

On codim4 $\mathbb{Q}$ -Fano 3 Folds with Fano index 2

Kaori SUZUKI

Interactive Research Center of Science, Tokyo Institute of technology, Japan

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1 Main Results

We study Fano 3-folds with Fano index 2: that is, 3-folds X with Picard rank 1, $\mathbb{Q}$ -factorial terminal singularities and $-K_X = 2A$ for an ample Weil divisor A . We gave a first classification of all possible Hilbert series of such polarised varieties X, A and list 33 families that can be realised in codimension up to 4. See, Brown-Suzuki, *Fano 3-folds with divisible anticanonical class*, *Manuscripta Math.*123, 2007, pp.37–51.

We work over the complex number field $\mathbb{C}$.

2 The classification of $\mathbb{Q}$ -Fano 3-folds

The *Fano index* $f = f(X)$ of a Fano 3-fold X is the largest positive integer such that $-K_X = fA$ for some Weil divisor A . Equality of divisors denotes linear equivalence of some multiple. A Weil divisor A is called a *primitive ample divisor*.

History

1. Smooth Iskovskikh (1980)
2. Toric Borisov-Borisov (1993)
3. $f=1$
MMP $\longrightarrow$ Mori, Mukai, Takagi (many)
Graded Ring Method
 $\longrightarrow$ Altinok, Brown, Corti, Fletcher, Reid
4. $f \geq 2$ and codimension up to 3
Graded Ring Method $\longrightarrow$ Suzuki [Su1]

3 The graded ring method

A Fano 3-fold X with primitive ample divisor A has a graded ring $R(X, A) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nA))$. This graded ring is finitely generated. The *Hilbert series* of X, A is $P(X, A) = \sum_{n=0}^{\infty} \chi(\theta_X(nA))t^n$. Each $\chi(\theta_X(nA))$ is given by the singular Riemann-Roch formula ([YPG, Su2]) and then

Thm 1 ([Su2])

$$P_{X,A}(t) = \frac{1}{1-t} + \frac{t}{(1-t)^4} A^3 + \frac{t}{(1-t)^2} \frac{AC_2(X)}{12} + \sum_{p \in B} c_p(t)$$

where, $c_p(t)$ is equal to

$$\frac{1}{1-t^r} \left(\sum_{k=1}^{r-1} \left(\frac{-ik(r^2-1)}{12r} + \sum_{j=1}^{i_k-1} \frac{\overline{bj}(r-\overline{bj})}{2r} \right) t^k \right).$$

1. $\bar{x}$: the smallest residue mod r
2. $B := \left\{ \frac{1}{r_n}(a_n, r_n - a_n, 2) \right\} (n = 1, 2, 3 \dots)$.
3. i : the smallest positive integer which satisfies $nA \sim_{\mathbb{Q}} iK_X$.

A choice of minimal homogeneous generating set $x_0, \dots, x_N \in R(X, A)$ determines an embedding $X \hookrightarrow \mathbb{P}^N = \mathbb{P}(\alpha_0, \dots, \alpha_N)$ for some weighted projective space $\mathbb{P}^N$, where $x_i \in H^0(X, \mathcal{O}_X(\alpha_i A))$. With this embedding, we say that X has *codimension* $N - 3$.

Ex. 1 Codimension 1 hypersurface case

$$B = \left\{ \frac{1}{3}(1, 2, 2), \frac{1}{5}(1, 4, 2), \frac{1}{11}(3, 8, 2) \right\}$$

$$P(X, t) = \frac{1-t^{38}}{(1-t^2)(1-t^3)(1-t^5)(1-t^{11})(1-t^{19})}$$

$$\implies X_{38} \subset \mathbb{P}(2, 3, 5, 11, 19)$$

It is known that the embedding of codimension 2 [resp.3] is defined by the complete intersection [resp. five maximal Pfaffians of a 5×5 skew matrix]. 36 can be realized by (X, A) of codimension up to 3.

Next we consider the embedding of codimension 4. If $|-K_X|$ contains a K3 surface S , we may compare with those of [B] to the constructions we might make.

Consider $X \subset \mathbb{P}(2, 2, 3, 5, 5, 7, 12, 17)$ with $B = \{ \frac{1}{17}(5, 12, 2) \}$ as an example.

Ex.2 Codimension 4 case

$$\begin{array}{ccc} S & \subset & \mathbb{P}(2, 3, 5, 5, 7, 12, 17) \\ \downarrow \text{Type I projection from } 1/17(5, 12) & & \\ T_{10,12,14,15,17} & \subset & \mathbb{P}(2, 3, 5, 5, 7, 12) \\ \downarrow \text{Type I projection from } 1/12(5, 7) & & \\ Z_{10,12} & \subset & \mathbb{P}(2, 2, 3, 5, 5, 7) \end{array}$$

Using Type II unprojection, we construct from the example above

$$\mathbb{P}(2, 5, 12) \subset Y_{10,12,14,15,17} \subset \mathbb{P}(2, 2, 3, 5, 5, 7, 12)$$

as a linear subspace in the codimension 3 weak Fano 3-fold. By projection and unprojection again, we have $\mathbb{Q}$ -Fano 3-fold in codimension 4.

4 Database

We use computer programs by Magma at the final stage.

If there is a $\mathbb{Q}$ -Fano 3-fold X of Fano index 2, there must be integer solutions r_k, a_k of the following equations and inequalities:

Conditions

1. $\sum_{k=1}^m (r_k - \frac{1}{r_k}) < 24, (r_k, a_k) = 1$
and $(2, r_k) = 1$ for all k
2. $0 \leq A^3 \leq 4/5 AC_2$ (Kawamata Boundedness)
3. $\chi(\mathcal{O}_X(-A)) = 0$

Then we have whole possible candidates of B and their Hilbert series. As a result, we have the following list of $\mathbb{Q}$ -Fano 3-folds in codimension 4.

Remark Lists of examples available at

<http://malham.kent.ac.uk/grdb/FanoForm.php>

Question

The existence problem of two remaining candidates

1. $B = \{ 5 \times 1/3(1, 2, 2), 1/5(1, 4, 2) \}, A^3 = 1/15$
2. $B = \{ 3 \times 1/3(1, 2, 2), 1/5(2, 3, 2), 1/7(1, 6, 2) \}, A^3 = 1/35$

is still open, but both are expected not to exist.

Ambient $\mathbb{P}^7$	Basket B	A^3
$\mathbb{P}(1, 1, 1, 1, 1, 1, 1, 1)$	nonsingular	6
$\mathbb{P}(1, 1, 1, 1, 1, 2, 2, 3)$	$\frac{1}{3}$	10/3
$\mathbb{P}(1, 1, 1, 2, 2, 2, 2, 3, 3)$	$2 \times \frac{1}{3}$	5/3
$\mathbb{P}(1, 1, 1, 2, 2, 2, 3, 5)$	$\frac{1}{3}(2, 3, 2)$	8/5
$\mathbb{P}(1, 1, 1, 2, 2, 3, 4, 5)$	$\frac{1}{3}(1, 4, 2)$	7/5
$\mathbb{P}(1, 1, 2, 2, 2, 3, 3, 3)$	$3 \times \frac{1}{3}$	1
$\mathbb{P}(1, 1, 2, 2, 2, 3, 3, 5)$	$\frac{1}{3}, \frac{1}{3}(2, 3, 2)$	14/15
$\mathbb{P}(1, 1, 2, 2, 2, 3, 5, 7)$	$\frac{1}{3}(2, 5, 2)$	6/7
$\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 5)$	$\frac{1}{3}, \frac{1}{3}(1, 4, 2)$	11/15
$\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 7)$	$\frac{1}{3}(3, 4, 2)$	5/7
$\mathbb{P}(1, 1, 2, 3, 4, 5, 6, 7)$	$\frac{1}{3}(1, 6, 2)$	3/7
$\mathbb{P}(1, 2, 2, 3, 3, 3, 4, 5)$	$3 \times \frac{1}{3}, \frac{1}{3}(1, 4, 2)$	2/5
$\mathbb{P}(1, 2, 2, 3, 3, 3, 4, 7)$	$2 \times \frac{1}{3}, \frac{1}{3}(3, 4, 2)$	8/21
$\mathbb{P}(1, 2, 2, 3, 3, 4, 5, 5)$	$\frac{1}{3}, \frac{1}{3}(1, 4, 2), \frac{1}{3}(2, 3, 2)$	1/3
$\mathbb{P}(1, 2, 2, 3, 3, 4, 5, 7)$	$\frac{1}{3}(2, 3, 2), \frac{1}{3}(3, 4, 2)$	11/35
$\mathbb{P}(1, 2, 2, 3, 3, 5, 8, 11)$	$\frac{1}{11}(3, 8, 2)$	3/11
$\mathbb{P}(1, 2, 2, 3, 4, 5, 5, 7)$	$\frac{1}{3}(1, 4, 2), \frac{1}{3}(2, 5, 2)$	9/35
$\mathbb{P}(1, 2, 3, 4, 4, 5, 5, 5)$	$3 \times \frac{1}{3}(1, 4, 2)$	1/5
$\mathbb{P}(1, 2, 3, 4, 4, 5, 5, 9)$	$\frac{1}{3}(1, 4, 2), \frac{1}{3}(4, 5, 2)$	8/45
$\mathbb{P}(1, 2, 3, 4, 4, 5, 9, 13)$	$\frac{1}{13}(4, 9, 2)$	2/13
$\mathbb{P}(1, 2, 3, 4, 5, 5, 6, 7)$	$\frac{1}{3}, \frac{1}{3}(1, 4, 2), \frac{1}{3}(1, 6, 2)$	17/105
$\mathbb{P}(1, 2, 3, 4, 5, 5, 6, 11)$	$\frac{1}{3}, \frac{1}{11}(5, 6, 2)$	5/33
$\mathbb{P}(1, 2, 3, 4, 5, 6, 7, 7)$	$\frac{1}{7}(1, 6, 2), \frac{1}{7}(3, 4, 2)$	1/7
$\mathbb{P}(1, 2, 3, 5, 6, 7, 8, 9)$	$2 \times \frac{1}{3}, \frac{1}{9}(1, 10, 2)$	1/9
$\mathbb{P}(1, 2, 5, 7, 8, 9, 10, 11)$	$\frac{1}{3}(2, 3, 2), \frac{1}{11}(1, 10, 2)$	3/5
$\mathbb{P}(2, 2, 3, 3, 4, 5, 5, 5)$	$\frac{1}{3}, 3 \times \frac{1}{3}(2, 3, 2)$	2/15
$\mathbb{P}(2, 2, 3, 3, 4, 5, 5, 7)$	$2 \times \frac{1}{3}, \frac{1}{3}(2, 3, 2), \frac{1}{3}(2, 5, 2)$	13/105
$\mathbb{P}(2, 2, 3, 3, 4, 5, 7, 9)$	$3 \times \frac{1}{3}, \frac{1}{9}(2, 7, 2)$	1/9
$\mathbb{P}(2, 2, 3, 5, 5, 7, 12, 17)$	$\frac{1}{17}(5, 12, 2)$	1/17
$\mathbb{P}(2, 3, 3, 4, 5, 5, 6, 7)$	$5 \times \frac{1}{3}, \frac{1}{3}(1, 4, 2)$	1/15
$\mathbb{P}(2, 3, 3, 4, 5, 7, 10, 13)$	$2 \times \frac{1}{3}, \frac{1}{13}(3, 10, 2)$	
$\mathbb{P}(2, 3, 4, 4, 5, 5, 6, 7, 7)$	$\frac{1}{3}, \frac{1}{3}(1, 4, 2), \frac{1}{3}(2, 3, 2), \frac{1}{3}(3, 4, 2)$	1/21
$\mathbb{P}(2, 3, 4, 5, 5, 6, 7, 9)$	$2 \times \frac{1}{3}, \frac{1}{3}(2, 3, 2), \frac{1}{9}(4, 5, 2)$	2/45
$\mathbb{P}(2, 3, 5, 6, 7, 7, 8, 9)$	$3 \times \frac{1}{3}, \frac{1}{3}(2, 3, 2), \frac{1}{3}(1, 6, 2)$	1/35
$\mathbb{P}(2, 5, 5, 6, 7, 8, 9, 11)$	$2 \times \frac{1}{3}(2, 3, 2), \frac{1}{11}(5, 6, 2)$	1/55

Table 1: $\mathbb{Q}$ -Fano 3-folds in codimension 4

References

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- [YPG] M. Reid, *Young person's guide to canonical singularities*, *Algebraic Geometry*(1985), ed. S.Bloch, *Proc. of Sym. Pure Math.* 46, A.M.S. (1987), vol.1, 345–414.