

Anyons と自発的磁場生成

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I. Anyons と Chern-Simons 理論

Non-relativistic quantum mechanics of anyons and Chern-Simons gauge theory are “almost”, but not quite equivalent.[1] We show it by several steps.

a. エニオン

Under a clockwise or counter-clockwise interchange of two identical particles, say a and b , a Schrödinger wave function picks up a phase

$$\begin{aligned} P_{\pm}(a, b)\Phi(1, \dots, N) &= -e^{\pm i\theta_s}\Phi(1, \dots, N) \\ \Phi(1, \dots, N)|_{\mathbf{x}_a=\mathbf{x}_b} &= 0 \quad \text{for} \quad -e^{\pm i\theta_s} \neq 1 \end{aligned} \quad (1.1)$$

The second equation says that Pauli's principle holds for anyons except for the case of bosons. Free anyons are described by a Hamiltonian

$$H = \sum_{a=1}^N -\frac{\hbar^2}{2m} \nabla_a^2 \quad (1.2)$$

with the boundary condition (1.1)

b. フェルミオン表示

We perform a singular gauge transformation:

$$\begin{aligned} \Phi^f &= \Omega_{\text{singular}} \Phi \\ \Omega_{\text{singular}} &= \exp \left\{ i \frac{\theta_s}{\pi} \sum_{a < b} \tan^{-1} \frac{(x_a - x_b)_1}{(x_a - x_b)_2} \right\} \end{aligned} \quad (1.3)$$

Φ^f satisfies

$$\begin{aligned} i \frac{\partial}{\partial t} \Phi^f &= \sum_{a=1}^N -\frac{\hbar^2}{2m} \left\{ \nabla_a^j + i \frac{\theta_s}{\pi} \sum_{a \neq b} \epsilon^{jk} \frac{(\mathbf{x}_a - \mathbf{x}_b)_k}{(\mathbf{x}_a - \mathbf{x}_b)^2} \right\}^2 \Phi^f \\ P_{\pm}(a, b) \Phi^f(1, \dots, N) &= -\Phi^f(1, \dots, N) \end{aligned} \quad (1.4)$$

i.e. Φ^f is a wave function in the fermion representation.

Non-trivial Hamiltonian in (1.5) implies that *free anyons are not free*, and that they are equivalent to a fermion system with a particular interaction.

c. Aharonov-Bohm 効果

The above interaction is interpreted as an Aharonov-Bohm effect. Suppose that each particle carries both a charge e and a magnetic flux μ . Then a vector potential $\vec{A}$ at the location of particle a generated by other particles is

$$\frac{e}{\hbar c} A^j(\mathbf{x}_a) = -\frac{e\mu}{2\pi\hbar c} \sum_{b \neq a} \epsilon^{jk} \frac{(\mathbf{x}_a - \mathbf{x}_b)_k}{(\mathbf{x}_a - \mathbf{x}_b)^2}.$$

Upon identifying

$$\theta_s = \frac{e\mu}{2\hbar c} \quad (1.5)$$

one sees that the interaction is nothing but an Aharonov-Bohm (charge-flux) interaction. Hence one can phrase that

$$\text{anyon} = \begin{cases} \text{charge} & 1 \\ \text{flux} & 2\hbar c\theta_s \end{cases}. \quad (1.6)$$

However it's not exactly a Maxwell interaction. There is no charge-charge interaction.

d. Chern-Simons ゲージ理論

Consider a nonrelativistic matter field ψ coupled to Chern-Simons gauge fields:

$$\mathcal{L} = -\frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + i\psi^\dagger D_0 \psi - \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) \quad (1.7)$$

where $D_0 = \partial_0 + ia_0$ and $D_k = \partial_k - ia^k$. ($\hbar = c = 1$ hereafter.) Solving the Euler equation $-(\kappa/4\pi)\epsilon^{\mu\nu\rho} f_{\nu\rho} = j^\mu$ in the radiation gauge $\partial_k a^k = 0$, one can eliminate the Chern-Simons fields. The resultant theory of nonrelativistic (fermionic) matter field is described by

$$\begin{aligned} i\frac{\partial}{\partial t} \psi &= [\psi, H] \\ H[\psi, \psi^\dagger] &= \int d\mathbf{x} \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) \\ \text{where } a^j(x) &= -\frac{1}{\kappa} \int d\mathbf{y} \epsilon^{jk} \frac{(\mathbf{x} - \mathbf{y})_k}{(\mathbf{x} - \mathbf{y})^2} \cdot \psi^\dagger \psi(y) \end{aligned} \quad (1.8)$$

The connection to quantum mechanics of a finite number of particles is made by identifying the Schrödinger wave function in the fermion representation as a matrix element of a string of field operators:

$$\Phi^f(1, \dots, N) = \langle 0 | \psi(1) \cdots \psi(N) | \Psi_N \rangle \quad (1.9)$$

where $|0\rangle$ and $|\Psi_N\rangle$ are the vacuum and N -particle state.

The equation satisfied by the above wave function is

$$\begin{aligned}
 i \frac{\partial}{\partial t} \Phi^f &= \sum_{a=1}^N \langle 0 | \psi(1) \cdots [\psi(a), H] \cdots \psi(N) | \Psi_N \rangle \\
 &\vdots \\
 &= \sum_{a=1}^N -\frac{\hbar^2}{2m} \left\{ \nabla_a^j + \frac{i}{\kappa} \sum_{a \neq b} \epsilon^{jk} \frac{(\mathbf{x}_a - \mathbf{x}_b)_k}{(\mathbf{x}_a - \mathbf{x}_b)^2} \right\}^2 \Phi^f .
 \end{aligned} \tag{1.10}$$

This is exactly the same as eq. (1.4), provided that

$$\kappa = \frac{\theta_s}{\pi} . \tag{1.11}$$

e. ほとんど等価なこと

We have shown a series of equivalence

$$\begin{aligned}
 &\mathcal{L}^{\text{CS}}[a_\mu, \psi, \psi^\dagger] \quad (1.7) \\
 &\Updownarrow \quad a_\mu \text{ の除去} \\
 &H[\psi, \psi^\dagger] \quad (1.8) \\
 &\Updownarrow \quad \theta_s = \frac{\pi}{\kappa}, \quad \Phi^f = \langle 0 | \psi(1) \cdots \psi(N) | \Psi_N \rangle \quad (\text{Equiv}) \\
 &\text{Fermion 表示の QM} \quad (1.4) \\
 &\Updownarrow \quad \Omega_{\text{singular}} \\
 &\text{自由な anyon 系} \quad (1.1), (1.2)
 \end{aligned}$$

However, this does not necessarily mean that the role of Chern-Simons fields is just to change the statistics which is defined *mod* 2π .

To illustrate it, let us compare two models:

$$\begin{aligned}
 \mathcal{L}_0 &= i\psi^\dagger \dot{\psi} - \frac{1}{2m} |(\partial_k - ieA_{\text{ext}}^k)\psi|^2 \\
 \mathcal{L}_p &= -\frac{1}{4\pi p} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + i\psi^\dagger (\partial_0 + ia_0)\psi - \frac{1}{2m} |(\partial_k - ia^k - ieA_{\text{ext}}^k)\psi|^2 .
 \end{aligned} \tag{1.12}$$

In the latter, statistics phase is changed by $\theta_s = p\pi$ so that fermions are transformed into fermions for an even integer $p = 2n$. Are the two theories $\mathcal{L}_0$ and $\mathcal{L}_{2n}$ really equivalent? In the literature in QHE the equivalence is implicit.

Subtlety lies in the boundary condition. The corresponding Schrödinger wave functions in the fermion representation are related by

$$\Phi_{p=\pm 2n}^f(1, \dots, N) = \prod_{a < b} \left(\frac{z_a - z_b}{\bar{z}_a - \bar{z}_b} \right)^{\pm n} \cdot \Phi_0^f(1, \dots, N) \tag{1.13}$$

where $z_a = x_a + iy_a$. Although both Φ^f 's are single-valued, the transformation factor is singular under differentiation. Hence, even if Φ_0^f belongs to the Hilbert space in the $\mathcal{L}_0$ theory, the corresponding $\Phi_{\pm 2n}^f$ may not do so in the $\mathcal{L}_{\pm 2n}$ theory.

This subtlety is more profound when fermions are transformed into bosons, as in the case of $\mathcal{L}_p$ with an odd integer p . Although transformed particles are bosons, their wave functions must vanish when coordinates of two particles coincide. In other words resultant bosons have hard cores.

To conclude, Chern-Simons gauge theory is equivalent to naive anyon quantum mechanics up to the subtlety in the boundary condition on wave functions.

II. 磁場の自発的生成

In relativistic theory the existence of a bare Chern-Simons term can induce spontaneous magnetization.[2] To be more specific, we shall show that in a model described by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \sum \bar{\psi} \{ \gamma^\mu (i\partial_\mu + qA_\mu) - m \} \psi, \quad (2.1)$$

a magnetic field $B \neq 0$ is spontaneously generated if a certain condition is satisfied. It implies that the Lorentz invariance also is spontaneously broken.

There are two types of two-component Dirac fermions in 2+1 dimensions. They are characterized by "chirality" defined by $\eta = \frac{i}{2} \text{Tr} \gamma^0 \gamma^1 \gamma^2 = \pm 1$. Hence the fermion content is specified with $\{\eta_a, q_a, m_a\}$.

In the presence of a spontaneously generated uniform magnetic field B , there appear Landau levels whose energies are quantized as $E^2 = m^2 + (2n/l^2)$ where $n = 0, 1, 2, \dots$ and l is the magnetic length ($|qB| = 1/l^2$). There results asymmetry at the lowest Landau level ($n = 0$). For $qB > 0$ ($qB < 0$) there are only positive (negative) energy solutions at $n = 0$. This asymmetry is responsible for the non-vanishing charge density

$$\langle j^0 \rangle = \sum_a \frac{1}{2\pi} \eta_a q_a^2 \left(\nu_a - \frac{1}{2} \right) B \quad (2.1)$$

where ν_a is the filling factor at the lowest Landau level. $\nu_a = 0$ or 1 if the level is empty or completely filled, respectively.

A variational ground state is specified by B and $\{\nu_a = 0 \text{ or } 1\}$. We denote its energy density by $\mathcal{E}(B, \{\nu_a\})$, whereas the energy density of the perturbative vacuum is given by $\mathcal{E}_0$.

One of the Euler equations implies that

$$\kappa = \sum_a \frac{1}{2\pi} \eta_a q_a^2 \left(\nu_a - \frac{1}{2} \right) = \Pi_1(0) \quad \text{in order for } B \neq 0. \quad (2.2)$$

Here $\Pi_1(p)$ is one of the invariant functions appearing in the vacuum polarization tensor. $-\Pi_1(0)$ represents the induced Chern-Simons term. In other words, to have a non-vanishing B , the bare Chern-Simons term must be exactly cancelled by the induced Chern-Simons term. It can be shown that the condition (2.2) is related to the Nambu-Goldstone theorem associated with the spontaneous breaking of the Lorentz invariance.

As an example we consider a chirally symmetric model in which all fermions have $q_a = e$ and $m_a = 0$. We prepare $2N$ species of fermions, a half of which have $\eta=+$, and the other half of which have $\eta=-$. Further we suppose that the bare CS coefficient is $\kappa = Ne^2/2\pi$. We pick a variational ground state in which $\nu_a=1$ (0) for $\eta=+$ ($-$) so that the condition (2.2) is satisfied.

A detailed computation of quantum fluctuations shows that

$$\mathcal{E}(B, \{\nu_a\}) - \mathcal{E}_0 = -\frac{N_f e^2}{2\pi^3} \tan^{-1} \frac{4}{\pi} \cdot |eB| + O(|B|^{3/2}) \quad (2.3)$$

for small $|B|$. The sign in the linear term is negative. For large B the Maxwell energy $\frac{1}{2}B^2$ dominates. There develops a minimum at $B \neq 0$.

A major factor in reducing the energy density by $B \neq 0$ is in the shift in zero-point energies of photons. In perturbation theory a photon is originally topologically massive ($m_{\text{ph}} = \kappa$). In the $B \neq 0$ state the condition (2.2) implies that $m_{\text{ph}}(\vec{p}=0) = 0$. One can show that $m_{\text{ph}}(\vec{p})$ changes from 0 to κ with the crossover around $|\vec{p}| \sim l^{-1}$.

To summarize we have found a model in which a non-vanishing magnetic field is spontaneously generated. This is probably the first consistent renormalizable theory where the Lorentz invariance is spontaneously broken.

References

- [1] Y. Hosotani, *Int. J. Mod. Phys. B* **7** (1993) 2219 - 2323.
- [2] Y. Hosotani, "Spontaneously broken Lorentz invariance in three-dimensional gauge theories", UMN-TH-1211/93, to appear in *Phys. Lett. B*.