\overline{\mathbb{Q}}/F')$ の λ -adic 表現

から得られる。ゆえに、 $\text{Gal}(\overline{\mathbb{Q}}/F)$ の表現から $\text{Gal}(\overline{\mathbb{Q}}/F')$ の表現を構成する自然な方法があるだろう、と推定される。誘導表現の概念を拡張した対応 $\sigma_\lambda \rightarrow \bigotimes_{\mathbb{Q}_\ell} \text{Inf}_{H'}^{\lambda'} \sigma_\lambda$, $H = \text{Gal}(\overline{\mathbb{Q}}/F)$, $H' = \text{Gal}(\overline{\mathbb{Q}}/F')$ を構成することにより、 $L(\lambda, \pi, r, \ell)$ に attach した λ -adic 表現が explicit に書けることを §1 で示した。

S_K の zeta 函数の特殊値が、Shimura [16] の周期不変量 $P(\chi, \delta, r)$, $Q(\chi, \delta)$ で表わされるであろう、という予想を志村先生が 1989 年に筆者に伝えられた。この志村予想が本稿を書く主要な motivation になった。

$Z(\lambda, S_K/F')$ の特殊値を motive の見地から扱ったのが §2 ~ §5 である。対応 $\sigma_\lambda \rightarrow \bigotimes_{\mathbb{Q}_\ell} \text{Inf}_{H'}^{\lambda'} \sigma_\lambda$ は、 λ -adic 表現の compatible system を compatible system に写す。元の表現 σ_λ が motive から得られているれば、 $\bigotimes_{\mathbb{Q}_\ell} \text{Inf}_{H'}^{\lambda'} \sigma_\lambda$ も motive から得られるであろう、と予想できる。この様にして restriction of scalar を拡張した、 F 上の motive を F' 上の motive に写す functor $M \rightarrow \bigotimes_{\mathbb{Q}_\ell} \text{Res}_{F/F'} M$ を得る。そこで Deligne の予想 [6] を適用すれば、 $L(\lambda, \bigotimes_{\mathbb{Q}_\ell} \text{Res}_{F/F'} M)$ の特殊値が予想の level ではわかる。 M が Hilbert modular form に対応する motive で、 $|\delta| = r \geq 2$ のとき、実際幾つかの $Q(\chi, \delta)$ と π の積で、特殊値の超越部分が表わさ

れる。

この計算で本質的なのは、 M が F 上の motive のとき、Deligne の periods $C^{\pm}(\text{Res}_{F/\mathbb{Q}}(M))$ を、 $\tau \in \text{Hom}(F, \mathbb{C})$ で index づけられる τ -periods の積として表わしておくことである。Deligne は $L(\omega, M) = L(\omega, \text{Res}_{F/\mathbb{Q}}(M))$ であるから、 $C^{\pm}(\text{Res}_{F/\mathbb{Q}}(M))$ を主に考えたが、tensor 積等の操作による period の変化を見るには、この分解が重要である。このような分解は Shimura [16] にある周期の定義から suggest されたものである。

§6 では 志村不変量 $Q(\chi, \delta)$ を予想は仮定せずに考察した。主要結果は $Q(\chi, \delta)$ が consistent に定義できることである。

記号 1° F を有限次代数体とする。 F から $\mathbb{C}$ の中への non-trivial homomorphisms 全体の集合を J_F により表わす。 I_F により、 J_F で生成される free $\mathbb{Z}$ -module を表わす。

2° 体 k 上の有限次元 vector spaces の成す category を $V(k)$ により表わす。

3° 環 R に係数をもつ、 n 行 m 列行列全体の集合を $M_{n,m}(R)$ とかく。 $M_{m,n}(R)$ を $M_n(R)$ と略す。

4° 群 N, H と準同型 $H \rightarrow \text{Aut}(N)$ が与えられたとき N と H の半直積を $N \rtimes H$ とかく。

5° 加群 M に群 G が作用しているとき、 M^G により M の G 不変元の成す加群を表わす。

6° $\mathbb{H}$ は複素上半平面を表わす。

7° $a, b \in \mathbb{C}$ とする。 a, b の少なくとも 하나가 non-zero であつて、その商 ($a \neq 0$ のときは b/a) が $\overline{\mathbb{Q}}$ に入るとき、 $a \sim b$ とかく。

§1. New functors $\otimes_{\Omega} \text{Ind}$, $\otimes_{\Omega} \text{Res}$ and zeta functions of Shimura varieties

Let G be a group and H be a subgroup of index $n < \infty$. Let k be a field and V be a vector space over k of dimension $d < \infty$. Let σ be a representation of H into $GL(V)$. Let $G = \cup_{i=1}^n s_i H$ be a coset decomposition and put $\Omega = \cup_{i=1}^r s_i H$, where r is any integer such that $1 \leq r \leq n$. Let H' be the stabilizer of Ω under the natural action of G on G/H :

$$(1.1) \quad H' = \{g \in G \mid g\Omega = \Omega\}.$$

We can construct a representation τ of H' in the following way. For every i , $1 \leq i \leq r$, we prepare a vector space $s_i V$ over k which is isomorphic to V . Take $g \in H'$. Then we have

$$gs_i = s_{j(i)} h_i, \quad 1 \leq i \leq r, \quad h_i \in H.$$

Here $i \rightarrow j(i)$ is a permutation on r -letters. Put $W = \otimes_{i=1}^r s_i V$ and set

$$(1.2) \quad \tau(g)(\otimes_{i=1}^r s_i v_i) = \otimes_{i=1}^r s_{j(i)} \sigma(h_i) v_i, \quad s_i v_i \in s_i V_i.$$

Extending (1.2) k -linearly to whole W , we can easily verify that τ defines a representation of H' on W .

The definition (1.2) is somewhat informal. We can rewrite it as follows. Let V_i , $1 \leq i \leq r$ be a vector space over k isomorphic to V . Put $W_1 = \otimes_{i=1}^r V_i$. For $g \in H'$, set

$$g^{-1} s_i = s_{k(i)} h_i^*, \quad 1 \leq i \leq r, \quad h_i^* \in H.$$

Then we find $i \rightarrow k(i)$ is a permutation on r -letters and that

$$j(k(i)) = i, \quad h_{k(i)} = (h_i^*)^{-1} = s_i^{-1} g s_{k(i)}.$$

Put

$$(1.3) \quad \tau_1(g)(\otimes_{i=1}^r v_i) = \otimes_{i=1}^r \sigma(s_i^{-1} g s_{k(i)}) v_{k(i)}, \quad v_i \in V_i.$$

This is merely a reformulation of (1.2) identifying $s_i V$ with V_i . Thus, by (1.3), τ_1 defines a representation of H' on W_1 which is equivalent to τ . We see easily that the equivalence class of τ does not depend on a choice of $\{s_i\}$. We denote τ_1 by $\otimes_{\Omega} \text{Ind}_H^{H'} \sigma$ or $\otimes_{\Omega} \text{Ind}(\sigma; H \rightarrow H')$. We can perform similar construction replacing $\otimes$ by $\oplus$. The representation constructed using $\oplus$ instead of $\otimes$ in (1.3) shall be denoted by $\oplus_{\Omega} \text{Ind}_H^{H'} \sigma$ or $\oplus_{\Omega} \text{Ind}(\sigma; H \rightarrow H')$. We have

$$(1.4) \quad \dim(\otimes_{\Omega} \text{Ind}_H^{H'} \sigma) = (\dim \sigma)^r, \quad \dim(\oplus_{\Omega} \text{Ind}_H^{H'} \sigma) = r(\dim \sigma).$$

Examples. (1) If $\Omega = G$, then $H' = G$. Clearly $\oplus_G \text{Ind}_H^G \sigma$ is the usual induced representation.

(2) If $\Omega = H$, then $H' = H$. We have $\oplus_H \text{Ind}_H^H \sigma \cong \oplus_H \text{Ind}_H^H \sigma \cong \sigma$.

(3) Assume $\Omega = G$, $\dim \sigma = 1$. Then $\sigma \longrightarrow \oplus_G \text{Ind}_H^G \sigma$ is the dual map of the transfer map $G/[G, G] \longrightarrow H/[H, H]$.

Let $\tau = \oplus_\Omega \text{Ind}_H^{H'} \sigma$ be realized by (1.3). Let χ_σ and χ_τ denote characters of σ and τ respectively. We can express χ_τ in the following way. Let $\{e_1, \dots, e_d\}$ be a basis of V over k . Put

$$\sigma(h)e_i = \sum_{j=1}^d \sigma_{ji}(h)e_j, \quad h \in H, \quad 1 \leq i \leq d.$$

Then $\{e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_r}\}$ make a basis of $\otimes_{i=1}^r V_i$ when $j_1, \dots, j_r$ run over $[1, d]^r$. We have

$$\begin{aligned} \tau(g)(\otimes_{i=1}^r e_{j_i}) &= \otimes_{i=1}^r \sigma(s_i^{-1} g s_{k(i)}) e_{j_{k(i)}}, \\ \sigma(s_i^{-1} g s_{k(i)}) e_{j_{k(i)}} &= \sum_{l=1}^d \sigma_{lj_{k(i)}}(s_i^{-1} g s_{k(i)}) e_l. \end{aligned}$$

Hence $\otimes_{i=1}^r e_{j_i}$ contributes

$$\prod_{i=1}^r \sigma_{j_i j_{k(i)}}(s_i^{-1} g s_{k(i)})$$

to the trace. Therefore we obtain

$$(1.5) \quad \chi_\tau(g) = \sum_{j_1, \dots, j_r \in [1, d]^r} \prod_{i=1}^r \sigma_{j_i j_{k(i)}}(s_i^{-1} g s_{k(i)}), \quad g \in H'.$$

If $g \in \cap_{i=1}^r s_i H s_i^{-1} \subseteq H'$, then (1.5) simplifies to

$$(1.6) \quad \chi_\tau(g) = \prod_{i=1}^r \chi_\sigma(s_i^{-1} g s_i), \quad g \in \cap_{i=1}^r s_i H s_i^{-1}.$$

The above construction $\oplus_\Omega \text{Ind}_H^{H'} \sigma$ applies also to the case where σ is a λ -adic representation of a Galois group or σ is a representation of a Weil group. In other words, the continuity condition of $\oplus_\Omega \text{Ind}_H^{H'} \sigma$ can easily be derived from that of σ .

We shall consider the case of λ -adic representation in more detail. Let F and E be algebraic number fields of finite degree. Let λ be a finite place of E and let

$$\sigma_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(V)$$

be a λ -adic representation. Here V is a $d < \infty$ dimensional vector space over E_λ . Take $G = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, $H = \text{Gal}(\overline{\mathbf{Q}}/F)$, $G = \cup_{i=1}^r s_i H$, $\Omega = \cup_{i=1}^r s_i H$ and apply the

above construction. Let F' be the fixed field of H' and $\tilde{F}$ be the normal closure of F . Put $K = \text{Gal}(\overline{\mathbf{Q}}/\tilde{F})$. We have $H' = \text{Gal}(\overline{\mathbf{Q}}/F') \supseteq K$. Then $\tau_\lambda = \bigotimes_{\Omega} \text{Ind}_H^{H'} \sigma_\lambda$ defines a λ -adic representation

$$\tau_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow GL(W)$$

where W is a d' -dimensional vector space over E_λ .

Let S be a finite set of prime ideals of F . We assume that σ_λ is unramified outside of S . If $\mathfrak{p}$ is a prime ideal of F such that $\mathfrak{p} \notin S$, we set

$$(1.7) \quad f_{\mathfrak{p}}(\sigma_\lambda, X) = \det(1 - \sigma_\lambda(F_{\mathfrak{p}})X) \in E_\lambda[X],$$

where $F_{\mathfrak{p}}$ denotes a representative from the Frobenius conjugacy class of $\mathfrak{p}$ in $\text{Gal}(\overline{\mathbf{Q}}/F)$.

Let $\mathfrak{p}'$ be a prime ideal of F' and let $\mathfrak{P}'$ be a prime divisor of $\mathfrak{p}'$ in $\tilde{F}$. For $1 \leq i \leq r$, let $\mathfrak{p}_i$ be the restriction of $s_i^{-1}\mathfrak{P}'$ to F . Then the set of prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\}$ does not depend on the choice of $\mathfrak{P}'$ and $\{s_i\}$. Let S' be the set of $\mathfrak{p}'$ such that either one of $\mathfrak{p}_i$ ramifies in $F/\mathbf{Q}$ or that σ_λ ramifies at one of $\mathfrak{p}_i$.

THEOREM 1.1. *Let the notation be the same as above. Then τ_λ is unramified outside of S' . If $f_{\mathfrak{p}}(\sigma_\lambda, X) \in E[X]$ whenever $\mathfrak{p} \notin S$, then for any prime ideal $\mathfrak{p}' \notin S'$ of F' , we have $f_{\mathfrak{p}}(\tau_\lambda, X) \in E[X]$. Furthermore if λ' is another finite place of E and $\sigma_{\lambda'}$ is a λ' -representation of $\text{Gal}(\overline{\mathbf{Q}}/F)$ unramified outside of S such that $f_{\mathfrak{p}}(\sigma_\lambda, X) = f_{\mathfrak{p}}(\sigma_{\lambda'}, X)$ if $\mathfrak{p} \notin S$, then we have $f_{\mathfrak{p}}(\tau_\lambda, X) = f_{\mathfrak{p}}(\tau_{\lambda'}, X)$ for $\mathfrak{p}' \notin S'$.*

PROOF: From the realization of τ_λ by (1.3), we have

$$(1.8) \quad \text{Ker}(\tau_\lambda) \supseteq \bigcap_{i=1}^r s_i \text{Ker}(\sigma_\lambda) s_i^{-1}.$$

Assume $\mathfrak{p}' \notin S'$. First we shall show that τ_λ is unramified at $\mathfrak{p}'$. Let $\tilde{\mathfrak{P}}'$ be a prime divisor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$ and let $I_{\tilde{\mathfrak{P}}'}$ be the inertia group of $\tilde{\mathfrak{P}}'$. It suffices to show $\tau_\lambda(I_{\tilde{\mathfrak{P}}'}) = \{1\}$. By (1.8), this assertion follows if we could show

$$(1.9) \quad s_i^{-1} I_{\tilde{\mathfrak{P}}', s_i} \subseteq H,$$

$$(1.10) \quad \sigma_\lambda(s_i^{-1} I_{\tilde{\mathfrak{P}}', s_i}) = \{1\}$$

for every $1 \leq i \leq r$. Let $\mathfrak{P}'$ be a prime ideal of $\tilde{F}$ which lies under $\tilde{\mathfrak{P}}'$. Since $s_i^{-1} I_{\tilde{\mathfrak{P}}', s_i} = I_{s_i^{-1}\mathfrak{P}'}$, (1.9) is equivalent to $I_{s_i^{-1}\mathfrak{P}'} \subseteq \text{Gal}(\tilde{F}/F)$, where $I_{s_i^{-1}\mathfrak{P}'}$ is the inertia group of $s_i^{-1}\mathfrak{P}'$ in $\text{Gal}(\tilde{F}/\mathbf{Q})$. This condition is equivalent to that $s_i^{-1}\mathfrak{P}'$ is unramified in $F/\mathbf{Q}$, i.e., $\mathfrak{p}_i$ is unramified in $F/\mathbf{Q}$. Hence (1.9) is verified. Since $s_i^{-1}\tilde{\mathfrak{P}}'$ is a prime divisor of $\mathfrak{p}_i$ in $\overline{\mathbf{Q}}$, (1.10) follows from the assumption that σ_λ is unramified at $\mathfrak{p}_i$.

Next we shall show E -rationality for $\mathfrak{p}' \notin S'$. Put

$$\begin{aligned} K &= \text{Gal}(\overline{\mathbf{Q}}/\tilde{F}), & \overline{G} &= G/K, & \overline{H} &= H/K, \\ \overline{\Omega} &= \Omega/K, & \overline{F}_{\mathfrak{p}'} &= F_{\mathfrak{p}'} \bmod K \in \overline{G}. \end{aligned}$$

Let U be the cyclic subgroup of $\overline{G}$ generated by $\overline{F}_{\mathfrak{p}'}$. Let

$$(1.11) \quad \overline{\Omega} = \cup_{j=1}^m U \overline{t}_j \overline{H}, \quad \overline{t}_j \in \overline{G}$$

be a double coset decomposition of $\overline{\Omega}$. For every j , let n_j be the minimal positive integer a such that $\overline{F}_{\mathfrak{p}'}^a \in \overline{t}_j \overline{H} \overline{t}_j^{-1}$. Then

$$(1.12) \quad \overline{\Omega} = \cup_{j=1}^m \cup_{i=0}^{n_j-1} \overline{F}_{\mathfrak{p}'}^i \overline{t}_j \overline{H}$$

is a coset decomposition of $\overline{\Omega}$. Take $t_j \in G$ so that $t_j \bmod K = \overline{t}_j$. Then

$$(1.13) \quad \Omega = \cup_{j=1}^m \cup_{i=0}^{n_j-1} s_{ij} H, \quad s_{ij} = F_{\mathfrak{p}'}^i t_j$$

is a coset decomposition of Ω . We may realize τ_λ using (1.13) and (1.3) on

$$W = \bigotimes_{j=1}^m \bigotimes_{i=0}^{n_j-1} V_{ij}, \quad V_{ij} \cong V.$$

Since

$$F_{\mathfrak{p}'}^{-1} s_{ij} H = \begin{cases} s_{n_j-1,j} H & \text{if } i = 0, \\ s_{i-1,j} H & \text{if } 1 \leq i \leq n_j - 1, \end{cases}$$

we have

$$\tau_\lambda(F_{\mathfrak{p}'}) (\otimes_{j=1}^m \otimes_{i=1}^{n_j-1} v_{ij}) = \otimes_{j=1}^m (\sigma_\lambda(t_j^{-1} F_{\mathfrak{p}'}^{n_j} t_j) v_{n_j-1,j} \otimes (\otimes_{i=1}^{n_j-1} v_{i-1,j})).$$

For $1 \leq j \leq m$, let A_j be the linear operator on $\otimes_{i=0}^{n_j-1} V_{ij}$ defined by

$$A_j(\otimes_{i=1}^{n_j-1} v_{ij}) = \sigma_\lambda(t_j^{-1} F_{\mathfrak{p}'}^{n_j} t_j) v_{n_j-1,j} \otimes (\otimes_{i=1}^{n_j-1} v_{i-1,j}).$$

Then we have $\tau_\lambda(F_{\mathfrak{p}'}) = \otimes_{j=1}^m A_j$. Therefore it suffices to show

$$(1.14) \quad \det(1 - A_j X) \in E[X], \quad 1 \leq j \leq m.$$

Let $\tilde{\mathfrak{p}}'$ be a prime divisor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$ and let $F_{\tilde{\mathfrak{p}'}} \in \text{Gal}(\overline{\mathbf{Q}}/F')$ be a Frobenius element of $\tilde{\mathfrak{p}}'$. We may take $F_{\mathfrak{p}'} = F_{\tilde{\mathfrak{p}'}}$. We have $t_j^{-1} F_{\mathfrak{p}'}^{n_j} t_j = F_{t_j^{-1} \tilde{\mathfrak{p}'}}^{n_j}$. Let $\mathfrak{p}_j$ (resp. $\mathfrak{p}'_j$) be the restriction of $t_j^{-1} \tilde{\mathfrak{p}}'$ to F (resp. F'). Let f_j (resp. f'_j) be the degree of

$\mathfrak{p}_j$ (resp. $\mathfrak{p}'_j$) in $F/\mathbf{Q}$ (resp. $F'/\mathbf{Q}$). Let $\tilde{F}_{t_j^{-1}\tilde{\mathfrak{p}}'}$ be a Frobenius element of $t_j^{-1}\tilde{\mathfrak{p}}'$ in $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We may assume

$$(\tilde{F}_{t_j^{-1}\tilde{\mathfrak{p}}'})^{f_j} = F_{\mathfrak{p}_j}, \quad (\tilde{F}_{t_j^{-1}\tilde{\mathfrak{p}}'})^{f'_j} = F_{\mathfrak{p}'_j} = F_{t_j^{-1}\tilde{\mathfrak{p}}'}.$$

Hence we have

$$t_j^{-1}F_{\mathfrak{p}'_j}^{n_j}t_j = (\tilde{F}_{t_j^{-1}\tilde{\mathfrak{p}}'})^{n_j f'_j} \in H.$$

Therefore f_j must divide $n_j f'_j$ and we obtain

$$t_j^{-1}F_{\mathfrak{p}'_j}^{n_j}t_j = F_{\mathfrak{p}_j}^{n_j f'_j / f_j}.$$

Now the assertion (1.14) and also the last assertion of Theorem 1.1 follows from the next Lemma.

LEMMA 1.2. *Let V be a finite dimensional vector space over a field k and let $A \in \text{End}(V)$. Let $W = \bigotimes_{i=0}^{n-1} V_i$, $V_i \cong V$. Define $A_1 \in \text{End}(W)$ by*

$$A_1(\bigotimes_{i=0}^{n-1} v_i) = Av_{n-1} \otimes (\bigotimes_{i=1}^{n-1} v_{i-1}).$$

Put $f_A(X) = \det(1 - AX)$, $f_{A_1}(X) = \det(1 - A_1X)$. Then $f_{A_1}(X)$ depends only on $f_A(X)$. Furthermore if k_0 is a subfield of k such that $f_A(X) \in k_0[X]$, then we have $f_{A_1}(X) \in k_0[X]$.

The proof is omitted since it is easy. This completes the proof of Theorem 1.1.

Let $\sigma_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(V)$ and $\tau_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow GL(W)$ be as above. If σ_λ is of Hodge-Tate type, then we can show that τ_λ is also of Hodge-Tate type and its type can be determined.

Let p be a rational prime which lies under λ . Set

$$\mu_{p^\infty} = \{ z \in \mathbf{Q} \mid z^{p^a} = 1 \text{ for some } 1 \leq a \in \mathbf{Z} \}.$$

Define a homomorphism χ of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\mathbf{Z}_p^\times$ by

$$g(z) = z^{\chi(g)}, \quad g \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), \quad z \in \mu_{p^\infty}.$$

Let $\mathfrak{p}$ be a prime factor of p in F and take a prime divisor $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$ in $\overline{\mathbf{Q}}$. We identify $\overline{\mathbf{Q}}_{\mathfrak{p}}$ with $\overline{\mathbf{Q}}_{\tilde{\mathfrak{p}}}$ and consider μ_{p^∞} as a subgroup of $\overline{\mathbf{Q}}_{\mathfrak{p}}^\times$. We regard E_λ as a subfield of $\overline{\mathbf{Q}}_{\mathfrak{p}}$. Let $\mathbf{C}_p = \hat{\overline{\mathbf{Q}}}_{\mathfrak{p}}$ be the completion of $\overline{\mathbf{Q}}_{\mathfrak{p}}$. Put $V_{\mathbf{C}_p} = \mathbf{C}_p \otimes_{E_\lambda} V$. Then $\text{Gal}(\overline{\mathbf{Q}}_{\tilde{\mathfrak{p}}}/F_{\mathfrak{p}} \vee E_\lambda) \ni g$ acts on $V_{\mathbf{C}_p}$ by

$$g(c \otimes v) = g(c) \otimes \sigma_\lambda(g)v, \quad c \in \mathbf{C}_p, \quad v \in V.$$

For every $i \in \mathbf{Z}$, let

$$V^i = \{ v \in V_{\mathbf{C}_p} \mid g(v) = \chi(g)^i v \text{ for every } g \in \text{Gal}(\overline{\mathbf{Q}}_{\mathfrak{p}}/F_{\mathfrak{p}} \vee E_{\lambda}) \}$$

and put $V(i) = \mathbf{C}_p \otimes_{F_{\mathfrak{p}} \vee E_{\lambda}} V^i$. Then $\oplus_{i \in \mathbf{Z}} V(i)$ can be considered as a sub $\mathbf{C}_p$ -vector space of $V_{\mathbf{C}_p}$ (cf. Serre [10], III-6). We call σ_{λ} is of Hodge-Tate type at $\mathfrak{p}$ if

$$(1.15) \quad V_{\mathbf{C}_p} = \oplus_{i \in \mathbf{Z}} V(i_{\mathfrak{p}}).$$

PROPOSITION 1.3. *Assume that σ_{λ} is of Hodge-Tate type (1.15) at every prime factor $\mathfrak{p}$ of p in F . Let $\mathfrak{p}'$ be any prime factor of p in F' . Define prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of F as above. Then τ_{λ} is of Hodge-Tate type at $\mathfrak{p}'$ such that*

$$\mathbf{C}_p \otimes_{E_{\lambda}} W = \oplus_{i_{\mathfrak{p}_1}, \dots, i_{\mathfrak{p}_r} \in \mathbf{Z}^r} W(i_{\mathfrak{p}_1} + i_{\mathfrak{p}_2} + \dots + i_{\mathfrak{p}_r}).$$

PROOF: Let $\mathfrak{P}'$ be a prime factor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$. Let $\tilde{F}$ be the normal closure of F in $\overline{\mathbf{Q}}$ and $\mathfrak{P}'$ be the restriction of $\mathfrak{P}'$ to $\tilde{F}$. For every $1 \leq i \leq r$, take $v_i \in V^{i_{\mathfrak{p}_i}}$ so that

$$g v_i = \chi(g)^{i_{\mathfrak{p}_i}} v_i, \quad g \in \text{Gal}(\overline{\mathbf{Q}}_{\mathfrak{P}'}/F_{\mathfrak{p}_i} \vee E_{\lambda}).$$

Then $v_1 \otimes \dots \otimes v_r \in \mathbf{C}_p \otimes_{E_{\lambda}} W$ and we can easily verify that

$$g(\otimes_{i=1}^r v_i) = \chi(g)^{i_{\mathfrak{p}_1} + \dots + i_{\mathfrak{p}_r}} (\otimes_{i=1}^r v_i) \quad \text{if } g \in \text{Gal}(\overline{\mathbf{Q}}_{\mathfrak{P}'}/\tilde{F}_{\mathfrak{P}'} \vee E_{\lambda}).$$

In view of the injectivity result of [10], III-6 and III-31, Theorem 1, the assertion follows immediately.

Let $\sigma_{\lambda} : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow GL(V)$ be as before. We define L -series $L(s, \sigma_{\lambda})$ attached to σ_{λ} by

$$L(s, \sigma_{\lambda}) = \prod_{\mathfrak{p} \notin S} \det(1 - \sigma_{\lambda}(F_{\mathfrak{p}}) N(\mathfrak{p})^{-s})^{-1},$$

a formal Dirichlet series with coefficients in E_{λ} . We assume that F is normal over $\mathbf{Q}$ until (1.17). For $1 \leq i \leq r$, put

$$\sigma_{\lambda}^i(h) = \sigma_{\lambda}(s_i^{-1} h s_i), \quad h \in H = \text{Gal}(\overline{\mathbf{Q}}/F).$$

Put $\tau_{\lambda} = \otimes_{\Omega} \text{Ind}_H^{H'} \sigma_{\lambda}$. Then we have

$$(1.16) \quad \tau_{\lambda}|_H \cong \otimes_{i=1}^r \sigma_{\lambda}^i.$$

Therefore we obtain

$$\text{Ind}_H^G(\otimes_{i=1}^r \sigma_{\lambda}^i) \cong \oplus_{\chi \in \widehat{G/H}} (\tau_{\lambda} \otimes \chi).$$

Here χ extends over irreducible representations of G/H and we have assumed that E_λ is sufficiently large so that every χ is realized over E_λ . Now the well known property of L -series yields

$$(1.17) \quad L(s, \otimes_{i=1}^r \sigma_\lambda^i) = \prod_{\chi \in \widehat{G/H}} L(s, \tau_\lambda \otimes \chi)$$

up to finitely many Euler factors.

We are going to consider a relation between $\otimes_\Omega \text{Ind}_H^{H'} \sigma_\lambda$ and the Langlands L -function used to express the zeta functions of certain Shimura varieties (cf. Langlands [8]). Let F be a totally real algebraic number field and B be a quaternion algebra over F . Set $H = \text{Gal}(\overline{\mathbf{Q}}/F)$. Fix an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$. Then J_F can be identified with $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$. Let $G = \text{Res}_{F/\mathbf{Q}}(B^\times)$. Then the L -group ${}^L G$ of G is given by

$${}^L G = GL_2(\mathbf{C})^{J_F} \times_s \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$$

where the multiplication is defined by

$$(g_1, \sigma_1)(g_2, \sigma_2) = (g_1 \sigma_1(g_2), \sigma_1 \sigma_2), \quad g_1, g_2 \in GL_2(\mathbf{C})^{J_F}, \quad \sigma_1, \sigma_2 \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}).$$

Here we take the action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $GL_2(\mathbf{C})^{J_F}$ by

$$\sigma(g) = (g_{\sigma^{-1}\tau})_{\tau \in J_F} \quad \text{for} \quad g = (g_\tau)_{\tau \in J_F}, \quad g_\tau \in GL_2(\mathbf{C}).$$

Put ${}^L G^0 = GL_2(\mathbf{C})^{J_F}$. We shall define two representations of ${}^L G$. Let $V = \oplus_{\tau \in J_F} V_\tau$, $V_\tau \cong \mathbf{C}^2$. Let r_0^* be the standard representation of ${}^L G^0$ on V , i.e.,

$$g(\oplus_{\tau \in J_F} v_\tau) = \oplus_{\tau \in J_F} g_\tau v_\tau, \quad g = (g_\tau) \in {}^L G^0.$$

For $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, define $I_\sigma \in GL(V)$ by

$$I_\sigma(\oplus_{\tau \in J_F} v_\tau) = \oplus_{\tau \in J_F} v_{\sigma^{-1}\tau}.$$

Then we can verify

$$(1.18) \quad I_{\sigma_1 \sigma_2} = I_{\sigma_1} I_{\sigma_2}, \quad \sigma_1, \sigma_2 \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

$$(1.19) \quad I_\sigma r_0^*(g) = r_0^*(\sigma(g)) I_\sigma, \quad \sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), \quad g \in {}^L G^0.$$

Put

$$r_0((g, \sigma)) = r_0^*(g) I_\sigma, \quad (g, \sigma) \in {}^L G.$$

By (1.18) and (1.19), we see easily that r_0 defines a representation of ${}^L G$ on V .

Let δ be a subset of J_F at which B splits and B ramifies at $J_F \setminus \delta$. We assume that δ is not empty. Since J_F is identified with $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$, δ can be identified with a subset Ω of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$. Let

$$H' = \{g \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \mid g\Omega = \Omega\}$$

and let F' be the subfield of $\overline{\mathbf{Q}}$ which corresponds to H' . Let $W = \bigotimes_{\tau \in \delta} V_\tau$, $V_\tau \cong \mathbf{C}^2$. Let r_1^* be the representation of ${}^L G^0$ on W defined by

$$g(\bigotimes_{\tau \in \delta} v_\tau) = \bigotimes_{\tau \in \delta} g_\tau v_\tau, \quad g = (g_\tau) \in {}^L G^0.$$

For $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/F')$, define $J_\sigma \in GL(W)$ by

$$J_\sigma(\bigotimes_{\tau \in \delta} v_\tau) = \bigotimes_{\tau \in \delta} v_{\sigma^{-1}\tau}.$$

Then we have

$$J_{\sigma_1 \sigma_2} = J_{\sigma_1} J_{\sigma_2}, \quad J_\sigma r_1^*(g) = r_1^*(\sigma(g)) J_\sigma$$

for $\sigma_1, \sigma_2, \sigma \in \text{Gal}(\overline{\mathbf{Q}}/F')$, $g \in {}^L G^0$. Therefore we can define a representation $r_1^{(0)}$ of $GL_2(\mathbf{C})^{J_F} \times_s \text{Gal}(\overline{\mathbf{Q}}/F')$ by

$$r_1^{(0)}((g, \sigma)) = r_1^*(g) J_\sigma, \quad g \in {}^L G^0, \quad \sigma \in \text{Gal}(\overline{\mathbf{Q}}/F').$$

Then we let

$$r_1 = \text{Ind}(r_1^{(0)}; {}^L G^0 \times_s \text{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow {}^L G).$$

THEOREM 1.4. Let π be an automorphic representation of G_A . Let E be an algebraic number field of finite degree and λ be a finite place of E . Let $\sigma_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL_2(E_\lambda)$ be a λ -adic representation. We assume that $L(s, \pi, r_0) = L(s, \sigma_\lambda)$ holds up to finitely many Euler factors, when we fix an embedding of E_λ into $\mathbf{C}$ and consider two L -series as Euler products over rational primes. Then we have

$$L(s, \pi, r_1) = L(s, \bigotimes_{\Omega} \text{Ind}_H^{H'} \sigma_\lambda)$$

up to finitely many Euler factors.

PROOF: Let $\iota : E_\lambda \subset \mathbf{C}$ be the fixed embedding. Then $\iota \circ \sigma_\lambda$ defines a homomorphism of $\text{Gal}(\overline{\mathbf{Q}}/F)$ into $GL_2(\mathbf{C})$. Put $\rho = \iota \circ \sigma_\lambda$. Let $G = \bigcup_{i=1}^n s_i H$, $\Omega = \bigcup_{i=1}^r s_i H$. For $g \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, set

$$(1.20) \quad \tilde{\rho}(g) = ((\rho(s_i^{-1} g s_{k(i)})), g) \in {}^L G.$$

Here the meaning of $(\rho(s_i^{-1} g s_{k(i)})) \in {}^L G^0$ is as follows. We identify J_F with $\{s_i | F; 1 \leq i \leq n\}$. Then the s_i -component of $(\rho(s_i^{-1} g s_{k(i)}))$ is $\rho(s_i^{-1} g s_{k(i)}) \in GL_2(\mathbf{C})$. We have set

$$g^{-1} s_i = s_{k(i)} h_i^*, \quad 1 \leq i \leq n, \quad h_i^* \in H.$$

It can be verified that $\tilde{\rho}$ defines a homomorphism of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into ${}^L G$. By the definition of $r_1^{(0)}$, we obtain

$$(1.21) \quad (r_1^{(0)} \circ \tilde{\rho})(g)(\otimes_{i=1}^r v_{s_i}) = \otimes_{i=1}^r \rho(s_i^{-1} g s_{k(i)}) v_{k(i)}, \quad g \in \text{Gal}(\overline{\mathbf{Q}}/F').$$

Comparing (1.21) with (1.3), we get

$$(1.22) \quad \iota \circ \tau_\lambda \cong r_1^{(0)} \circ (\tilde{\rho}|_{\text{Gal}(\overline{\mathbf{Q}}/F')}),$$

where $\tau_\lambda = \otimes_{\Omega} \text{Ind}_H^{H'} \sigma_\lambda$. Now let us show

$$(1.23) \quad \iota \circ \text{Ind}_H^G \tau_\lambda \cong r_1 \circ \tilde{\rho}.$$

Let $G = \cup_{i=1}^m t_i H'$. We realize $\text{Ind}_H^G \tau_\lambda$ by the similar formula to (1.3). Thus $\text{Ind}_H^G \tau_\lambda$ is realized on $\oplus_{i=1}^m W_i$, $W_i \cong W$, where W is the representation space of τ_λ . Put, for $g \in G$,

$$g^{-1} t_i = t_{l(i)} h'_i, \quad 1 \leq i \leq m, \quad h'_i \in H'.$$

Then we have

$$(\text{Ind}_H^G \tau_\lambda)(g)(\oplus_{i=1}^m w_i) = \oplus_{i=1}^m \tau_\lambda(t_i^{-1} g t_{l(i)}) w_{l(i)}, \quad g \in G.$$

On the other hand, take a coset decomposition

$${}^L G = \cup_{i=1}^m \tilde{\rho}(t_i)({}^L G^0 \times_s H')$$

and realize r_1 on $\oplus_{i=1}^m W'_i$, $W'_i \cong W'$ where W' is the representation space of $r_1^{(0)}$. Then we have

$$r_1(\tilde{\rho}(g))(\oplus_{i=1}^m w'_i) = \oplus_{i=1}^m r_1^{(0)}(\tilde{\rho}(t_i^{-1} g t_{l(i)})) w'_{l(i)}, \quad w'_i \in W'_i, \quad g \in G.$$

Since we may take $W'_i = W_i \otimes_{E_\lambda} \mathbf{C}$, (1.23) follows from (1.22). Let p be a rational prime at which π , $\text{Ind}_H^G \sigma_\lambda$ and $\text{Ind}_H^G (\otimes_{\Omega} \text{Ind}_H^{H'} \sigma_\lambda)$ are unramified and also

$$(1.24) \quad L(s, \pi, r_0) = L(s, \sigma_\lambda)$$

holds at Euler p -factors. Fix a Frobenius element $F_p \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of p . Let $\pi = \otimes_p \pi_p \otimes \pi_\infty$ and let $(g_p, F_p) \in {}^L G$ be the Langlands class of π_p . By (1.24), we have

$$(1.25) \quad \tilde{\rho}(F_p) = (g_p, F_p).$$

Therefore we obtain

$$\begin{aligned} \det(1 - X r_1((g_p, F_p))) &= \det(1 - X r_1 \circ \tilde{\rho}(F_p)) \\ &= \det(1 - X(\iota \circ \text{Ind}_H^G \tau_\lambda)(F_p)) = \iota(\det(1 - X(\text{Ind}_H^G \tau_\lambda)(F_p))) \end{aligned}$$

by (1.23). This completes the proof.

Let M be a motive over F with coefficients in E . For every finite place λ of E , the λ -adic realization $H_\lambda(M)$ of M determines a λ -adic representation $\sigma_\lambda : \text{Gal}(\overline{\mathbf{Q}}/F) \rightarrow GL(H_\lambda(M))$; $\{\sigma_\lambda\}$ makes a compatible system of λ -adic representations. Let Ω be any non-empty subset of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/\text{Gal}(\overline{\mathbf{Q}}/F)$ and define F' as before. By Theorem 1.1, we have a compatible system of λ -adic representations $\{\otimes_{\Omega} \text{Ind}_{\text{Gal}(\overline{\mathbf{Q}}/F)}^{\text{Gal}(\overline{\mathbf{Q}}/F')} \sigma_\lambda\}$ of $\text{Gal}(\overline{\mathbf{Q}}/F')$. We conjecture that this system of representations is realized by a motive.

CONJECTURE 1.5. *There exists a motive M' over F' with coefficients in E such that the λ -adic representation of $\text{Gal}(\overline{\mathbf{Q}}/F')$ obtained from M' coincides with $\otimes_{\Omega} \text{Ind}_{\text{Gal}(\overline{\mathbf{Q}}/F)}^{\text{Gal}(\overline{\mathbf{Q}}/F')} \sigma_\lambda$ for every finite place λ of E .*

The rank of M' is $(\text{rank } M)^r$ where $r = |\Omega|$. In analogy with the case of induced representations, we denote the above M' by $\otimes_{\Omega} \text{Res}_{F/F'} M$. (Of course F' is not a subfield of F in general.) The computation of special values of the L -function attached to M' based on Deligne's conjecture shall be performed in §5 and shall be shown to be consistent with a conjecture and certain results of Shimura.

§2. Factorization of Deligne's period $c^{\pm}(M)$ of a motive M

Let E and F be algebraic number fields of finite degree. Let M be a motive over F with coefficients in E . Let λ be a finite place of E and consider the λ -adic realization $H_\lambda(M) \in V(E_\lambda)$ of M . For a prime ideal $\mathfrak{p}$ of F such that $(\lambda, \mathfrak{p}) = 1$, put

$$(2.1) \quad Z_{\mathfrak{p}}(M, X) = \det(1 - F_{\mathfrak{p}}X, H_\lambda(M)^{I_{\mathfrak{p}}})^{-1},$$

where $F_{\mathfrak{p}}$ denotes a geometric Frobenius of $\mathfrak{p}$. It is conjectured that $Z_{\mathfrak{p}}(M, X) \in E[X]$ independently of λ . We shall assume this conjecture. For $\sigma \in J_E$, put

$$(2.2) \quad L_{\mathfrak{p}}(\sigma, M, s) = \sigma Z_{\mathfrak{p}}(M, N(\mathfrak{p})^{-s}),$$

$$(2.3) \quad L(\sigma, M, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\sigma, M, s).$$

Let $\text{Res}_{F/\mathbf{Q}}(M) = R_{F/\mathbf{Q}}(M)$ denote the motive over $\mathbf{Q}$ with coefficients in E obtained from M by the restriction of scalar. Then we have

$$(2.4) \quad L(\sigma, M, s) = L(\sigma, R_{F/\mathbf{Q}}(M), s)$$

for every $\sigma \in J_E$. Since $E \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathbf{C}^{J_E}$, we can define a function $L^*(M, s)$ taking values in $E \otimes_{\mathbf{Q}} \mathbf{C}$ by arranging $L(\sigma, M, s)$. Deligne's conjecture predicts

$$(2.5) \quad L^*(M, 0)/c^+(R_{F/\mathbf{Q}}(M)) \in E$$

if 0 is critical for $R_{F/\mathbf{Q}}(M)$ (which is assumed to be homogeneous) with $E \subset E \otimes_{\mathbf{Q}} \mathbf{C}$ canonically. Here the period $c^+(R_{F/\mathbf{Q}}(M)) \in (E \otimes_{\mathbf{Q}} \mathbf{C})^\times$ is defined as follows.

Let M be a motive over $\mathbf{Q}$ with coefficients in E . Let $H_B(M) \in V(E)$ denote the Betti realization of M . Then the complex conjugation F_∞ acts on $H_B(M)$. We have

$$(2.6) \quad H_B(M) = H_B^+(M) \oplus H_B^-(M),$$

where $H_B^\pm(M)$ denotes the eigenspaces of $H_B(M)$ with eigenvalues ± 1 . We assume that M is homogeneous of weight w . Then we have

$$H_B(M) \otimes_{E, \sigma} \mathbf{C} = \bigoplus_{p+q=w} H^{pq}(\sigma, M), \quad \sigma \in J_E.$$

In view of the Gamma factor of the conjectural functional equation of $L^*(M, s)$, we find that if 0 is critical for M , then:

$$(2.7) \quad \text{Whenever } H^{pq}(\sigma, M) \neq \{0\} \text{ and } p < q, \quad p < 0, q > -1 \text{ must hold.}$$

If w is odd, (2.7) is sufficient for 0 to be critical. If w is even, F_∞ must act on $\bigoplus_{\sigma \in J_E} H^{pp}(\sigma, M)$, $p = w/2$ by scalar. Put

$$F_\infty = (-1)^{p+\epsilon}, \quad \epsilon = 0 \text{ or } 1 \text{ on } H^{pp}(\sigma, M).$$

Then

$$(2.8) \quad \begin{cases} p < \epsilon & \text{if } p + \epsilon \text{ is even,} \\ -\epsilon - 1 < p & \text{if } p + \epsilon \text{ is odd,} \end{cases}$$

must be satisfied; (2.7) and (2.8) are sufficient for 0 to be critical.

Remark. We see that $n \in \mathbf{Z}$ is critical for M hence the transcendental part of $L^*(M, n)$ is predictable by Deligne's conjecture if and only if the following conditions are satisfied. (Of course, we admit the conjectural functional equation for $L^*(M, s)$.)

$$(2.9) \quad p < n \leq q \quad \text{if} \quad H^{pq}(\sigma, M) \neq \{0\}, \quad p < q.$$

$$(2.10) \quad \begin{cases} n > p - \epsilon & \text{if } p + \epsilon + n \text{ is even,} \\ n < p + \epsilon + 1 & \text{if } p + \epsilon + n \text{ is odd,} \end{cases}$$

if $F_\infty = (-1)^{p+\epsilon}$, $\epsilon = 0$ or 1 on $H^{pp}(\sigma, M) \neq \{0\}$.

Let $H_{DR}(M) \in V(E)$ be the de Rham realization of M . We have the canonical isomorphism

$$I : H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{\mathbf{Q}} \mathbf{C}$$

as $E \otimes_{\mathbf{Q}} \mathbf{C}$ -modules. We choose $F^{\pm}(M) \in V(E)$ as certain subspaces of $H_{DR}(M)$ obtained from the Hodge filtration; explicitly we have

$$I^{-1}(F^{+}(M) \otimes_{E, \sigma} \mathbf{C}) = \begin{cases} \oplus_{p \geq q} H^{pq}(\sigma, M) & \text{if } F_{\infty} = 1 \text{ on } H^{pp}(\sigma, M), \\ \oplus_{p > q} H^{pq}(\sigma, M) & \text{if } F_{\infty} = -1 \text{ on } H^{pp}(\sigma, M). \end{cases}$$

$$I^{-1}(F^{-}(M) \otimes_{E, \sigma} \mathbf{C}) = \begin{cases} \oplus_{p > q} H^{pq}(\sigma, M) & \text{if } F_{\infty} = 1 \text{ on } H^{pp}(\sigma, M), \\ \oplus_{p \geq q} H^{pq}(\sigma, M) & \text{if } F_{\infty} = -1 \text{ on } H^{pp}(\sigma, M). \end{cases}$$

Put $H_{DR}^{\pm}(M) = H_{DR}(M)/F^{\mp}(M)$. We have the canonical isomorphisms

$$I^{\pm} : H_B^{\pm}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}^{\pm}(M) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Let $\delta(M) = \det(I)$, $c^{\pm}(M) = \det(I^{\pm})$ be the determinants calculated by E -rational basis. Then $\delta(M)$, $c^{\pm}(M) \in (E \otimes_{\mathbf{Q}} \mathbf{C})^{\times}$ are determined up to multiplications by elements of E .

Now going back to the general case, let M be a motive over F with coefficients in E . We assume that F is totally real. For every $\tau \in J_F$, we have the Betti realization $H_{\tau, B}(M) \in V(E)$ of M and the complex conjugation $F_{\infty, \tau}$ associated with τ acts on $H_{\tau, B}(M)$. Similarly to (2.6), we have

$$(2.11) \quad H_{\tau, B}(M) = H_{\tau, B}^{+}(M) \oplus H_{\tau, B}^{-}(M),$$

$H_{\tau, B}^{\pm}(M) \in V(E)$. We assume that $R_{F/\mathbf{Q}}(M)$ is homogeneous of weight w . Then we have

$$H_{\tau, B}(M) \otimes_{E, \sigma} \mathbf{C} = \oplus_{p+q=w} H^{pq}(\tau, \sigma, M), \quad \sigma \in J_E.$$

If w is even, we assume that $\oplus_{\tau \in J_F} F_{\infty, \tau}$ acts on $\oplus_{\tau} \oplus_{\sigma} H^{pp}(\tau, \sigma, M)$, $p = w/2$ by scalar. The de Rham realization $H_{DR}(M) \in V(E)$ has the structure of a free $E \otimes_{\mathbf{Q}} F$ -module. We have the canonical isomorphism

$$I_{\tau} : H_{\tau, B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F, \tau} \mathbf{C}$$

as $(E \otimes_{\mathbf{Q}} \mathbf{C})$ -modules. By the Hodge filtration obtained from the convergence of the spectral sequence

$$E_1^{pq} = H^q(M, \Omega^p) \implies H_{DR}^{p+q}(M),$$

we can define subspaces $F^{\pm}(M) \in V(E)$ of $H_{DR}(M)$ as in the case $F = \mathbf{Q}$; $F^{\pm}(M)$ has the structure of a vector space over F . We have

$$I_{\tau}^{-1}(F^{+}(M) \otimes_{F, \tau} \mathbf{C}) = \begin{cases} \oplus_{\sigma \in J_E} \oplus_{p \geq q} H^{pq}(\tau, \sigma, M) & \text{if } F_{\infty, \tau} = 1 \text{ on } H^{pp}(\tau, \sigma, M), \\ \oplus_{\sigma \in J_E} \oplus_{p > q} H^{pq}(\tau, \sigma, M) & \text{if } F_{\infty, \tau} = -1 \text{ on } H^{pp}(\tau, \sigma, M). \end{cases}$$

$$I_{\tau}^{-1}(F^{-}(M) \otimes_{F, \tau} \mathbb{C}) \\ = \begin{cases} \oplus_{\sigma \in J_E} \oplus_{p > q} H^{pq}(\tau, \sigma, M) & \text{if } F_{\infty \tau} = 1 \text{ on } H^{pp}(\tau, \sigma, M), \\ \oplus_{\sigma \in J_E} \oplus_{p \geq q} H^{pq}(\tau, \sigma, M) & \text{if } F_{\infty \tau} = -1 \text{ on } H^{pp}(\tau, \sigma, M). \end{cases}$$

Put $H_{DR}^{\pm}(M) = H_{DR}(M)/F^{\mp}(M)$. We have the canonical isomorphisms

$$I_{\tau}^{\pm} : H_{\tau, B}^{\pm}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}^{\pm}(M) \otimes_{F, \tau} \mathbb{C}$$

as $(E \otimes_{\mathbb{Q}} \mathbb{C})$ -modules. Let $\delta_{\tau}(M) = \det(I_{\tau})$, $c_{\tau}^{\pm}(M) = \det(I_{\tau}^{\pm})$ be the determinant calculated by E -rational basis of the left hand side and by $E \otimes_{\mathbb{Q}} F$ -basis (since they are free $E \otimes_{\mathbb{Q}} F$ -modules) of the right hand side modules. Then $\delta_{\tau}(M)$, $c_{\tau}^{\pm}(M) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$ are determined up to multiplications by elements of $(E \vee \tilde{F})$. Here $\tilde{F}$ denotes the normal closure of F in $\mathbb{Q}$.

PROPOSITION 2.1. *Let the notation be the same as above. We have*

$$c^{+}(R_{F/\mathbb{Q}}(M)) = \prod_{\tau \in J_F} c_{\tau}^{+}(M), \quad c^{-}(R_{F/\mathbb{Q}}(M)) = \prod_{\tau \in J_F} c_{\tau}^{-}(M), \\ \delta(R_{F/\mathbb{Q}}(M)) = \prod_{\tau \in J_F} \delta_{\tau}(M),$$

up to multiplications by elements of $E \vee \tilde{F}$.

PROOF: It is known (cf. Deligne [6]) that $H_B(R_{F/\mathbb{Q}}(M)) = \oplus_{\tau \in J_F} H_{\tau, B}(M)$ as vector spaces over E and that $H_{DR}(R_{F/\mathbb{Q}}(M))$ can be identified with $H_{DR}(M)$ forgetting its structure as a vector space over F . We see that

$$H_B^{+}(R_{F/\mathbb{Q}}(M)) = \oplus_{\tau \in J_F} H_{\tau, B}^{+}(M), \quad H_B^{-}(R_{F/\mathbb{Q}}(M)) = \oplus_{\tau \in J_F} H_{\tau, B}^{-}(M),$$

and that $H_{DR}^{\pm}(R_{F/\mathbb{Q}}(M))$ is identified with $H_{DR}^{\pm}(M)$ forgetting the structure of a vector space over F . We have the isomorphism of $(E \otimes_{\mathbb{Q}} \mathbb{C})$ -modules

$$I^{+} : H_B^{+}(R_{F/\mathbb{Q}}(M)) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}^{+}(R_{F/\mathbb{Q}}(M)) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Since

$$(2.12) \quad H_{DR}^{+}(R_{F/\mathbb{Q}}(M)) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}^{+}(R_{F/\mathbb{Q}}(M)) \otimes_F F \otimes_{\mathbb{Q}} \mathbb{C} \\ \cong \oplus_{\tau \in J_F} (H_{DR}^{+}(M) \otimes_{F, \tau} \mathbb{C}),$$

I^{+} may be written as

$$I^{+} : \oplus_{\tau \in J_F} (H_{\tau, B}^{+}(M) \otimes_{\mathbb{Q}} \mathbb{C}) \cong \oplus_{\tau \in J_F} (H_{DR}^{+}(M) \otimes_{F, \tau} \mathbb{C}).$$

Restricting I^{+} to a direct factor, we obtain

$$I_{\tau}^{+} : H_{\tau, B}^{+}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}^{+}(M) \otimes_{F, \tau} \mathbb{C}.$$

The isomorphism (2.12) does not preserve E -structure but preserve $E \vee \tilde{F}$ -structure on both sides. Hence we obtain the first assertion. The second and the last assertions can be proved in similar way. This completes the proof.

§3. Variations of periods $c_\tau^\pm(M)$ under standard operations

(I) Let M and N be motives over F with coefficients in E . Let $d(M)$ and $d(N)$ be the ranks of M and N respectively. For example, we have $d(M) = \dim_E H_{\tau,B}(M)$ for every $\tau \in J_F$. We assume that $R_{F/\mathbf{Q}}(M)$ and $R_{F/\mathbf{Q}}(N)$ are homogeneous of weights w and w' respectively. For $\tau \in J_F$, we obviously have

$$(3.1) \quad \begin{aligned} H_{\tau,B}(M \otimes N) &= H_{\tau,B}(M) \otimes_E H_{\tau,B}(N), \\ H_{\tau,B}^+(M \otimes N) &= (H_{\tau,B}^+(M) \otimes_E H_{\tau,B}^+(N)) \oplus (H_{\tau,B}^-(M) \otimes_E H_{\tau,B}^-(N)), \\ H_{\tau,B}^-(M \otimes N) &= (H_{\tau,B}^+(M) \otimes_E H_{\tau,B}^-(N)) \oplus (H_{\tau,B}^-(M) \otimes_E H_{\tau,B}^+(N)), \\ H_{DR}(M \otimes N) &= H_{DR}(M) \otimes_{(E \otimes_{\mathbf{Q}} F)} H_{DR}(N). \end{aligned}$$

Since

$$\begin{aligned} H_{\tau,B}(M) \otimes_E H_{\tau,B}(N) \otimes_{\mathbf{Q}} \mathbf{C} &\cong (H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C}) \otimes_{E \otimes \mathbf{C}} (H_{\tau,B}(N) \otimes_{\mathbf{Q}} \mathbf{C}), \\ &\quad H_{DR}(M) \otimes_{(E \otimes_{\mathbf{Q}} F)} H_{DR}(N) \otimes_{F,\tau} \mathbf{C} \\ &\cong (H_{DR}(M) \otimes_{F,\tau} \mathbf{C}) \otimes_{E \otimes \mathbf{C}} (H_{DR}(N) \otimes_{F,\tau} \mathbf{C}), \end{aligned}$$

we have

$$(3.2) \quad \delta_\tau(M \otimes N) = \delta_\tau(M)^{d(N)} \delta_\tau(N)^{d(M)}.$$

Assume $d(N) = 1$, w' is even and put $p' = w'/2$. Assume further that $H_B(R_{F/\mathbf{Q}}(N)) \otimes_{\mathbf{Q}} \mathbf{C}$ is of Hodge type (p', p') . If $H_B(R_{F/\mathbf{Q}}(N)) \otimes_{\mathbf{Q}} \mathbf{C}$ does not have a component of Hodge type (p, p) , we have

$$F^\pm(M \otimes N) = F^\pm(M) \otimes_{E \otimes F} H_{DR}(N).$$

In view of (3.1), we immediately obtain

$$(3.3) \quad c_\tau^\pm(M \otimes N) = c_\tau^{\pm \epsilon_\tau}(M) \delta_\tau(N)^{d_\tau^\pm(M)},$$

where $F_{\infty, \tau} = (-1)^{\epsilon_\tau}$ on $H_{\tau,B}(N)$ and $d_\tau^\pm(M) = \dim_E H_{\tau,B}^\pm(M)$. If $H_B(R_{F/\mathbf{Q}}(M)) \otimes_{\mathbf{Q}} \mathbf{C}$ has a component of type (p, p) , we assume that F_∞ acts on both of $\oplus_{\sigma \in J_E} H^{pp}(\sigma, R_{F/\mathbf{Q}}(M))$ and $\oplus_{\sigma \in J_E} H^{p'p'}(\sigma, R_{F/\mathbf{Q}}(N))$ by scalar. Then we obtain

$$(3.4) \quad c_\tau^\pm(M \otimes N) = \begin{cases} c_\tau^\pm(M) \delta_\tau(N)^{d_\tau^\pm(M)} & \text{if } F_\infty = 1 \text{ on } H_B(R_{F/\mathbf{Q}}(N)), \\ c_\tau^\mp(M) \delta_\tau(N)^{d_\tau^\mp(M)} & \text{if } F_\infty = -1 \text{ on } H_B(R_{F/\mathbf{Q}}(N)). \end{cases}$$

For $n \in \mathbf{Z}$, let $T(n)$ denote the Tate motive over F . We have

$$(3.5) \quad L^*(M \otimes T(n), s) = L^*(M, s + n),$$

$$(3.6) \quad \delta_\tau(T(n)) = (2\pi\sqrt{-1})^n,$$

$$(3.7) \quad F_\infty \text{ acts on } H_B(R_{F/\mathbf{Q}}(T(n))) \text{ by } (-1)^n.$$

Hence, if F_∞ acts on $\oplus_{\sigma \in J_E} H^{pp}(\sigma, R_{F/\mathbf{Q}}(M))$ by scalar, we obtain

$$(3.8) \quad \begin{aligned} c_\tau^\pm(M(n)) &= \begin{cases} (2\pi\sqrt{-1})^{nd_\tau^\pm(M)} c_\tau^\pm(M) & \text{if } n \text{ is even,} \\ (2\pi\sqrt{-1})^{nd_\tau^\mp(M)} c_\tau^\mp(M) & \text{if } n \text{ is odd,} \end{cases} \\ \delta_\tau(M(n)) &= (2\pi\sqrt{-1})^{nd(M)} \delta_\tau(M), \end{aligned}$$

where $M(n) = M \otimes T(n)$.

(II) Let M, N and related notations be the same as in the beginning of (I).

PROPOSITION 3.1. We assume $\oplus_{\sigma \in J_E} H^{pp}(\sigma, R_{F/\mathbf{Q}}(M)) = \{0\}$. We further assume that if

$$H^{pq}(\tau, \sigma, M) \neq \{0\}, \quad p > q \quad \text{and} \quad H^{p'q'}(\tau, \sigma, M) \neq \{0\}, \quad p' \geq q'$$

for $\tau \in J_F, \sigma \in J_E$, then $p - q > p' - q'$ holds. Then we have

$$\begin{aligned} c_\tau^+(M \otimes N) &= c_\tau^+(M)^{d_\tau^+(N)} c_\tau^-(M)^{d_\tau^-(N)} \delta_\tau(N)^{d_\tau^+(M)}, \\ c_\tau^-(M \otimes N) &= c_\tau^+(M)^{d_\tau^-(N)} c_\tau^-(M)^{d_\tau^+(N)} \delta_\tau(N)^{d_\tau^-(M)}. \end{aligned}$$

PROOF: By the assumption, we immediately obtain

$$F^+(M) = F^-(M), \quad F^\pm(M \otimes N) = F^\pm(M) \otimes_{E \otimes F} H_{DR}(N).$$

Let

$$\begin{aligned} I_\tau^M &: H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F,\tau} \mathbf{C} \\ I_\tau^N &: H_{\tau,B}(N) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(N) \otimes_{F,\tau} \mathbf{C} \end{aligned}$$

be canonical isomorphisms. Let $u_1^\pm, \dots, u_n^\pm$ (resp. $v_1, \dots, v_m$) be a basis of $H_{\tau,B}^\pm(M)$ (resp. $H_{\tau,B}(N)$) over E where $n = d_\tau^+(M)$, $m = d_\tau(N)$. Let $e_1^-, \dots, e_n^-$ be a basis of $F^-(M)$ as free $E \otimes F$ -module. Take $e_1^+, \dots, e_n^+$ so that $e_1^+, \dots, e_n^+, e_1^-, \dots, e_n^-$ becomes a basis of $H_{DR}(M)$ as free $E \otimes F$ -module. Let $d_1, \dots, d_m$ be a basis of $H_{DR}(N)$ as free $E \otimes F$ -module. Put

$$\begin{aligned} I_\tau^M(u_i^\pm) &= \sum_{j=1}^n x_{ij}^{+,\pm} e_j^+ + \sum_{j=1}^n x_{ij}^{-,\pm} e_j^-, \\ I_\tau^N(v_i) &= \sum_{j=1}^m y_{ij} d_j \end{aligned}$$

with $x_{ij}^{\pm, \pm}, y_{ij} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Put

$$X_{11} = (x_{ij}^{+,+}), \quad X_{12} = (x_{ij}^{-,+}), \quad X_{21} = (x_{ij}^{+,-}), \quad X_{22} = (x_{ij}^{-,-}) \in M_n(E \otimes_{\mathbf{Q}} \mathbf{C}),$$

$$Y = (y_{ij}) \in M_m(E \otimes_{\mathbf{Q}} \mathbf{C}).$$

Then we have

$$c_{\tau}^{+}(M) = \det(X_{11}), \quad c_{\tau}^{-}(M) = \det(X_{21}), \quad \delta_{\tau}(N) = \det(Y).$$

We may assume that $v_1, \dots, v_t$ (resp. $v_{t+1}, \dots, v_m$) is a basis of $H_{\tau, B}^{+}(N)$ (resp. $H_{\tau, B}^{-}(N)$) where $t = d_{\tau}^{+}(N)$. We have

$$(I_{\tau}^M \otimes I_{\tau}^N)(u_i^{+} \otimes v_j) = \left(\sum_{k=1}^n x_{ik}^{+,+} e_k^{+} \right) \otimes \left(\sum_{l=1}^m y_{jl} d_l \right)$$

$$(I_{\tau}^M \otimes I_{\tau}^N)(u_i^{-} \otimes v_j) = \left(\sum_{k=1}^n x_{ik}^{+,-} e_k^{+} \right) \otimes \left(\sum_{l=1}^m y_{jl} d_l \right)$$

modulo $F^{-}(M \otimes N)$. Therefore we have

$$c_{\tau}^{+}(M) = \det \begin{pmatrix} X_{11} \otimes Y_1 \\ X_{21} \otimes Y_2 \end{pmatrix}, \quad c_{\tau}^{-}(M) = \det \begin{pmatrix} X_{11} \otimes Y_2 \\ X_{21} \otimes Y_1 \end{pmatrix},$$

where $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ with $Y_1 \in M_{t,m}(E \otimes_{\mathbf{Q}} \mathbf{C})$, $Y_2 \in M_{m-t,m}(E \otimes_{\mathbf{Q}} \mathbf{C})$. Hence we obtain

$$c_{\tau}^{+}(M) = \det(X_{11})^t \det(X_{21})^{m-t} (\det Y)^n,$$

$$c_{\tau}^{-}(M) = \det(X_{11})^{m-t} \det(X_{21})^t (\det Y)^n,$$

and the assertion follows.

(III) Let $n \geq 2$ and suppose that we are given motives M_i over F with coefficients in E for $1 \leq i \leq n$. We assume that M_i is of rank 2 for every i and let

$$H_{\tau, B}(M_i) \otimes_{E, \sigma} \mathbf{C} = H^{a_i(\tau, +), a_i(\tau, -)}(\tau, \sigma, M_i) \oplus H^{a_i(\tau, -), a_i(\tau, +)}(\tau, \sigma, M_i),$$

$1 \leq i \leq n$, $\tau \in J_F$. We assume that $a_i(\tau, +) > a_i(\tau, -)$ for every $\tau \in J_F$ and i . We shall give a formula for $c_{\tau}^{\pm}(M_1 \otimes M_2 \otimes \dots \otimes M_n)$, which is suggested by Blasius [2]. Let Λ be the set of all maps from $\{1, 2, \dots, n\}$ to $\{\pm 1\}$. Set

$$\Lambda_{\pm} = \{ \lambda \in \Lambda \mid \prod_{i=1}^n \lambda(i) = \pm 1 \},$$

$$\Lambda^{+} = \{ \lambda \in \Lambda \mid \sum_{i=1}^n a_i(\tau, \lambda(i)) > \sum_{i=1}^n a_i(\tau, -\lambda(i)) \}.$$

We have $|\Lambda_{\pm}| = 2^{n-1}$. We assume that

$$(3.9) \quad \sum_{i=1}^n a_i(\tau, \lambda(i)) \neq \sum_{i=1}^n a_i(\tau, -\lambda(i)) \quad \text{for every } \lambda \in \Lambda.$$

We note that if (3.9) is not satisfied, then the action of $F_{\infty\tau}$ on $H^{pp}(\tau, \sigma, M_1 \otimes \cdots \otimes M_n)$ is not a scalar. By (3.9), we have $|\Lambda^+| = 2^{n-1}$ since $\lambda \in \Lambda^+$ is equivalent to $-\lambda \notin \Lambda^+$. Let n_i (resp. m_i) be the number of $\lambda \in \Lambda^+$ such that $\lambda(i) = 1$ (resp. $\lambda(i) = -1$). We have

$$(3.10) \quad n_i + m_i = 2^{n-1}.$$

PROPOSITION 3.2. We assume that (3.9) holds for every $\tau \in J_F$. Then we have

$$c_{\tau}^{\pm}(M_1 \otimes M_2 \otimes \cdots \otimes M_n) = \prod_{i=1}^n (c_{\tau}^{+}(M_i) c_{\tau}^{-}(M_i))^{(n_i - m_i)/2} \delta_{\tau}(M_i)^{m_i}.$$

PROOF: Take $u_i^{\pm}$ so that

$$Eu_i^{\pm} = H_{\tau, B}^{\pm}(M_i), \quad 1 \leq i \leq n.$$

Choose d_i^{-} so that

$$(E \otimes F)d_i^{-} = F^{-}(M_i) = F^{+}(M_i)$$

and choose d_i^{+} so that

$$H_{DR}(M_i) = (E \otimes F)d_i^{+} + (E \otimes F)d_i^{-}, \quad 1 \leq i \leq n.$$

Let

$$I_{\tau}^{M_i} : H_{\tau, B}(M_i) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M_i) \otimes_{F, \tau} \mathbf{C}$$

be the canonical isomorphism and put

$$I_{\tau}^{M_i}(u_i^{\pm}) = x_i^{+, \pm} d_i^{+} + x_i^{-, \pm} d_i^{-}, \quad 1 \leq i \leq n$$

with $x_i^{\pm, \pm} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Then we have

$$c_{\tau}^{\pm}(M_i) = x_i^{+, \pm}, \quad \delta_{\tau}(M_i) = \det \begin{pmatrix} x_i^{+, +} & x_i^{-, +} \\ x_i^{+, -} & x_i^{-, -} \end{pmatrix}.$$

A basis of $H_{\tau, B}^{\pm}(M_1 \otimes \cdots \otimes M_n)$ over E is given by $\otimes_{i=1}^n u_i^{\epsilon(i)}$ when ϵ extends over $\Lambda_{\pm}$. Also we see easily that a basis of $H_{DR}^{\pm}(M_1 \otimes \cdots \otimes M_n)$ is given by $\otimes_{i=1}^n d_i^{\lambda(i)}$ mod $F^{-}(M_1 \otimes \cdots \otimes M_n)$ when λ extends over Λ^+ . Since

$$\otimes_i I_{\tau}^{M_i}(\otimes_i u_i^{\epsilon(i)}) = \otimes_i (x_i^{+, \epsilon(i)} d_i^{+} + x_i^{-, \epsilon(i)} d_i^{-}) = \sum_{\lambda \in \Lambda^+} \prod_{i=1}^n x_i^{\lambda(i), \epsilon(i)} \otimes_i d_i^{\lambda(i)}$$

mod $F^-(M_1 \otimes M_2 \otimes \cdots \otimes M_n)$, we have

$$c_r^\pm(M_1 \otimes M_2 \otimes \cdots \otimes M_n) = \det(X^\pm),$$

where $X^\pm$ is the $2^{n-1} \times 2^{n-1}$ -matrix whose (λ, ϵ) -entry for $\lambda \in \Lambda^+$, $\epsilon \in \Lambda_\pm$ is given by $\prod_{i=1}^n x_i^{\lambda(i), \epsilon(i)}$. We shall prove the formula for c_r^+ since the other case can be shown similarly.

It suffices to show

$$(3.11) \quad \det(X^+) = c \prod_{i=1}^n (x_i^{+,+} x_i^{+,-})^{(n_i - m_i)/2} (x_i^{+,+} x_i^{-,-} - x_i^{-,+} x_i^{+,-})^{m_i}, \quad c \in \mathbb{Q},$$

regarding $x_i^{\pm, \pm}$, $1 \leq i \leq n$ as indeterminates. It is obvious that $\det(X^+)$ is a homogeneous polynomial of degree $2^{n-1}n$ with $\mathbb{Z}$ -coefficients of $4n$ -variables $x_i^{\pm, \pm}$. Fix i , $1 \leq i \leq n$. If we change variables $x_i^{\pm, \pm} \rightarrow \mu x_i^{\pm, \pm}$ with $\mu \in \mathbb{C}$, then every (λ, ϵ) -entry of X^+ with $\lambda(i) = 1$ is multiplied by μ . Hence $\det(X^+)$ is multiplied by μ^{n_i} . Therefore we have

$$(3.12) \quad \det(X^+) = \sum_{a+b=n_i} (x_i^{+,+})^a (x_i^{+,-})^b Q_{a,b}$$

where $Q_{a,b}$ is a polynomial which does not contain the variables $x_i^{\pm, \pm}$. Suppose $\lambda \in \Lambda^+$, $\lambda(i) = -1$. Put $\lambda'(j) = \lambda(j)$, $j \neq i$, $\lambda'(i) = 1$. Then $\lambda' \in \Lambda^+$ since $a_i(\tau, +) > a_i(\tau, -)$. Thus we may set

$$X^+ = \begin{pmatrix} x_i^{+,+} A & x_i^{+,-} B \\ x_i^{+,-} C & x_i^{-,-} D \\ x_i^{-,+} C & x_i^{-,-} D \end{pmatrix}$$

where A , B , C and D are $(n_i - m_i) \times 2^{n-2}$, $(n_i - m_i) \times 2^{n-2}$, $m_i \times 2^{n-2}$ and $m_i \times 2^{n-2}$ matrices respectively which does not contain the variables $x_i^{\pm, \pm}$. By standard operations on matrices, we have

$$\begin{aligned} \det(X^+) &= \det \begin{pmatrix} x_i^{+,+} A & x_i^{+,-} B \\ x_i^{+,-} C & x_i^{-,-} D \\ 0 & x_i^{-,-} D - (x_i^{-,+} x_i^{+,-} / x_i^{+,+}) D \end{pmatrix} \\ &= \det \begin{pmatrix} A & x_i^{+,+} x_i^{+,-} B \\ C & 0 \\ 0 & (x_i^{+,+} x_i^{-,-} - x_i^{-,+} x_i^{+,-}) D \end{pmatrix} \\ &= (x_i^{+,+} x_i^{-,-} - x_i^{-,+} x_i^{+,-})^{m_i} \det \begin{pmatrix} A & x_i^{+,+} x_i^{+,-} B \\ C & 0 \\ 0 & D \end{pmatrix}. \end{aligned}$$

Hence we have

$$\det(X^+) = (x_i^{+,+} x_i^{-,-} - x_i^{-,+} x_i^{+,-})^{m_i} \sum_j (x_i^{+,+} x_i^{+,-})^j P_j$$

where P_j is a polynomial which does not contain the variables $x_i^{\pm,\pm}$. By (3.12), $P_j = 0$ except for $m_i + 2j = n_i$. Therefore we have

$$\det(X^+) = (x_i^{+,+} x_i^{-,-} - x_i^{-,+} x_i^{+,-})^{m_i} (x_i^{+,+} x_i^{+,-})^{(n_i - m_i)/2} Q$$

where Q is a polynomial with $\mathbf{Q}$ -coefficients which does not contain the variables $x_i^{\pm,\pm}$. Since this expression holds for arbitrary i , we obtain (3.11). This completes the proof.

§4. On motives attached to Hilbert modular forms and Shimura's invariants

Let F be a totally real algebraic number field of degree n over $\mathbf{Q}$. Let $k = (k(\tau)) \in \mathbf{Z}^{J(F)}$ be a weight. By the Hilbert modular cusp form of weight k , we understand an element of $S_k(\mathfrak{c}, \psi)$ in the notation of Shimura [12], p. 649. Assume that f is a non-zero common eigenfunction of all Hecke operators. We attach Dirichlet series $D(s, f)$ to f by (2.25) of [12]. Now the form of the Gamma factor and the functional equation of $D(s, f)$ (cf. (2.47), (2.48) of [12]) suggest the following conjecture.

CONJECTURE 4.1. Assume $k(\tau) \bmod 2$ is independent of τ and put $k_0 = \max_{\tau \in J_F} k(\tau)$. Let E be the algebraic number field of finite degree generated by eigenvalues of Hecke operators of f (cf. [12], Prop. 2.8.). Then there exists a motive M_f over F with coefficients in E which satisfies the following conditions.

- (1) $L(\sigma, M_f, s) = D(s, f^\sigma)$ for every $\sigma \in J_E$.
- (2) $H_{\tau, B}(M_f) \otimes_{E, \sigma} \mathbf{C} \cong H^{(k_0 + k(\tau))/2 - 1, (k_0 - k(\tau))/2}(\tau, \sigma, M_f) \oplus H^{(k_0 - k(\tau))/2, (k_0 + k(\tau))/2 - 1}(\tau, \sigma, M_f)$, $\sigma \in J_E$, $\tau \in J_F$.
- (3) $\wedge^2 M_f \cong \text{Art}_{\psi^{-1}}(1 - k_0)$ where $\text{Art}_{\psi^{-1}}$ denotes the Artin motive attached to ψ .

Let χ be a Hecke character of F of finite order. Let $\mathfrak{c}$ be the conductor of χ and $\mathbf{Q}(\chi)$ be the field generated over $\mathbf{Q}$ by values of χ . As in [6], §6, we can attach a motive $\text{Art}_\chi = N_\chi$ over F with coefficients in $\mathbf{Q}(\chi)$ such that $L(s, \chi^\sigma) = L(\sigma, N_\chi, s)$ for every $\sigma \in J_{\mathbf{Q}(\chi)}$. The rank of N_χ is 1 and the Hodge type of $H_{\tau, B}(N_\chi) \otimes_{\mathbf{Q}} \mathbf{C}$ is $(0, 0)$ for every $\tau \in J_F$. For the real archimedean place ∞_τ corresponding to $\tau \in J_F$, we have

$$(4.1) \quad \chi_{\infty_\tau}(x) = \text{sgn}(x)^{m_\tau}, \quad x \in k_{\infty_\tau}^\times \cong \mathbf{R}^\times, \quad m_\tau = 0 \quad \text{or} \quad 1.$$

If $m_\tau = 0$ (resp. $m_\tau = 1$), then F_{∞_τ} acts on $H_{\tau, B}(N_\chi)$ by 1 (resp. -1). We are going to calculate $\delta(R_{F/\mathbf{Q}}(N_\chi))$.

For this purpose, let us recall the following facts concerning an Artin motive M over $\mathbf{Q}$. Let ρ be a representation of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $GL(V)$ where V is a vector space over E of finite dimension m . Then there exists an Artin motive $M = \text{Art}_\rho$ over $\mathbf{Q}$ with coefficients in E such that (cf. [6])

$$(4.2) \quad L(s, \rho^\sigma) = L(\sigma, M, s) \quad \text{for every } \sigma \in J_E,$$

$$(4.3) \quad H_B(M) = V, \quad H_{DR}(M) = (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}.$$

Obviously $\delta(M) = \delta(\wedge^m M)$ and $\wedge^m M$ is the Artin motive attached to the representation $\det \rho$ of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. For a Dirichlet character η of $\mathbf{Q}$ of conductor (f) , $f > 0$, put

$$(4.4) \quad g_0(\eta) = \sum_{u=1}^f \eta(u) \exp(2\pi\sqrt{-1}u/f).$$

Then, as is shown in [6], §6, we have

$$(4.5) \quad \delta(\wedge^m M) = g_0((\det \rho)_*)^{-1}$$

where $(\det \rho)_*$ denotes the Dirichlet character associated to $\det \rho$.

We may regard χ as a character of $\text{Gal}(K/F)$ where K is a finite Galois extension of $\mathbf{Q}$. Put $\rho = \text{Ind}(\chi; \text{Gal}(K/F) \rightarrow \text{Gal}(K/\mathbf{Q}))$. Then $R_{F/\mathbf{Q}}(N_\chi)$ is the Artin motive associated with ρ . We have (cf. [5], Prop. 1.2)

$$(\det \rho)(\sigma) = \chi(t(\sigma))\epsilon(\sigma), \quad \sigma \in \text{Gal}(K/\mathbf{Q})$$

where t denotes the transfer map from $\text{Gal}(K/\mathbf{Q})^{ab}$ to $\text{Gal}(K/F)^{ab}$ and ϵ denotes the determinant of the left regular representation of $\text{Gal}(K/\mathbf{Q})$ on $\text{Gal}(K/\mathbf{Q})/\text{Gal}(K/F)$. Let χ_* denote the character of ideal class group of conductor $\mathfrak{c}$ of F associated with χ and let ϵ_* denote the Dirichlet character associated with ϵ . We have

$$(\det \rho)_*(n) = \chi_*(n)\epsilon_*(n), \quad n \in \mathbf{Z}, \quad n > 0.$$

Define a Gauss sum by

$$(4.6) \quad g(\chi) = \sum_{x \in \mathfrak{c}^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1}, x \gg 0} \chi_*(x\mathfrak{c}\mathfrak{d}_F) \exp(2\pi\sqrt{-1}\text{Tr}_{F/\mathbf{Q}}(x))$$

where $\mathfrak{d}_F$ denotes the different of F .

LEMMA 4.2. Put $E = \mathbf{Q}(\chi)$. We have $g(\chi)/g_0(\det \rho) \in E \vee \tilde{F}$.

We omit the proof which is not difficult.

Let $M = M_f$ and $N = N_\chi$ be as above. Define $m_\tau = 0, 1$ by (4.1) and let $\epsilon_\tau = +$ (resp. $-$) if $m_\tau = 0$ (resp. 1) for $\tau \in J_F$. We assume that $k(\tau) \geq 2$ for every τ . Let E denote the number field generated by the eigenvalues of f under Hecke operators and the values of χ . By (3.4), we have

$$c_\tau^+(M \otimes N) = c_\tau^{\epsilon_\tau}(M) \delta_\tau(N), \quad c_\tau^-(M \otimes N) = c_\tau^{-\epsilon_\tau}(M) \delta_\tau(N).$$

By Proposition 2.1, we have

$$c^+(R_{F/\mathbf{Q}}(M \otimes N)) = \prod_\tau c_\tau^{\epsilon_\tau}(M) \delta_\tau(N), \quad c^-(R_{F/\mathbf{Q}}(M \otimes N)) = \prod_\tau c_\tau^{-\epsilon_\tau}(M) \delta_\tau(N)$$

modulo $(E \vee \tilde{F})^\times$. We have, by Lemma 4.2,

$$\prod_\tau \delta_\tau(N) = \delta(R_{F/\mathbf{Q}}(N)) = g_0(\det \rho)^{-1} = g(\chi)^{-1}$$

modulo $(E \vee \tilde{F})^\times$. By (3.8) we obtain

$$c^+((R_{F/\mathbf{Q}}(M \otimes N))(m)) = \begin{cases} (2\pi\sqrt{-1})^{nm} \prod c_\tau^{\epsilon_\tau}(M) g(\chi)^{-1} & \text{if } m \text{ is even} \\ (2\pi\sqrt{-1})^{nm} \prod c_\tau^{-\epsilon_\tau}(M) g(\chi)^{-1} & \text{if } m \text{ is odd} \end{cases}$$

modulo $(E \vee \tilde{F})^\times$. Put

$$D(s, f, \chi^{-1}) = \sum_{\mathfrak{n}} c(\mathfrak{n}, f) \chi(\mathfrak{n})^{-1} N(\mathfrak{n})^{-s}.$$

Then Deligne's conjecture predicts

$$(4.7) \quad D(m, f, \chi^{-1}) / ((2\pi\sqrt{-1})^{nm} \prod_{\tau \in J_F} c_\tau^{(-1)^{m_\tau}}(M) g(\chi)^{-1}) \in E \vee \tilde{F}$$

if $m \in \mathbf{Z}$ is critical for $R_{F/\mathbf{Q}}(M \otimes N)$, that is

$$(k_0 - \min_{\tau \in J_F} k(\tau))/2 < m \leq (k_0 + \min_{\tau \in J_F} k(\tau))/2 - 1.$$

(cf. (2.9) in §2.)

We see easily that (4.7) is consistent with Theorem 4.3, (I) of [12] by putting

$$(4.8) \quad u(r, f) = \prod_\tau c_\tau^{\epsilon_\tau}(M_f), \quad r = (m_\tau).$$

However Shimura's result is more precise in two points. First it is shown that the quantity on the left of (4.7) belongs to E . Secondly it transforms covariantly under

$\sigma \in J_E$. We note one more important fact which cannot be derived from Deligne's conjecture. Define

$$I(f^\sigma) = (2\pi\sqrt{-1})^{n(1-k_0)} \pi^{\sum_{\tau \in J_F} k(\tau)} g(\psi)^{-1} \langle f^\sigma, f^\sigma \rangle, \quad \sigma \in J_E$$

where E denotes the field generated by eigenvalues of Hecke operators of f . Consider $\{I(f^\sigma)\}$ as an element of $(E \otimes_{\mathbf{Q}} \mathbf{C})^\times$. Then Theorem 4.3, (II) of [12] suggests

$$(4.9) \quad \begin{aligned} & c^+(R_{F/\mathbf{Q}}(M_f)) c^-(R_{F/\mathbf{Q}}(M_f)) \\ &= \prod_{\tau \in J_F} c_\tau^+(M_f) c_\tau^-(M_f) = \{I(f^\sigma)\} \pmod{E \vee \tilde{F}}. \end{aligned}$$

Now let $f \in S_k(\mathfrak{c}, \psi)$, $g \in S_l(\mathfrak{c}, \varphi)$ which are common eigenfunctions of all Hecke operators. Let

$$\begin{aligned} D(s, f) &= \sum_{\mathfrak{n}} c(\mathfrak{n}, f) N(\mathfrak{n})^{-s}, & D(s, g) &= \sum_{\mathfrak{n}} c(\mathfrak{n}, g) N(\mathfrak{n})^{-s}, \\ k_0 &= \max_{\tau \in J_F} k(\tau), & l_0 &= \max_{\tau \in J_F} l(\tau). \end{aligned}$$

Put

$$\begin{aligned} D(s, f, g) &= \sum_{\mathfrak{n}} c(\mathfrak{n}, f) c(\mathfrak{n}, g) N(\mathfrak{n})^{-s}, \\ \mathfrak{D}_\epsilon(s, f, g) &= L_\epsilon(2s + 2 - k_0 - l_0, \psi\varphi) D(s, f, g). \end{aligned}$$

Here L_ϵ denotes the L -function whose Euler $\mathfrak{p}$ -factors are dropped for $\mathfrak{p}|\mathfrak{c}$. Then $\mathfrak{D}_\epsilon(s, f, g)$ coincides with the L -function $L(id., M_f \otimes M_g, s)$ up to finitely many Euler $\mathfrak{p}$ -factors. By Proposition 3.2, we have

$$(4.10) \quad \begin{aligned} & c_\tau^+(M_f \otimes M_g) = c_\tau^-(M_f \otimes M_g) \\ &= \begin{cases} c_\tau^+(M_f) c_\tau^-(M_g) \delta_\tau(M_g) & \text{if } k(\tau) > l(\tau), \\ c_\tau^+(M_g) c_\tau^-(M_f) \delta_\tau(M_f) & \text{if } k(\tau) < l(\tau). \end{cases} \end{aligned}$$

Hence, by (3.8), we obtain

$$(4.11) \quad \begin{aligned} & c_\tau^+((M_f \otimes M_g)(m)) = c_\tau^-((M_f \otimes M_g)(m)) \\ &= \begin{cases} (2\pi\sqrt{-1})^{2m} c_\tau^+(M_f) c_\tau^-(M_g) \delta_\tau(M_g) & \text{if } k(\tau) > l(\tau), \\ (2\pi\sqrt{-1})^{2m} c_\tau^+(M_g) c_\tau^-(M_f) \delta_\tau(M_f) & \text{if } k(\tau) < l(\tau). \end{cases} \end{aligned}$$

Let E be the field generated by eigenvalues of Hecke operators of f and g .

First we assume that

$$k(\tau) > l(\tau) \quad \text{for every } \tau \in J_F.$$

By Proposition 3.2 and (4.11), we obtain

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)(m))) \\ &= (2\pi\sqrt{-1})^{2mn} c^+(R_{F/\mathbf{Q}}(M_f)) c^-(R_{F/\mathbf{Q}}(M_f)) \delta(R_{F/\mathbf{Q}}(M_g)) \mod (E \vee \tilde{F})^\times. \end{aligned}$$

Since $\bigwedge^2 M_g \cong \text{Art}_{\varphi^{-1}}(1 - l_0)$, we have $\delta(R_{F/\mathbf{Q}}(M_g)) = (2\pi\sqrt{-1})^{n(1-l_0)} g(\varphi)$. Thus we have shown

$$(4.12) \quad \begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)(m))) \\ &= (2\pi\sqrt{-1})^{n(2m+1-l_0)} g(\varphi) \{I(f^\sigma)\} \mod (E \vee \tilde{F})^\times. \end{aligned}$$

From (4.12), we see easily that Deligne's conjecture is consistent with Shimura [12], Theorem 4.2. However Shimura's result is more precise in two points mentioned above and also in that the condition on weights is less restrictive.

Next assume that

$$k(\tau) > l(\tau) \quad \text{for } \tau \in \delta, \quad k(\tau) < l(\tau) \quad \text{for } \tau \in \delta'$$

where δ and δ' are subsets of J_F such that $\delta \cup \delta' = J_F$, $\delta \cap \delta' = \emptyset$. By Proposition 3.2, we have

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g))) = c^-(R_{F/\mathbf{Q}}((M_f \otimes M_g))) = \prod_{\tau \in \delta} c_\tau^+(M_f) c_\tau^-(M_f) \\ & \times \prod_{\tau \in \delta'} c_\tau^+(M_g) c_\tau^-(M_g) \prod_{\tau \in \delta} \delta_\tau(M_f) \prod_{\tau \in \delta'} \delta_\tau(M_g) \mod (E \vee \tilde{F})^\times. \end{aligned}$$

Since

$$\bigwedge^2 M_f \cong \text{Art}_{\psi^{-1}}(1 - k_0), \quad \bigwedge^2 M_g \cong \text{Art}_{\varphi^{-1}}(1 - l_0),$$

we obtain

$$(4.13) \quad \begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)(m))) = (2\pi\sqrt{-1})^{2mn+(1-k_0)|\delta'|+(1-l_0)|\delta|} \\ & \times \prod_{\tau \in \delta} c_\tau^+(M_f) c_\tau^-(M_f) \prod_{\tau \in \delta'} c_\tau^+(M_g) c_\tau^-(M_g) \prod_{\tau \in \delta} \delta_\tau(\text{Art}_{\varphi^{-1}}) \prod_{\tau \in \delta'} \delta_\tau(\text{Art}_{\psi^{-1}}) \end{aligned}$$

by (3.8). Since $\delta_\tau(\text{Art}_{\varphi^{-1}}) \sim 1$, $\delta_\tau(\text{Art}_{\psi^{-1}}) \sim 1$, we have

$$(4.14) \quad \begin{aligned} & \mathfrak{D}_c(m, f, g) \sim \pi^{2mn+(1-k_0)|\delta'|+(1-l_0)|\delta|} \\ & \prod_{\tau \in \delta} c_\tau^+(M_f) c_\tau^-(M_f) \prod_{\tau \in \delta'} c_\tau^+(M_g) c_\tau^-(M_g) \end{aligned}$$

if m is critical for $\text{Res}_{F/\mathbf{Q}}(M_f \otimes M_g)$, that is

$$\frac{k_0 + l_0}{2} - \frac{|k(\tau) - l(\tau)|}{2} - 1 < m < \frac{k_0 + l_0}{2} + \frac{|k(\tau) - l(\tau)|}{2}$$

for all $\tau \in J_F$.

We shall show that (4.14) is consistent with Shimura's results and a part of his conjectures. (The part involving modular forms of half integral weight shall remain mysterious.)

Let χ (resp. χ') be the system of eigenvalues of Hecke operators attached to f (resp. g). Set

$$(4.15) \quad Q(\chi, \delta) = \pi^{(k_0-1)|\delta| - \sum_{\tau \in \delta} k(\tau)} \prod_{\tau \in \delta} c_{\tau}^{+}(M_f) c_{\tau}^{-}(M_f).$$

We note that

$$D(s, \chi, \eta) = \sum_{\mathfrak{a}} \eta(\mathfrak{a}) \chi(\mathfrak{a}) N(\mathfrak{a})^{-s-1} = D(s + \frac{k_0}{2}, f, \eta)$$

when η is a Hecke character of finite order of F ,

$$D(s, \chi, \chi') = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \chi'(\mathfrak{a}) N(\mathfrak{a})^{-s} = D(s + \frac{k_0 + l_0}{2} - 2, f, g)$$

in the normalization of Shimura [15], [16]. By (4.14) and (4.15), we obtain

$$\begin{aligned} D(\frac{t}{2}, \chi, \chi') &= D(\frac{t}{2} + \frac{k_0 + l_0}{2} - 2, f, g) \\ &\sim \pi^{tn + \sum_{\tau \in \delta} k(\tau) + \sum_{\tau \in \delta'} l(\tau)} \pi^{-2n} Q(\chi, \delta) Q(\chi', \delta') / L_{\epsilon}(t-2, \psi\varphi) \end{aligned}$$

if $t \in \mathbf{Z}$ satisfies

$$\begin{aligned} t &\equiv k_0 + l_0 \pmod{2}, \\ -\frac{|k(\tau) - l(\tau)|}{2} + 1 &< \frac{t}{2} \leq \frac{|k(\tau) - l(\tau)|}{2} + 1 \quad \text{for every } \tau \in J_F. \end{aligned}$$

Since $L_{\epsilon}(t-2, \psi\varphi) \sim \pi^{(t-2)n}$, we obtain

$$(4.16) \quad D(\frac{t}{2}, \chi, \chi') \sim \pi^{\sum_{\tau \in \delta} k(\tau) + \sum_{\tau \in \delta'} l(\tau)} Q(\chi, \delta) Q(\chi', \delta').$$

This is consistent with Theorem 5.3 of [15] and the definition of $Q(\chi, \delta)$ in [16].

Next we assume that $k(\tau)$ is even for every $\tau \in J_F$ and put, for $r \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$, $\delta \subseteq J_F$,

$$(4.17) \quad V(\chi, r) = \pi^{nk_0/2} \prod_{\tau \in J_F} c_{\tau}^{(-1)^{k_0/2} r_{\tau}}(M_f),$$

$$(4.18) \quad P(\chi, \delta, r) = \pi^{-\sum_{\tau \in \delta} k(\tau)/2} \pi^{k_0|\delta|/2} \pi^{-|\delta|} \prod_{\tau \in \delta} c_{\tau}^{(-1)^{k_0/2} r_{\tau}}(M_f).$$

If t is an integer such that $|t| < k(\tau)/2$ for every τ and η is a Hecke character of finite order of F such that

$$\eta(x) = \text{sgn}(x)^{t+r}, \quad x \in F_\infty^\times,$$

then we have

$$D(t, \chi, \eta) = D(t + \frac{k_0}{2}, f, \eta) \sim \pi^{n(t+k_0/2)} \prod_{\tau \in J_F} c_\tau^{(-1)^{k_0/2} r_\tau} (M_f) \sim \pi^{nt} V(\chi, r).$$

This is consistent with (8.2a) of [16].

We see easily that $\bar{\chi}$ attaches to $f \otimes \psi^{-1}$. Since we have assumed that $k(\tau)$ is even for every τ , $\psi(x) = 1$, $x \in F_\infty^\times$ holds. Therefore $c_\tau^\pm(M_f) \sim c_\tau^\pm(M_f \otimes \text{Art}_\psi)$ for every τ . Thus we may take $P(\bar{\chi}, \delta, r) \sim P(\chi, \delta, r)$ for every $\delta \subseteq J_F$, $r \in (\mathbb{Z}/2\mathbb{Z})^{J_F}$. Now (C3) and (C4) of [16], p. 293 state that

$$(4.19) \quad V(\chi, r) \sim \pi^{(\sum_{\tau \in J_F} k(\tau)/2) + n} P(\chi, \delta, r) P(\bar{\chi}, \iota - \delta, r - r\delta)$$

$$(4.20) \quad Q(\chi, \delta) \sim \pi^{|\delta|} P(\chi, \delta, r) P(\bar{\chi}, \delta, \delta - r)$$

for every $r \in (\mathbb{Z}/2\mathbb{Z})^{J_F}$, $\delta \subseteq J_F$. It is obvious that (4.19) and (4.20) follow from (4.15), (4.17) and (4.18) in view of $P(\chi, \delta, r) \sim P(\bar{\chi}, \delta, r)$.

§5. Special values of zeta functions of Shimura varieties

Let M be a motive over F with coefficients in E . Let $M' = \bigotimes_{\Omega} \text{Res}_{F/F'} M$. Our first task is to compute τ' -periods, $\tau' \in J_{F'}$ of M' from τ -periods, $\tau \in J_F$ of M . For this purpose, we supplement Conjecture 1.5 with specifying Betti and de Rham realizations of M' .

Let $\tau' \in J_{F'}$ and take any $\tilde{\tau}' \in G$ such that $\tilde{\tau}'|_{F'} = \tau'$. Then $\{\tilde{\tau}' s_i | F \mid 1 \leq i \leq r\}$ defines a set of r -distinct elements of J_F which does not depend on the choice of $\tilde{\tau}'$ and s_i . We have

$$(5.1) \quad H_{\tau', B}(M') = \bigotimes_{i=1}^r H_{\tilde{\tau}' s_i | F, B}(M)$$

where $\bigotimes_{i=1}^r$ denotes the tensor product as E -modules. If

$$H_{\tau, B}(M) \otimes_{E, \sigma} \mathbb{C} = \bigoplus H^{pq}(\tau, \sigma, M), \quad \tau \in J_F, \quad \sigma \in J_E$$

is the Hodge decomposition,

$$(5.2) \quad H_{\tau', B}(M') \otimes_{E, \sigma} \mathbb{C} = \bigotimes_{i=1}^r H^{pq}(\tilde{\tau}' s_i | F, \sigma, M)$$

is the Hodge decomposition of $H_{\tau', B}(M')$.

The de Rham realization of M' is

$$(5.3) \quad H_{DR}(M') = (\otimes_{i=1}^r (H_{DR}(M) \otimes_{F, s_i} \mathbf{Q}))^{\text{Gal}(\overline{\mathbf{Q}}/F')},$$

where $\otimes_{i=1}^r$ denotes the tensor product as $E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ -modules and $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/F')$ acts as

$$\sigma(\otimes_{i=1}^r (v_i \otimes_{F, s_i} a_i)) = \otimes_{i=1}^r (v_i \otimes_{F, \sigma s_i} \sigma(a_i)), \quad v_i \in H_{DR}(M), \quad a_i \in \overline{\mathbf{Q}}.$$

Since $H_{DR}(M)$ has a structure of free $E \otimes_{\mathbf{Q}} F$ -module, we can verify that the right hand side of (5.3) has a natural structure of free $E \otimes_{\mathbf{Q}} F'$ -module of rank = $\text{rank}(M)^r$. From (5.3), we obtain an isomorphism as $E \otimes_{\mathbf{Q}} \mathbf{C}$ -modules

$$(5.4) \quad H_{DR}(M') \otimes_{F', \tau'} \mathbf{C} \cong \otimes_{i=1}^r (H_{DR}(M) \otimes_{F, \tilde{\tau}' s_i|F} \mathbf{C}), \quad \tau' \in J_{F'}.$$

Here the isomorphism is given by

$$(\otimes_{i=1}^r (v_i \otimes_{F, s_i} a_i)) \otimes_{F', \tau'} 1 \longrightarrow (\otimes_{i=1}^r (v_i \otimes_{F, \tilde{\tau}' s_i|F} \tilde{\tau}'(a_i))), \quad v_i \in H_{DR}(M), \quad a_i \in \overline{\mathbf{Q}}.$$

For $\tau \in J_F$, let

$$I_{\tau} : H_{\tau, B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F, \tau} \mathbf{C}$$

be the canonical isomorphism of $E \otimes_{\mathbf{Q}} \mathbf{C}$ -modules. Let $\tau' \in J_{F'}$. In view of (5.1) and (5.4), we can take the canonical isomorphism

$$I_{\tau'} : H_{\tau', B}(M') \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F', \tau'} \mathbf{C}$$

as

$$(5.5) \quad I_{\tau'} = \otimes_{i=1}^r I_{\tilde{\tau}' s_i|F}.$$

PROPOSITION 5.1. *Let M be a motive over F with coefficients in E . We assume that M is of rank 2 and let*

$$H_{\tau, B}(M) \otimes_{E, \sigma} \mathbf{C} = H^{a(\tau, +), a(\tau, -)}(\tau, \sigma, M) \oplus H^{a(\tau, -), a(\tau, +)}(\tau, \sigma, M),$$

$\tau \in J_F, \sigma \in J_E$ be the Hodge decomposition. We assume

$$(5.6) \quad a(\tau, +) > a(\tau, -).$$

Let Ω be any subset of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/\text{Gal}(\overline{\mathbf{Q}}/F)$ such that $|\Omega| = r \geq 2$. Define F' and $M' = \otimes_{\Omega} \text{Res}_{F/F'} M$ as above. Let Λ be the set of all maps from $\{1, 2, \dots, r\}$ to $\{\pm 1\}$. We assume that

$$(5.7) \quad \sum_{i=1}^r a(\tilde{\tau}' s_i|F, \lambda(i)) \neq \sum_{i=1}^r a(\tilde{\tau}' s_i|F, -\lambda(i))$$

holds for every $\lambda \in \Lambda$. Set

$$\Lambda^+ = \{\lambda \in \Lambda \mid \sum_{i=1}^r a(\tilde{\tau}' s_i | F, \lambda(i)) > \sum_{i=1}^r a(\tilde{\tau}' s_i | F, -\lambda(i))\}.$$

and let n_i (resp. m_i) be the number of $\lambda \in \Lambda^+$ such that $\lambda(i) = 1$ (resp. $\lambda(i) = -1$), $1 \leq i \leq r$. (n_i and m_i depend on τ' .) Then we have, modulo $E \vee \tilde{F}$,

$$(5.8) \quad c_{\tau'}^{\pm}(M') = \prod_{i=1}^r (c_{\tilde{\tau}' s_i | F}^+(M) c_{\tilde{\tau}' s_i | F}^-(M))^{(n_i - m_i)/2} \delta_{\tilde{\tau}' s_i | F}(M)^{m_i}$$

for every $\tau' \in J_{F'}$.

PROOF: For every $\tau \in J_F$, take $u_{\tau}^{\pm}$ so that

$$Eu_{\tau}^{\pm} = H_{\tau, B}^{\pm}(M).$$

Take d^- so that

$$(E \otimes F)d^- = F^-(M) = F^+(M),$$

and then take d^+ so that

$$H_{DR}(M) = (E \otimes F)d^+ + (E \otimes F)d^-.$$

Put

$$I_{\tilde{\tau}' s_i | F}(u_{\tilde{\tau}' s_i | F}^{\pm}) = x_i^{+, \pm} d^+ + x_i^{-, \pm} d^-,$$

with $x_i^{\pm, \pm} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Then we have

$$c_{\tilde{\tau}' s_i | F}^{\pm}(M) = x_i^{+, \pm}, \quad \delta_{\tilde{\tau}' s_i | F}(M) = \det \begin{pmatrix} x_i^{+, +} & x_i^{-, +} \\ x_i^{+, -} & x_i^{-, -} \end{pmatrix}.$$

These quantities are elements of $(E \otimes_{\mathbf{Q}} \mathbf{C})^{\times}$ determined modulo multiplications by $E \vee \tilde{F}$. Set

$$\Lambda_{\pm} = \{\lambda \in \Lambda \mid \prod_{i=1}^r \lambda(i) = \pm 1\}.$$

By (5.1), a basis of $H_{\tau', B}^{\pm}(M')$ over E is given by $\otimes_{i=1}^n u_{\tilde{\tau}' s_i | F}^{\epsilon(i)}$ when ϵ extends over $\Lambda_{\pm}$. By (5.2) and (5.7), we have $F^+(M') = F^-(M')$. A basis of $H_{DR}(M') \otimes_{F', \tau'} \tilde{F}$ modulo $F^-(M') \otimes_{F', \tau'} \tilde{F}$ over $E \otimes_{\mathbf{Q}} \tilde{F}$ is given by $\otimes_{i=1}^r d^{\lambda(i)}$ where λ extends over Λ^+ . We have

$$\otimes_{i=1}^r I_{\tilde{\tau}' s_i | F}(\otimes_{i=1}^r u_{\tilde{\tau}' s_i | F}^{\epsilon(i)}) = \otimes_{i=1}^r (x_i^{+, \epsilon(i)} d^+ + x_i^{-, \epsilon(i)} d^-) = \sum_{\lambda \in \Lambda^+} \prod_{i=1}^r x_i^{\lambda(i), \epsilon(i)} \otimes_{i=1}^r d_i^{\lambda(i)}$$

mod $F^-(M') \otimes_{F', \tau'} \tilde{F}$. Therefore we obtain

$$c_{\tau'}^{\pm}(M') = \det(X^{\pm})$$

where $X^{\pm}$ is the $2^{r-1} \times 2^{r-1}$ -matrix whose (λ, ϵ) -entry for $\lambda \in \Lambda^+$, $\epsilon \in \Lambda_{\pm}$ is given by $\prod_{i=1}^r x_i^{\lambda(i), \epsilon(i)}$. Now (5.8) follows from (3.10) (and from the same formula for $\det(X^-)$). This completes the proof.

By (3.8), we obtain

$$(5.9) \quad c_{\tau'}^{\pm}(M'(m)) = (2\pi\sqrt{-1})^{2^{r-1}m} \prod_{i=1}^r (c_{\tau' s_i|F}^+(M) c_{\tau' s_i|F}^-(M))^{(n_i - m_i)/2} \delta_{\tau' s_i|F}(M)^{m_i},$$

and by Proposition 2.1, we have

$$(5.10) \quad c^{\pm}(\text{Res}_{F'/\mathbf{Q}}(M')(m)) = (2\pi\sqrt{-1})^{2^{r-1}mn'} \times \prod_{\tau' \in J_{F'}} \prod_{i=1}^r (c_{\tau' s_i|F}^+(M) c_{\tau' s_i|F}^-(M))^{(n_i - m_i)/2} \delta_{\tau' s_i|F}(M)^{m_i},$$

mod $E \vee \tilde{F}$, where $n' = [F' : \mathbf{Q}]$.

Now let $f \in S_k(\tau, \psi)$ be a new form and let M_f be the motive over F with coefficients in E attached to f as is given in Conjecture 4.1. We set $M'_f = \otimes_{\Omega} \text{Res}_{F'/F}(M_f)$. We regard $c^{\pm}(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m))$ as a complex number fixing an embedding $E \subset \mathbf{C}$. Since $\Lambda^2 M_f \cong \text{Art}_{\psi^{-1}}(1 - k_0)$, we have $\delta_{\tau}(M_f) \sim \pi^{1-k_0}$ for every $\tau \in J_F$. Then (5.10) yields

$$(5.11) \quad c^{\pm}(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m)) \sim \pi^{2^{r-1}mn'} \prod_{\tau' \in J_{F'}} (\pi^{1-k_0} \sum_{i=1}^r m_i \prod_{i=1}^r (c_{\tau' s_i|F}^+(M_f) c_{\tau' s_i|F}^-(M_f))^{(n_i - m_i)/2}),$$

Deligne's conjecture predicts

$$(5.12) \quad L(m, M_f) \sim c^{\pm}(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m))$$

if $m \in \mathbf{Z}$ is critical for M'_f . In view of the relation (4.15) of $Q(\chi, \delta)$ with $c_{\tau}^{\pm}(M_f)$, $c^{\pm}(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m))$ can be expressed as a product of π and $Q(\chi, \delta)$. (Note that we have assumed $r \geq 2$). Here χ denotes the system of eigenvalues of Hecke operators attached to f .

We shall show that (5.11) and (5.12) are in perfect accordance with a certain result of Shimura [14], II. Assume that F contains a subfield F_0 such that $[F : F_0] = 2$. Set $\Omega = \text{Gal}(\overline{\mathbf{Q}}/F_0)/\text{Gal}(\overline{\mathbf{Q}}/F)$. Then we have $H' = \text{Gal}(\overline{\mathbf{Q}}/F_0)$, $F' = F_0$.

Apply the above construction. We obtain a motive M'_f over F' with coefficients in E . Set

$$J_{F'} = \{\tau_1, \tau_2, \dots, \tau_{n/2}\}$$

and choose $\tilde{\tau}_i \in J_F$ so that $\tilde{\tau}_i|_{F'} = \tau_i$, $1 \leq i \leq n/2$. Let σ be the generator of $\text{Gal}(F/F')$. We assume that the weight of f satisfies

$$(5.13) \quad k(\tilde{\tau}_i|F) > k(\tilde{\tau}_i\sigma|F), \quad 1 \leq i \leq n/2.$$

Then we have $n_1 = 2$, $m_1 = 0$, $n_2 = 1$, $m_2 = 1$ for every $\tau' \in J_{F'}$. Hence we get

$$(5.14) \quad c^\pm(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m)) \sim \pi^{mn} \pi^{(1-k_0)n/2} \prod_{i=1}^{n/2} (c_{\tilde{\tau}_i}^+(M_f) c_{\tilde{\tau}_i}^-(M_f))$$

by (5.11). Set

$$\delta = \{\tilde{\tau}_1, \tilde{\tau}_2, \dots, \tilde{\tau}_{n/2}\} \subset J_F.$$

Since

$$Q(\chi, \delta) \sim \pi^{(k_0-1)n/2 - \sum_{\tau \in \delta} k(\tau)} \prod_{i=1}^n (c_{\tilde{\tau}_i|F}^+(M_f) c_{\tilde{\tau}_i|F}^-(M_f)),$$

we obtain

$$c^\pm(\text{Res}_{F'/\mathbf{Q}}(M'_f)(m)) \sim \pi^{mn} \pi^{(1-k_0)n} \pi^{\sum_{\tau \in \delta} k(\tau)} Q(\chi, \delta).$$

By (2.9), Conjecture 4.1 and (5.2), we see easily that $m \in \mathbf{Z}$ is critical for M'_f if and only if

$$k_0 - \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} - 1 < m \leq k_0 + \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} - 1$$

holds for every $1 \leq i \leq n/2$. Put

$$L^*(s, \chi) = L(s + k_0 - 2, M'_f).$$

Then if $m \in \mathbf{Z}$ satisfies

$$(5.15) \quad 1 - \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} < m \leq 1 + \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2}$$

for every $1 \leq i \leq n/2$,

$$(5.16) \quad L^*(m, \chi) \sim \pi^{(m-1)n} \pi^{\sum_{\tau \in \delta} k(\tau)} Q(\chi, \delta)$$

is predicted.

We see easily that (5.16) is in accordance with [14], II, Theorem 3.11. ($C(s)$ there essentially coincides with $L^*(s, \chi)$.) Furthermore Shimura obtained the result without assuming that $k(\tau) \bmod 2$ is independent of $\tau \in J_F$. In this sense, Shimura's result is more general than (5.16).

§ 6. 志村の不変量 $Q(X, \delta)$ について

F は $\mathbb{Q}$ 上 n 次の総実体、 B は F 上の quaternion algebra とする。
 B は $\delta \subset J_F$ で split し、 $\delta' = J_F \setminus \delta$ で split しないとする。
 従、 r

$$(6.1) \quad B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{n-r}, \quad r = |\delta|.$$

ここに $\mathbb{H}$ は Hamilton quaternion algebra を表わす。このとき
 B を signature (δ, δ') の quaternion algebra という。以下、
 $r \geq 1$ と仮定する。

$G = \text{Res}_{F/\mathbb{Q}}(B^\times)$ とおく。 G は $\mathbb{Q}$ 上の代数群であり。
 $G_K = (B \otimes_{\mathbb{Q}} K)^\times$ が任意の $\mathbb{Q}$ -algebra K に対して成立つ。
 G_A により G の adélization, G_{A_f} , G_∞ はそれぞれ G_A の
 finite part, archimedean part を表わす。

$$G_\infty \cong GL(2, \mathbb{R})^r \times (\mathbb{H}^\times)^{n-r}$$

である。 $G_{\infty+}$ により G_∞ の単位元の連結成分を表わす。
 Z を G の center とする。

Shimura [14], II に従い、保型因子、保型形式、
 arithmeticity を定義する。一部を復習しておこう。

表現 $\sigma_m: \mathbb{H}^\times \longrightarrow GL_{m+1}(\mathbb{C})$ を、埋込み $\mathbb{H}^\times \subset GL_2(\mathbb{C})$
 と、 m 次の symmetric tensor 表現 $GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ をつ
 ないで定義する。(arithmeticity の定義には、 $\mathbb{Q}$ -structure を
 こゝたとり方が必要である。cf. [14])

$k = \sum_{\tau \in \delta} k(\tau)\tau$, $\kappa = \sum_{\tau \in \delta'} \kappa(\tau)\tau \in I_F$, $k(\tau), \kappa(\tau) \geq 0$
 を weight とし, $d = \prod_{\tau \in \delta'} (k(\tau) + 1)$ とおく。 $\mathbb{C}^d$ を
 $\bigotimes_{\tau \in \delta'} \sigma_{k(\tau)}$ の表現空間とみる。 $\lambda \in G_{A+} = G_{A_f} \times G_{\infty+}$,
 $z = (z_\tau)_{\tau \in \delta} \in \mathbb{S}^r$ とする。

$$(6.2) \quad J(\lambda, z) = \prod_{\tau \in \delta} (c_\tau z_\tau + d_\tau)^{k(\tau)} N(\lambda_\tau)^{-k(\tau)/2}$$

$$\bigotimes_{\tau \in \delta'} \sigma_{\kappa(\tau)}(\lambda_\tau) \prod_{\tau \in \delta'} N(\lambda_\tau)^{-\kappa(\tau)/2},$$

$$\lambda_\infty = (\lambda_\tau)_{\tau \in J_F}, \quad \lambda_\tau = \begin{pmatrix} * & * \\ c_\tau & d_\tau \end{pmatrix}, \quad \tau \in \delta$$

とおく。ここに N は reduced norm を表わす。 $\mathbb{S}^r$ 上の
 $\mathbb{C}^d$ -valued function f に対し

$$(6.3) \quad (f \|_{k, \kappa}(\lambda))(z) = f(\lambda(z)) J(\lambda, z)^{-1},$$

$z \in \mathbb{S}^r$ とおく。 Γ を $G_{\mathbb{Q}+} = G_{\mathbb{Q}} \cap G_{A+}$ の congruence
 subgroup とする。

$S_{k, \kappa}(\Gamma) = \{ f \mid f \text{ は } \mathbb{S}^r \text{ 上の } \mathbb{C}^d\text{-valued} \\ \text{holomorphic function で } f \|_{k, \kappa} \gamma = f, \forall \gamma \in \Gamma \text{ を} \\ \text{みたす} \}$

を, weight が (k, κ) の Γ に関する cusp form の空間と
 する。但し, $B = M_2(F)$ のときは, 更に周知の cusp に
 ついての条件が課せられているものとする。

$$S_{k, \kappa} = \bigcup_{\Gamma} S_{k, \kappa}(\Gamma)$$

とおく。ここに Γ は $G_{\mathbb{Q}+}$ の全ての合同部分群を走る。

W_0 を G_{A_f} の open compact subgroup とし $W = W_0 G_{\infty+} \subset G_A$ とおく. $\mathcal{S}_{k,k}(W)$ により, G_A 上の $\mathbb{C}^d$ -valued function f 7 次の条件 (6.4), (6.5) をみたすものの成す vector space とする.

$$(6.4) \quad f(\alpha x w) = f(x), \quad \alpha \in G_{\mathbb{Q}}, x \in G_A, w \in W_0.$$

$$(6.5) \quad \forall x \in G_{A_f}, \exists g_x \in S_{k,k} \text{ such that}$$

$$f(xy) = (g_x \|_{k,k} y)(i) \text{ for } \forall y \in G_{\infty+}.$$

$$\text{ここに } i = (\sqrt{-1}, \dots, \sqrt{-1}) \in S_{\mathbb{C}}.$$

$$G_A = \bigcup_{i=1}^h G_{\mathbb{Q}} x_i W, \quad I_i = x_i W x_i^{-1} \cap G_{\mathbb{Q}}$$

とおくと.

$$(6.6) \quad \begin{array}{ccc} \mathcal{S}_{k,k}(W) & \cong & \prod_{i=1}^h S_{k,k}(I_i) \\ \downarrow & & \downarrow \\ f & \longrightarrow & (f_1, \dots, f_h) \end{array}$$

が canonical に成立つ. $f, g \in \mathcal{S}_{k,k}(W)$ に対し

$$(6.7) \quad \langle f, g \rangle = \int_{Z_{\infty+} G_{\mathbb{Q}} \backslash G_A} \overline{f(x)} g(x) dx$$

とおく. ここに $Z_{\infty+}$ は Z_{∞} における 1 の連結成分を表わし, $\text{vol}(Z_{\infty+} G_{\mathbb{Q}} \backslash G_A) = 1$ となるように invariant measure を normalize しておく, この内積は [14], II, (1.12) で与えられているものと一致する.

B の maximal order R を 1 とする. $W_0 = \prod_V (W_0)_V$, $(W_0)_V$ は $B_V^{\times}$ の open compact subgroup で, $(W_0)_V = R_V^{\times}$ が殆んど全ての V に対して成立つ, という形の W_0 を全て考え

$S_{k, \kappa} = \bigcup_{W_0} S_{k, \kappa}(W)$, $W = W_0 \Gamma_{\infty+}$
 とおく。

$S_{k, \kappa}(\overline{\mathbb{Q}})$ は $\overline{\mathbb{Q}}$ -rational elements の成す $S_{k, \kappa}$ の subset とする。($B = M_2(F)$ のときは $\text{cusp}(i\infty)^r$ の回りでの Fourier 展開の係数が全て $\overline{\mathbb{Q}}$ に入ることと同値である。)

$S_{k, \kappa} \ni f$ が $\overline{\mathbb{Q}}$ -rational $\iff f_\lambda \in S_{k, \kappa}(\overline{\mathbb{Q}})$, $1 \leq \lambda \leq n$
 と定義する。(cf. (6.6))

(6.8) $S_{k, \kappa} = S_{k, \kappa} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, $S_{k, \kappa} = S_{k, \kappa}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$
 が成立つ。 B による依存性を強調する時は、上の記号に B を挿入して $(S_{k, \kappa}(B, \overline{\mathbb{Q}}))$ 等と) 表わすことにする。

L を $\mathbb{C}$ の subfield とする。 $\mathcal{H}_L(G_{A_f})$ は G_{A_f} 上の L -valued locally constant compactly supported functions が convolution について成す L 上の algebra とする。但し G_{A_f} の measure は, $\text{vol}(R_V^X) = 1$ をみたす B_V^X の measure の product measure から得られているとする。 G_{A_f} の open compact subgroup K に対し, $\mathcal{H}_L(G_{A_f}, K)$ は, K 両側不変な函数の成す $\mathcal{H}_L(G_{A_f})$ の subalgebra とする。

$f \in S_{k, \kappa}$, $\varphi \in \mathcal{H}_L(G_{A_f})$ に対して

$$(6.9) \quad \rho(\varphi)f = f * \check{\varphi}, \quad \check{\varphi}(x) = \varphi(x^{-1}), x \in G_{A_f}$$

とみると, $\rho(\varphi)f \in S_{k, \kappa}$ で ρ は $\mathcal{H}_L$ の表現を与える。

さらに, $L \subset \overline{\mathbb{Q}}$ で f が $\overline{\mathbb{Q}}$ -rational ならば, $\rho(\varphi)f$ も

$\overline{\mathbb{Q}}$ -rational である。([14], II, p. 577, l. 26~28 による。)

$$(6.10) \quad \langle \rho(\varphi) f, g \rangle = \langle f, \rho(\tilde{\varphi}) g \rangle,$$

$\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ が成立つ。 v を F の finite place,

$$f \in \mathcal{S}_{k, \kappa}(W), \quad W = W_0 \otimes_{\infty+}, \quad W_0 = \prod_w (W_0)_w, \quad (W_0)_v = R_v^{\times}$$

B は v で split すると仮定し. 同型 $B_v \cong M_2(F_v)$, $R_v \cong M_2(\mathcal{O}_v)$ を固定する。 φ を

$$\prod_{w \neq v} (W_0)_w \times R_v^{\times} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} R_v^{\times}$$

の characteristic function $\times \text{vol}(W_0)^{-1}$, ととることにより

Hecke operator $T(v)$ の f への作用 $T(v)f$ を定める。ここに,

$\mathcal{O}_v, \varpi_v$ は F_v の整数環, 素元である。

F の各 finite place v に $\chi(v) \in \mathbb{C}$ を対応させる写像 χ がある $f(\neq 0) \in \mathcal{S}_{k, \kappa}(B)$ について

$$T(v)f = \chi(v)f \quad \text{for almost all } v$$

をみたすとき, Hecke operator の eigenvalue の system χ は

$\mathcal{S}_{k, \kappa}(B)$ に occur するといひ, f は χ に属するといふ。

χ に属する $f \in \mathcal{S}_{k, \kappa}(B)$ 全体の vector space を $W(\chi, B)$ とかき。

$$W(\chi, B, \overline{\mathbb{Q}}) = W(\chi, B) \cap \mathcal{S}_{k, \kappa}(B, \overline{\mathbb{Q}})$$

とおく。

Theorem 6.1. χ を $\mathcal{S}_{k,\kappa}(B)$ に occur する Hecke operator の eigenvalue の system, $f, g, h \in W(B, \chi, \overline{\mathbb{Q}})$, $f \neq 0$ とする。このとき, $\langle g, h \rangle / \langle f, f \rangle \in \overline{\mathbb{Q}}$.

Proof. G_{A_f} の open compact subgroup W_0 を十分小さくとり, $f, g, h \in \mathcal{S}_{k,\kappa}(W)$, $W = W_0 G_{\infty f}$ とできる。 $H_{\mathbb{C}}(G_A)$ を Jacquet - Langlands [7] の意味での G_A の Hecke 環とし, V を f, g, h により, $H_{\mathbb{C}}(G_A)$ の作用で生成される $G_{\mathbb{Q}} \backslash G_A$ 上の $\mathbb{C}^d$ -valued functions の空間とする。このとき, Shimizu - Jacquet - Langlands 対応と, $GL(2)$ における strong multiplicity 1 theorem により, V は $H_{\mathbb{C}}(G_A)$ の既約表現空間になることがわかる。 $V_1 = V \cap \mathcal{S}_{k,\kappa}(B)$ とおくと, V_1 は $H_{\mathbb{C}}(G_{A_f})$ 不変であり, V_1 における $H_{\mathbb{C}}(G_{A_f})$ の表現は既約である。さらに, $V_1^{W_0} = \mathcal{S}_{k,\kappa}(W)$ であり, $V_1^{W_0}$ は $H_{\mathbb{C}}(G_{A_f}, W_0)$ -不変であるが, この表現は既約である。(cf. Bernstein - Zelevinski [1], p. 17.) よ, 7 次の Lemma が証明できればよい。

Lemma 6.2. k は complex conjugation τ stable な $\overline{\mathbb{Q}}$ の subfield, V は k 上の $d < \infty$ 次元 vector space とし。

$V = V_0 \otimes_k \mathbb{C}$ とおく。 A は k 上の associative algebra, ρ は A の V における既約表現で $\rho(A)V_0 \subseteq V_0$ をみたす

と仮定する。 $\langle \cdot, \cdot \rangle$ は V 上の positive definite hermitian form で、 $\forall u \in \mathcal{H}, \exists \tilde{u} \in \mathcal{H}$ such that

$$(6.10) \quad \langle \rho(u)x, y \rangle = \langle x, \rho(\tilde{u})y \rangle, \quad \forall x, y \in V$$

をみたすと仮定する。このとき、 $x_1, \dots, x_d$ が V_0 の k 上の (任意の) basis で、 $X = (\langle x_i, x_j \rangle) \in M_d(\mathbb{C})$ とおくと、 $c \in \mathbb{C}^x$ があり、 $c^{-1}X \in M_d(k)$ が成立つ。

Proof. basis $x_1, \dots, x_d$ により V を k^d と同一視し、 ρ を $\mathcal{H} \rightarrow M_d(k)$ とする。 (6.10) から

$$(*) \quad \rho(u)X = X \overline{\rho(\tilde{u})}, \quad \forall u \in \mathcal{H}$$

を得る。任意の $\sigma \in \text{Aut}(\mathbb{C}/k)$ に対して、 (*) から

$$(**) \quad \rho(u)X^\sigma = X^\sigma \overline{\rho(\tilde{u})}.$$

$\det X \neq 0$ ゆえ、 (*), (**) から

$$\rho(u)X^\sigma X^{-1} = X^\sigma X^{-1} \rho(u), \quad \forall u \in \mathcal{H}.$$

ゆえに $X^\sigma X^{-1}$ は scalar で

$$(***) \quad X^\sigma = d_\sigma X, \quad d_\sigma \in \mathbb{C}^x.$$

を得る。 $c = \langle x_1, x_1 \rangle$ ととり、 (***) から、

$$(1,1)\text{-成分をみて、 } d_\sigma = c^\sigma/c, \text{ ゆえに } X = cX_0.$$

とおくと、 $X_0^\sigma = X_0$ 。これが $\forall \sigma \in \text{Aut}(\mathbb{C}/k_0)$ に対して成立つから、 $X_0 \in M_d(k)$ 。 QED

$$f \in W(B, \chi, \overline{\mathbb{Q}}), \quad f \neq 0 \text{ をとり}$$

$$Q(X, B) = \langle f, f \rangle \pmod{\overline{\mathbb{Q}}^\times}$$

とおく。 $Q(X, B) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ は、Theorem 6.1 により f のとり方に依らないで定まる。同様の証明法は高次元の場合にも使える。(例えば [11] で扱われている case.) Shimura [15], Theorem 5, 6 は次の様に幾分精密化できる。

Theorem 6.3. B_1, B_2 は F 上の quaternion algebra で共に signature は (δ, δ') とする。Hecke operator の eigenvalue の system χ が $\mathcal{S}_{k, \kappa}(B_1), \mathcal{S}_{k, \kappa}(B_2)$ に共に occur すると仮定し、 $g \in W(\chi, B_1, \overline{\mathbb{Q}}), h \in W(\chi, B_2, \overline{\mathbb{Q}}), g \neq 0$ をとる。このとき、 $\forall \tau \in \delta$ に対して $k(\tau) \geq 2$ ならば、 $\langle g, g \rangle \sim \langle h, h \rangle$ 。

Proof. $f \in \mathcal{S}_{m, 0}(M_2(F), \overline{\mathbb{Q}}),$

$$m_\tau = \begin{cases} k(\tau), & \tau \in \delta \\ k(\tau) + 2, & \tau \in \delta' \end{cases}$$

を χ に属する non-zero form とし、 $\pi = \bigotimes_v \pi_v$ を f が生成する $A_{\mathbb{Q}}(GL_2(F_A))$ の automorphic representation とする。

Theorem 6.1 により、 B_1, B_2 は F 上の quaternion algebra として同型ではない、と仮定してよい。ゆえに B_1, B_2 の少くとも一方は、 F のある finite place v_0 で分岐する。

Jacquet-Langlands [7], Theorems 14.4, 15.1 により、

π_{v_0} は special 又は absolutely cuspidal. F 上の quaternion algebra B' で $\delta \cup \{v_0\}$ の外では不分裂で, δ では分裂するものが存在する. B' の signature は (δ', δ) .

$k'(\tau) = k(\tau) + 2$, $\tau \in \delta'$, $k'(\tau) = k(\tau) - 2$, $\tau \in \delta$ とおくと, [7], Theorem 16.1 により, χ は $\mathcal{S}_{k', k'}(B')$ に occur する. (B' が v_0 で分裂したとき, i.e., $|\delta|$ が odd の case, π_{v_0} の形が利く.) $\mathfrak{g}' \in W(\chi, B', \overline{\mathbb{Q}})$, $\mathfrak{g}' \neq 0$ をとると, [15], Theorem 5.4 により,

$\langle \mathfrak{g}, \mathfrak{g} \rangle \langle \mathfrak{g}', \mathfrak{g}' \rangle \sim \langle \mathfrak{f}, \mathfrak{f} \rangle \sim \langle \mathfrak{h}, \mathfrak{h} \rangle \langle \mathfrak{g}', \mathfrak{g}' \rangle$ が成立ち, 結論を得る.

Remark. $\mathcal{S}_{k, k}(B)$ の定義は [14], II におけるものより幾分一般にしている. 従, F 上の証明で [15], Theorem 5.4 をそのままには適用できないが, new form の理論と Theorem 6.1 により, この点は処理できる.

χ と J_F の non-empty subset δ が与えられたとき, ある signature (δ, δ') の F 上の quaternion algebra B があ, χ は $\mathcal{S}_{k, k}(B)$ に χ が occur すれば,

$$Q(\chi, \delta) = Q(\chi, B)$$

として invariant $Q(\chi, \delta) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ は B のとり方に依存せずに定義できる. これは Shimura [16], p. 286 にある

予想を、 $k(\tau) \geq 2$, $\forall \tau \in \delta$ の仮定の下に確かめたことになっている。

Theorem 6.4. χ は Hecke operator の eigenvalue の system,
 $f \in S_{m,0}(M_2(F)) \cap W(\chi, M_2(F), \overline{\mathbb{Q}})$, $f \neq 0$,
 $g \in S_{k,k}(B) \cap W(\chi, B, \overline{\mathbb{Q}})$, $g \neq 0$
 とする。 B の signature を (δ, δ') とする。 F_1 を F の l 次
 の totally real cyclic extension とし。 $B_1 = B \otimes_F F_1$,
 $\tau \in J_{F_1}$ に対し。

$\tilde{m}(\tau) = m(\tau|F)$, $\tilde{k}(\tau) = k(\tau|F)$, $\tilde{\kappa}(\tau) = \kappa(\tau|F)$
 とおく。 $\tilde{f} \in S_{\tilde{m},0}(M_2(F_1), \overline{\mathbb{Q}})$ を f の base change lift,
 $\tilde{f} \in W(\tilde{\chi}, M_2(F_1), \overline{\mathbb{Q}})$ とする。このとき, $\tilde{g} \in S_{\tilde{k},\tilde{\kappa}}(B_1)$,
 $\tilde{g} \in W(\tilde{\chi}, B_1, \overline{\mathbb{Q}})$ が存在して

$$\langle \tilde{g}, \tilde{g} \rangle \sim \langle g, g \rangle^l$$

 が, $\forall \tau \in \delta$ に対し $k(\tau) \geq 3$ ならば成立する。

証明は、 F の CM-extension K と $K_A^\times$ の Hecke character を
 とり、 [15], Theorem 5.7 を適用することを得られる。
 証明の最後の段階で、 [13], Theorem 1.1 にある CM-periods
 の関係を用いる。 $k(\tau) \geq 3$, $\forall \tau \in \delta$ の条件がつくのは、
 [15], Prop. 5.2 にある non-vanishing result を使うため。

この条件は多分不要である。また Theorem 6.4 はもう少し精密にできる。

χ が $S_{k,n}(B, \overline{\mathbb{Q}})$ に occur しなくても、適当に l -次拡大をとると ($l=2$ でよい) $\tilde{\chi}$ は $S_{k,n}(B_l, \overline{\mathbb{Q}})$ に occur するから

$$Q(\chi, \delta) = Q(\tilde{\chi}, \tilde{\delta})^{1/l} \quad (\text{in } \mathbb{C}^X / \overline{\mathbb{Q}}^X)$$

とおく。Theorems 6.3, 6.4 により、 $k(\tau) \geq 3$, $\forall \tau \in \delta$ の条件下で、 $Q(\chi, \delta) \in \mathbb{C}^X / \overline{\mathbb{Q}}^X$ は常に consistent に定義できることがわかる。(ここに $\tilde{\delta}$ は制限写像 $J_{F_l} \rightarrow J_F$ による δ の full inverse image である。)

References

- [1] I.N.Bernstein and A.V.Zelevinski, Representations of the group $GL(n, F)$, where F is a local non-archimedean field, Russian Math. Surveys 31(1976), 1–68.
- [2] D.Blasius, Appendix to Orloff: Critical values of certain tensor product L -functions, Inv. Math. 90(1987), 181–188.
- [3] A.Borel, Automorphic L -functions, Proc. Symposia Pure Math. 33(1979), part 2, 27–61.
- [4] H.Carayol, Sur les représentations l -adiques associées aux formes modulaires de Hilbert, Ann. Éc. Norm. Sup. 19(1986), 409–468.
- [5] P.Deligne, Les constantes des equations fonctionnelles des fonctions L , in Modular functions of one variable II, 501–597, Lecture notes in Math. 349, 1973, Springer Verlag.
- [6] P.Deligne, Valeurs de fonctions L et périodes d'intégrales, Proc. Symposia Pure Math. 33(1979), part 2, 313–346.
- [7] H.Jacquet and R.P.Langlands, Automorphic forms on $GL(2)$, Lecture notes in Math. 114, 1970, Springer Verlag.
- [8] R.P.Langlands, On the zeta-functions of some simple Shimura varieties, Can. J. Math. XXXI(1979), 1121–1216.
- [9] R.P.Langlands, Base change for $GL(2)$, Ann. of Math. Studies No. 96, Princeton University Press, 1980.
- [10] J-P.Serre, Abelian l -adic representations and elliptic curves, Benjamin, 1968.
- [11] G.Shimura, On the Fourier coefficients of modular forms of several variables, Göttingen Nachrichten (1975), Nr. 17, 1–8.
- [12] G.Shimura, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J. 45(1978), 637–679.
- [13] G.Shimura, The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, Ann. of Math. 111(1980), 313–375.
- [14] G.Shimura, On certain zeta functions attached to two Hilbert modular forms I, II, Ann. of Math. 114(1981), 127–164, 569–607.
- [15] G.Shimura, Algebraic relations between critical values of zeta functions and inner products, Amer. J. Math. 104(1983), 253–285.
- [16] G.Shimura, On the critical values of certain Dirichlet series and the periods of automorphic forms, Inv. Math. 94(1988), 245–305.

- [17] G.Shimura, On the fundamental periods of automorphic forms of arithmetic type, *Inv. Math.* 102(1990), 399–428.
- [18] R.Taylor, On Galois representations associated to Hilbert modular forms, *Inv. Math.* 98(1989), 265–280.

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