

Algebraic independence of the values of a certain function invariant under the action of the dihedral group

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1 Introduction

Liouville series $L(z) = \sum_{k=0}^{\infty} z^{k!}$ is known to have the following property:

Theorem 1.1 (Nishioka [3]). *Let $q_1, \dots, q_r$ be algebraic numbers with $0 < |q_i| < 1$ ($1 \leq i \leq r$). Then the values $L(q_1), \dots, L(q_r)$ are algebraically dependent if and only if there exist i, j with $1 \leq i < j \leq r$ such that q_i/q_j is a root of unity.*

In what follows, $\overline{\mathbb{Q}}^\times$ denotes the set of nonzero algebraic numbers. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and let μ_∞ be the multiplicative group consisting of all the roots of unity. Theorem 1.1 means that the points of $D \cap \overline{\mathbb{Q}}^\times$ at which $L(z)$ takes algebraically dependent values lie on the same orbit under the action of the group μ_∞ defined by $\mu_\infty \times D \ni (\zeta, z) \rightarrow \zeta z \in D$.

Although μ_∞ is an infinite group, in this paper we consider the functions of several variables with the following property: The points at which such a function takes the same value always lie on the identical orbit under the action of a finite group including non-commutative one, except trivial cases such as $f(X, Y) = g(X^N, Y^N)$ with the multiplicative action of the cyclic group $G = \langle (\zeta_N, \zeta_N) \rangle$, where ζ_N is a primitive N -th root of unity. Let $\{R_k\}_{k \geq 1}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \dots + c_n R_k \quad (k \geq 1), \quad (1)$$

where $n \geq 2$ and $c_1, \dots, c_n$ are nonnegative integers with $c_n \neq 0$. In this paper we assume that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for all $k \geq 1$ to exclude the trivial case above. The author [8] studied the two-variable function $E(x, q)$ defined by

$$E(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{xq^{R_l}}{1 - q^{R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1 + R_2 + \dots + R_k}}{(1 - q^{R_1})(1 - q^{R_2}) \dots (1 - q^{R_k})},$$

which may be regarded as an analogue of q -exponential function

$$E_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k q^{1+2+\dots+k}}{(1-q)(1-q^2) \dots (1-q^k)}$$

(cf. Gasper and Rahman [1]), if we replace k in the exponent of q in $E_q(x)$ with $\{R_k\}_{k \geq 1}$ defined by (1). Let

$$\Phi(X) = X^n - c_1 X^{n-1} - \dots - c_n.$$

The author proved the following:

Theorem 1.2 (Corollary 4 of [8]). *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(\pm 1) \neq 0$ and that the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for all $k \geq 1$. Then the infinite set of the values*

$$\left\{ E(x, q) \mid x, q \in \overline{\mathbb{Q}}^\times, |q| < 1 \right\}$$

is algebraically independent.

Remark 1.3. It is shown in Remark 2 of [5] that, if $\Phi(\pm 1) \neq 0$ and if the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, then the $\{R_k\}_{k \geq 1}$ defined by (1) is expressed as $R_k = c\rho^k + o(\rho^k)$ with $c > 0$ and $\rho > 1$, which guarantees the convergence of the series $E(x, q)$.

Remark 1.4. The condition that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for all $k \geq 1$ implies that the sequence $\{R_k\}_{k \geq 1}$ is not a geometric progression, which is a technically inevitable condition for the values $E(x, q)$ to be treated as special values of Mahler functions of several variables similarly to our poof in Section 3.

Example 1.5. Let $\{F_k\}_{k \geq 1}$ be the sequence of Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$ ($k \geq 1$). Since $\{F_k\}_{k \geq 1}$ satisfies all the conditions of Theorem 1.2, the infinite set of the values

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{F_1+F_2+\dots+F_k}}{(1-q^{F_1})(1-q^{F_2})\dots(1-q^{F_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| < 1 \right\}$$

is algebraically independent.

Since $R_k = c\rho^k + o(\rho^k)$ with $c > 0$ and $\rho > 1$ by Remak 1.3, the two-variable function $E(x, q)$ converges on the domain

$$(\mathbb{C} \times \{|q| < 1\}) \cup (\{|x| < 1\} \times \{|q| > 1\}) := \{(x, q) \in \mathbb{C}^2 \mid |q| < 1 \vee (|x| < 1 \wedge |q| > 1)\},$$

whereas a ‘balanced’ analogue

$$\Theta(x, q) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{xq^{R_l}}{1-q^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\dots(1-q^{2R_k})}$$

converges on the wider domain

$$\mathbb{C} \times \{|q| \neq 1\} := \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 1\}.$$

Indeed, if $q \neq 0$, then $\Theta(x, q)$ is invariant under the map

$$\sigma_1 : (x, q) \mapsto (-x, q^{-1}),$$

namely

$$\Theta(\sigma_1(x, q)) = \sum_{k=1}^{\infty} \frac{(-x)^k q^{-R_1-R_2-\dots-R_k}}{(1-q^{-2R_1})(1-q^{-2R_2})\dots(1-q^{-2R_k})} = \Theta(x, q)$$

and so $\Theta(x, q)$ converges on $\mathbb{C} \times \{|q| \neq 1\}$ by the similar reason to the convergence of $E(x, q)$ on $\mathbb{C} \times \{|q| < 1\}$.

Moreover, if $\{R_k\}_{k \geq 1}$ is a sequence of odd integers, then $\Theta(x, q)$ is invariant also under the maps

$$\sigma_2 : (x, q) \mapsto (-x, -q) \quad \text{and} \quad \sigma_3 : (x, q) \mapsto (x, -q^{-1}).$$

Since $\sigma_1 \circ \sigma_1 = \sigma_2 \circ \sigma_2 = \text{id}$ and $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \sigma_3$, we see that $G_4 = \{\text{id}, \sigma_1, \sigma_2, \sigma_3\}$ is Klein four-group. Therefore, $\Theta(x, q)$ can be regarded as a map defined on the set of orbits $(\mathbb{C} \times \{|q| \neq 0, 1\})/G_4$, where $\mathbb{C} \times \{|q| \neq 0, 1\} = \{(x, q) \in \mathbb{C}^2 \mid |q| \neq 0, 1\}$, namely the map

$$\tilde{\Theta} : (\mathbb{C} \times \{|q| \neq 0, 1\})/G_4 \longrightarrow \Theta(\mathbb{C} \times \{|q| \neq 0, 1\})$$

given by

$$\overline{(x, q)} \mapsto \Theta(x, q),$$

where $\overline{(x, q)}$ denotes the orbit represented by (x, q) , is a well-defined surjection. Hence the restriction to algebraic points

$$\tilde{\Theta} : \left((\mathbb{C} \times \{|q| \neq 0, 1\}) \cap (\overline{\mathbb{Q}}^\times)^2 \right) / G_4 \longrightarrow \Theta \left((\mathbb{C} \times \{|q| \neq 0, 1\}) \cap (\overline{\mathbb{Q}}^\times)^2 \right),$$

or equivalently

$$\tilde{\Theta} : \left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) / G_4 \longrightarrow \Theta \left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right)$$

is also a well-defined surjection. The author proved the following:

Theorem 1.6 (Theorem 1 of [9]). *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1). Suppose that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for all $k \geq 1$. Assume further that $\{R_k\}_{k \geq 1}$ is a sequence of odd integers. Then the infinite set of the values*

$$\tilde{\Theta} \left(\left(\overline{\mathbb{Q}}^\times \times (\overline{\mathbb{Q}}^\times \setminus \{|q| = 1\}) \right) / G_4 \right)$$

is algebraically independent.

Remark 1.7. It is easily seen that $\Phi(2) < 0$ implies that $\Phi(1) \neq 0$.

Corollary 1.8. *Let $\{R_k\}_{k \geq 1}$ be as in Theorem 1.6. Then the infinite set of the values*

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1-q^{2R_1})(1-q^{2R_2})\dots(1-q^{2R_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\}$$

is a ‘4-fold’ algebraically independent set, namely in this set each value appears 4 times and the infinite set consisting of all the distinct values of this set is algebraically independent.

Example 1.9. Let $\{P_k\}_{k \geq 1}$ be the sequence defined by $P_1 = P_2 = 1$ and $P_{k+2} = 2P_{k+1} + P_k$ ($k \geq 1$). Since $\{P_k\}_{k \geq 1}$ satisfies all the conditions of Theorem 1.6, the infinite set of the values

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{P_1+P_2+\dots+P_k}}{(1-q^{2P_1})(1-q^{2P_2})\dots(1-q^{2P_k})} \mid x, q \in \overline{\mathbb{Q}}^\times, |q| \neq 1 \right\}$$

is a ‘4-fold’ algebraically independent set.

2 Main results

The three-variable function

$$\mathcal{F}(x, a, q) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{xq^{R_l}}{1 - aq^{2R_l}} = \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1 - aq^{2R_1})(1 - aq^{2R_2}) \dots (1 - aq^{2R_k})} \quad (2)$$

converges on the domain

$$U := \{(x, a, q) \in \mathbb{C}^3 \mid |q| < 1, aq^{2R_k} \neq 1 \text{ for any } k \geq 1\} \\ \cup \{(x, a, q) \in \mathbb{C}^3 \mid a \neq 0, |q| > 1, aq^{2R_k} \neq 1 \text{ for any } k \geq 1\}.$$

If $a, q \neq 0$, then $\mathcal{F}(x, a, q)$ is invariant under the map

$$\tau : (x, a, q) \mapsto (-a^{-1}x, a^{-1}, q^{-1}), \quad (3)$$

namely

$$\mathcal{F}(\tau(x, a, q)) = \sum_{k=1}^{\infty} \frac{(-1)^k a^{-k} x^k q^{-R_1-R_2-\dots-R_k}}{(1 - a^{-1}q^{-2R_1})(1 - a^{-1}q^{-2R_2}) \dots (1 - a^{-1}q^{-2R_k})} = \mathcal{F}(x, a, q).$$

Here and hereafter, N denotes an integer greater than 1. Let ζ_N be a primitive N -th root of unity and let $\delta \in \{1, \dots, N-1\}$ be such that $\text{g.c.d.}(\delta, N) = 1$. If $R_k \equiv \delta \pmod{N}$ for all $k \geq 1$, then $\mathcal{F}(x, a, q)$ is invariant also under the map

$$\sigma : (x, a, q) \mapsto (\zeta_N^{-\delta}x, \zeta_N^{-2\delta}a, \zeta_N q), \quad (4)$$

namely $\mathcal{F}(\sigma(x, a, q)) = \mathcal{F}(x, a, q)$. In what follows, we denote by σ^i and $\sigma\tau$ the compositions of maps $\underbrace{\sigma \circ \dots \circ \sigma}_i$ and $\sigma \circ \tau$, respectively. By (3) and (4) we have $\sigma^N = \tau^2 = \text{id}$,

$$\sigma^2 : (x, a, q) \mapsto (\zeta_N^{-2\delta}x, \zeta_N^{-4\delta}a, \zeta_N^2 q), \dots, \sigma^{N-1} : (x, a, q) \mapsto (\zeta_N^{\delta}x, \zeta_N^{2\delta}a, \zeta_N^{-1} q), \\ \sigma\tau : (x, a, q) \mapsto (-\zeta_N^{-\delta}a^{-1}x, \zeta_N^{-2\delta}a^{-1}, \zeta_N q^{-1}), \\ \sigma^2\tau : (x, a, q) \mapsto (-\zeta_N^{-2\delta}a^{-1}x, \zeta_N^{-4\delta}a^{-1}, \zeta_N^2 q^{-1}),$$

and

$$\sigma^{N-1}\tau = \tau\sigma : (x, a, q) \mapsto (-\zeta_N^{\delta}a^{-1}x, \zeta_N^{2\delta}a^{-1}, \zeta_N^{-1} q^{-1}).$$

Hence $\{\text{id}, \sigma, \sigma^2, \dots, \sigma^{N-1}, \tau, \sigma\tau, \sigma^2\tau, \dots, \sigma^{N-1}\tau\}$ is isomorphic to the dihedral group D_{2N} of order $2N$. Since the equation $aq^{2R_k} = 1$ is invariant under σ and τ , the dihedral group D_{2N} acts on the set

$$U^* := \{(x, a, q) \in \mathbb{C}^3 \mid aq^{2R_k} \neq 1 \text{ for any } k \geq 1, a \neq 0, |q| \neq 0, 1\}.$$

Therefore, $\mathcal{F}(x, a, q)$ can be regarded as a map defined on the set of orbits U^*/D_{2N} , namely the map

$$\tilde{\mathcal{F}} : U^*/D_{2N} \longrightarrow \mathcal{F}(U^*)$$

given by

$$\overline{(x, a, q)} \mapsto \mathcal{F}(x, a, q),$$

where $\overline{(x, a, q)}$ denotes the orbit represented by (x, a, q) , is a well-defined surjection. In other words, the image of the function $\mathcal{F}(x, a, q)$ is a $2N$ -fold cover. Hence the restriction to algebraic points

$$\tilde{\mathcal{F}} : U_{\mathcal{A}}/D_{2N} \longrightarrow \mathcal{F}(U_{\mathcal{A}}),$$

is also a well-defined surjection, where

$$U_{\mathcal{A}} = U^* \cap \left(\overline{\mathbb{Q}^\times}\right)^3 = \{(x, a, q) \mid x, a, q \in \overline{\mathbb{Q}^\times}, aq^{2R_k} \neq 1 \text{ for any } k \geq 1, |q| \neq 1\}.$$

In this paper we prove the following:

Main Theorem 2.1. *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1) and let $\delta \in \{1, \dots, N-1\}$ be such that $\text{g.c.d.}(\delta, N) = 1$. Suppose that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Assume that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ and $R_k \equiv \delta \pmod{N}$ for all $k \geq 1$. Then the infinite set of the values $\tilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent if and only if there exist distinct orbits $\overline{(x_1, a_1, q_1)}, \dots, \overline{(x_s, a_s, q_s)} \in U_{\mathcal{A}}/D_{2N}$, algebraic numbers $c_1, \dots, c_s$, not all zero, and a sufficiently large positive integer u such that, for any $k \geq u$,*

$$a_1 q_1^{2R_k} = a_i q_i^{2R_k} \quad (2 \leq i \leq s) \quad \text{and} \quad \sum_{i=1}^s c_i x_i^{k-u+1} q_i^{R_u + \dots + R_k} = 0. \quad (5)$$

Remark 2.2. The equations (5) depend only on the orbits $\overline{(x_i, a_i, q_i)}$ but not on the choice of the representatives (x_i, a_i, q_i) , namely, under the action of the dihedral group D_{2N} , the equations (5) are invariant.

Main Theorem 2.3. *Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1), let N be an odd integer, and let $\delta \in \{1, \dots, N-1\}$ be such that $\text{g.c.d.}(\delta, N) = 1$. Suppose that $R_k \equiv \delta \pmod{N}$ and $\text{g.c.d.}(R_{k+1} - R_k, R_{k+2} - R_{k+1}, \dots, R_{k+n} - R_{k+n-1}) = N$ for all $k \geq 1$. Assume that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity. Then the infinite set of the values $\tilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent if and only if there exist distinct orbits $\overline{(x_1, a, q)}, \dots, \overline{(x_s, a, q)}, \overline{(x_{s+1}, a, -q)}, \dots, \overline{(x_{s+t}, a, -q)} \in U_{\mathcal{A}}/D_{2N}$, algebraic numbers $c_1, \dots, c_{s+t}$, not all zero, and a sufficiently large positive integer u such that, for any $k \geq u$,*

$$\sum_{i=1}^s c_i x_i^{k-u+1} + \sum_{i=s+1}^{s+t} c_i (-1)^{R_u + \dots + R_k} x_i^{k-u+1} = 0.$$

Remark 2.4. The condition of Main Theorem 2.3 that $R_k \equiv \delta \pmod{N}$ and $\text{g.c.d.}(R_{k+1} - R_k, R_{k+2} - R_{k+1}, \dots, R_{k+n} - R_{k+n-1}) = N$ for all $k \geq 1$ implies that $\text{g.c.d.}(R_k, R_{k+1}, \dots, R_{k+n-1}) = 1$ for all $k \geq 1$.

Example 2.5. Let $\{R_k\}_{k \geq 1}$ be the sequence defined by $R_1 = R_2 = 1$ and $R_{k+2} = NR_{k+1} + R_k$ ($k \geq 1$), where $N \geq 3$ is an odd integer. Then $\{R_k\}_{k \geq 1}$ satisfies all the conditions of Main Theorem 2.3 and the infinite set consisting of the distinct values of

$$\left\{ \sum_{k=1}^{\infty} \frac{x^k q^{R_1+R_2+\dots+R_k}}{(1-aq^{2R_1})(1-aq^{2R_2})\dots(1-aq^{2R_k})} \mid x, a, q \in \overline{\mathbb{Q}}^\times, aq^{2R_k} \neq 1 \ (k \geq 1), |q| \neq 1 \right\}$$

is algebraically dependent. In fact, there exists the linear relation

$$\mathcal{F}(x, a, q) - 2\zeta_3^2 \mathcal{F}(\zeta_3 x, a, q) - 2\zeta_3 \mathcal{F}(\zeta_3^2 x, a, q) - 3\mathcal{F}(x, a, -q) = 0$$

holds for $\mathcal{F}(x, a, q)$ of the form (2). Note that $\overline{(x, a, q)}$, $\overline{(\zeta_3 x, a, q)}$, $\overline{(\zeta_3^2 x, a, q)}$, and $\overline{(x, a, -q)}$ are the distinct orbits of U_A/D_{2N} , since $\mathcal{F}(x, a, q)$, $\mathcal{F}(\zeta_3 x, a, q)$, $\mathcal{F}(\zeta_3^2 x, a, q)$, and $\mathcal{F}(x, a, -q)$ are distinct as series in x, a , and q . Moreover, by Main Theorem 2.3 we have

$$\text{trans. deg}_{\mathbb{Q}} \mathbb{Q}(\mathcal{F}(x, a, q), \mathcal{F}(\zeta_3 x, a, q), \mathcal{F}(\zeta_3^2 x, a, q), \mathcal{F}(x, a, -q)) = 3,$$

or more precisely, any three of the numbers $\mathcal{F}(x, a, q)$, $\mathcal{F}(\zeta_3 x, a, q)$, $\mathcal{F}(\zeta_3^2 x, a, q)$, and $\mathcal{F}(x, a, -q)$ are algebraically independent.

3 Proof of the main theorems

Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries, where n is as in (1).

We define a multiplicative transformation $\Omega: \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\Omega \mathbf{z} = \left(\prod_{j=1}^n z_j^{\omega_{1j}}, \prod_{j=1}^n z_j^{\omega_{2j}}, \dots, \prod_{j=1}^n z_j^{\omega_{nj}} \right) \quad (6)$$

for any $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then $\Omega(\Omega^k \mathbf{z}) = \Omega^{k+1} \mathbf{z}$ ($k = 0, 1, 2, \dots$). Let $\{R_k\}_{k \geq 1}$ be a linear recurrence satisfying (1) and define a monomial

$$M(\mathbf{z}) = z_1^{R_n} \dots z_n^{R_1}. \quad (7)$$

Let

$$\Omega_1 = \begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \dots & \dots & 0 \end{pmatrix}. \quad (8)$$

It follows from (1), (6), and (7) that

$$M(\Omega_1^k \mathbf{z}) = z_1^{R_{k+n}} \dots z_n^{R_{k+1}} \quad (k \geq 0). \quad (9)$$

Sketch of the proof of Main Theorem 2.1. First we prove that, if there exist distinct orbits $\overline{(x_1, a_1, q_1)}, \dots, \overline{(x_s, a_s, q_s)} \in U_A/D_{2N}$, algebraic numbers $c_1, \dots, c_s$, not all zero, and a sufficiently large positive integer u such that the equations (5) hold for any $k \geq u$, then the values $\mathcal{F}(x_1, a_1, q_1), \dots, \mathcal{F}(x_s, a_s, q_s)$ are algebraically dependent, which means that the infinite set of the values $\widetilde{\mathcal{F}}(U_A/D_{2N})$ is algebraically dependent. Indeed, using (5) and letting $a_i q_i^{2R_k} = \eta_k$ ($1 \leq i \leq s$, $k \geq u$), we have

$$\sum_{i=1}^s c_i \prod_{k=1}^{u-1} \frac{1 - a_i q_i^{2R_k}}{x_i q_i^{R_k}} \left(\mathcal{F}(x_i, a_i, q_i) - \sum_{k=1}^{u-1} \prod_{l=1}^k \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}} \right)$$

$$\begin{aligned}
&= \sum_{i=1}^s c_i \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}} \\
&= \sum_{k=u}^{\infty} \sum_{i=1}^s c_i \frac{\prod_{l=u}^k x_i q_i^{R_l}}{\prod_{l=u}^k (1 - \eta_l)} = \sum_{k=u}^{\infty} \frac{\sum_{i=1}^s c_i x_i^{k-u+1} q_i^{R_u+\dots+R_k}}{\prod_{l=u}^k (1 - \eta_l)} = 0.
\end{aligned}$$

Next assume that the infinite set of the values $\tilde{\mathcal{F}}(U_{\mathcal{A}}/D_{2N})$ is algebraically dependent. Then there exist distinct orbits $(x_1, a_1, q_1), (x_2, a_2, q_2), \dots, (x_r, a_r, q_r) \in U_{\mathcal{A}}/D_{2N}$ such that the values

$$\theta_i := \mathcal{F}(x_i, a_i, q_i) = \sum_{k=1}^{\infty} \prod_{l=1}^k \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}} \quad (i = 1, \dots, r)$$

are algebraically dependent. Then there exist multiplicatively independent algebraic numbers $\beta_1, \dots, \beta_t$ with $0 < |\beta_j| < 1$ ($1 \leq j \leq t$) and a primitive N_1 -th root of unity ζ , which may be different from the ζ_N defined in Section 2, such that

$$q_i = \zeta^{m_i} \prod_{j=1}^t \beta_j^{e_{ij}} \quad (1 \leq i \leq r), \quad (10)$$

where $m_1, \dots, m_r$ are integers with $0 \leq m_i \leq N_1 - 1$ and e_{ij} ($1 \leq i \leq r$, $1 \leq j \leq t$) are nonnegative integers not all zero (cf. Loxton and van der Poorten [2], Nishioka [4]). We can choose a positive integer κ and a sufficiently large integer u such that $R_{k+\kappa} \equiv R_k \pmod{N_1}$ for any $k \geq u+1$, where u will be determined later. Let $p = \kappa N_1$. If necessary, we replace p with its multiple such that all the entries of Ω_1^p are positive, where Ω_1 is the matrix defined by (8). In fact, we can choose such a p . (For the proof see [5].) Let $y_{j\lambda}$ ($1 \leq j \leq t$, $1 \leq \lambda \leq n$) be variables and let $\mathbf{y}_j = (y_{j1}, \dots, y_{jn})$ ($1 \leq j \leq t$), $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_t)$. Define

$$f_i(\mathbf{y}) = \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \mathbf{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \mathbf{y}_j)^{e_{ij}} \right)^2} \quad (1 \leq i \leq r),$$

where $M(\mathbf{z})$ is defined by (7). Letting $\beta = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots, \beta_t, \underbrace{1, \dots, 1}_{n-1})$, we see by (9)

and (10) that

$$f_i(\beta) = \sum_{k=u}^{\infty} \prod_{l=u}^k \frac{x_i q_i^{R_{l+1}}}{1 - a_i q_i^{2R_{l+1}}} = \sum_{k=u+1}^{\infty} \prod_{l=u+1}^k \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}}$$

and so

$$\theta_i = \left(\prod_{k=1}^u \frac{x_i q_i^{R_k}}{1 - a_i q_i^{2R_k}} \right) f_i(\beta) + \sum_{k=1}^u \prod_{l=1}^k \frac{x_i q_i^{R_l}}{1 - a_i q_i^{2R_l}}.$$

Since $\theta_1, \dots, \theta_r$ are algebraically dependent, so are $f_i(\beta)$ ($1 \leq i \leq r$). Let

$$\Omega_2 = \text{diag}(\underbrace{\Omega_1^p, \dots, \Omega_1^p}_t).$$

Then, using the fact that $R_{k+p} \equiv R_k \pmod{N_1}$ for any $k \geq u+1$, we see that each $f_i(\mathbf{y})$ satisfies the functional equation

$$\begin{aligned} f_i(\mathbf{y}) &= \left(\prod_{k=u}^{p+u-1} \frac{x_i \zeta^{m_i R_{k+1}} \prod_{j=1}^t M(\Omega_1^k \mathbf{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{k+1}} \prod_{j=1}^t M(\Omega_1^k \mathbf{y}_j)^{e_{ij}} \right)^2} \right) f_i(\Omega_2 \mathbf{y}) \\ &\quad + \sum_{k=u}^{p+u-1} \prod_{l=u}^k \frac{x_i \zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \mathbf{y}_j)^{e_{ij}}}{1 - a_i \left(\zeta^{m_i R_{l+1}} \prod_{j=1}^t M(\Omega_1^l \mathbf{y}_j)^{e_{ij}} \right)^2}, \end{aligned}$$

where $\Omega_2 \mathbf{y} = (\Omega_1^p \mathbf{y}_1, \dots, \Omega_1^p \mathbf{y}_t)$. Let $D = |\det(\Omega_1 - E)|$, where E is the identity matrix. Put $M(D(\Omega_1 - E)^{-1} \Omega_1^k \mathbf{z}) = \mathbf{z}^{\mathbf{s}^{(k)}}$. By the condition that $\Phi(-1) \neq 0$, $\Phi(2) < 0$, and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity and by Remark 1.7, D is a positive integer and there exists a sufficiently large integer u such that we have the componentwise inequality among the vectors with integral components

$$D(R_{u+n}, \dots, R_{u+1}) > \mathbf{s}(u) > \mathbf{0}. \quad (11)$$

Changing the indices i if necessary, we may assume by Nishioka [4, p. 110, Theorem 3.5 and p. 115, Theorem 3.6.4], by [5, Lemma 4 and Proof of Theorem 2], by [6, Theorem 2], and by Remark 1.7 that there exist a positive integer $s \leq r$, nonzero algebraic numbers $\eta_u, \dots, \eta_{p+u-1}, \xi$, and nonnegative integers $e_1, \dots, e_t$ such that for any $i \in \{1, \dots, s\}$ we have $a_i \zeta^{2m_i R_{k+1}} = \eta_k$ ($u \leq k \leq p+u-1$), $(e_{i1}, \dots, e_{it}) = (e_1, \dots, e_t) \neq \mathbf{0}$, and $x_i^p \zeta^{m_i(R_{u+1}+\dots+R_{p+u})} = \xi$, and there exists a $G(\mathbf{z}) \in \overline{\mathbb{Q}}[z_1, \dots, z_n]$ satisfying the functional equation

$$\begin{aligned} G(\mathbf{z}) &= \xi \left(\prod_{k=u}^{p+u-1} \frac{M(\Omega_1^k \mathbf{z})^{2D\ell}}{1 - \eta_k M(\Omega_1^k \mathbf{z})^{2D\ell}} \right) G(\Omega_1^p \mathbf{z}) \\ &\quad + \mathbf{z}^{-\ell \mathbf{s}(u)} \sum_{k=u}^{p+u-1} \left(\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\dots+R_{k+1})} \right) \prod_{l=u}^k \frac{M(\Omega_1^l \mathbf{z})^{D\ell}}{1 - \eta_l M(\Omega_1^l \mathbf{z})^{2D\ell}}, \end{aligned}$$

where $\ell = \sum_{j=1}^t e_j \nu^j$ with ν a sufficiently large positive integer, $\mathbf{s}(u)$ is as in (11), and $c_1, \dots, c_s$ are algebraic numbers not all zero. Letting $G(\mathbf{z}) = A(\mathbf{z})/B(\mathbf{z})$, where $A(\mathbf{z})$ and $B(\mathbf{z})$ are coprime polynomials in $\overline{\mathbb{Q}}[z_1, \dots, z_n]$ with $B \not\equiv 0$, we have

$$\begin{aligned} &A(\mathbf{z})B(\Omega_1^p \mathbf{z})\mathbf{z}^{\ell \mathbf{s}(u)} \prod_{k=u}^{p+u-1} (1 - \eta_k M(\Omega_1^k \mathbf{z})^{2D\ell}) \\ &= \xi A(\Omega_1^p \mathbf{z})B(\mathbf{z})\mathbf{z}^{\ell \mathbf{s}(u)} \prod_{k=u}^{p+u-1} M(\Omega_1^k \mathbf{z})^{2D\ell} \\ &\quad + \sum_{k=u}^{p+u-1} \left(\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\dots+R_{k+1})} \right) B(\mathbf{z})B(\Omega_1^p \mathbf{z}) \prod_{l=u}^k M(\Omega_1^l \mathbf{z})^{D\ell} \\ &\quad \times \prod_{l'=k+1}^{p+u-1} (1 - \eta_{l'} M(\Omega_1^{l'} \mathbf{z})^{2D\ell}). \end{aligned} \quad (12)$$

We can put the greatest common divisor of $A(\Omega_1^p \mathbf{z})$ and $B(\Omega_1^p \mathbf{z})$ as $\mathbf{z}^{I(p)}$, where $I(p)$ is an n -dimensional vector with nonnegative integer components, by [4, Lemma 3.2.3]. Then $B(\Omega_1^p \mathbf{z})$ divides $B(\mathbf{z})\mathbf{z}^{\ell \mathbf{s}(u)+I(p)} \prod_{k=u}^{p+u-1} M(\Omega_1^k \mathbf{z})^{2D\ell}$ by (12). Therefore $B(\mathbf{z})$ is a monomial in $z_1, \dots, z_n$ by [6, Lemma 12] with Remark 1.7.

By (9) and (11) we see that $z_1 \cdots z_n \mathbf{z}^{\ell \mathbf{s}(u)}$ divides $M(\Omega_1^u \mathbf{z})^{D\ell}$. Since p and u are independent, the right-hand side of (12) is divisible by $z_1 \cdots z_n \mathbf{z}^{\ell \mathbf{s}(u)} B(\Omega_1^p \mathbf{z})$ for sufficiently large u and thus $A(\mathbf{z})$ is divisible by $z_1 \cdots z_n$. Since $A(\mathbf{z})$ and $B(\mathbf{z})$ are coprime, $B(\mathbf{z}) \in \overline{\mathbb{Q}}^\times$. If $A(\mathbf{z}) \notin \overline{\mathbb{Q}}$, then by [7, Lemma 6] and the fact that all the entries of Ω_1^p are positive, $\deg_{\mathbf{z}} A(\Omega_1^p \mathbf{z}) > \deg_{\mathbf{z}} A(\mathbf{z})$, which is a contradiction by comparing the total degrees of both sides of (12). Hence $A(\mathbf{z}) \in \overline{\mathbb{Q}}$ and so $A(\mathbf{z}) = 0$ by comparing the coefficients of $\mathbf{z}^{\ell \mathbf{s}(u)}$, the term of the smallest degree among the terms appearing in (12). Then again by (12), we see that $\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\cdots+R_{k+1})} = 0$ ($u \leq k \leq p+u-1$). Since $R_{k+p} \equiv R_k \pmod{N_1}$ for any $k \geq u+1$, we have $a_i \zeta^{2m_i R_{k+1}} = a_{i'} \zeta^{2m_{i'} R_{k+1}}$ ($k \geq u$) for any $i, i' \in \{1, \dots, s\}$. Noting that $p = \kappa N_1$ with $R_{k+\kappa} \equiv R_k \pmod{N_1}$ for any $k \geq u+1$, we see that $\sum_{l=k}^{k+p-1} R_{l+1} \equiv 0 \pmod{N_1}$ for any $k \geq u$ and so $\sum_{l=u}^k R_{l+1} \equiv \sum_{l=u}^{k+p\kappa} R_{l+1} \pmod{N_1}$ for any k with $u \leq k \leq p+u-1$ and for all $k' \geq 0$. Hence $\sum_{i=1}^s c_i x_i^{k-u+1} \zeta^{m_i(R_{u+1}+\cdots+R_{k+1})} = 0$ ($k \geq u$). By (10) with $(e_{i1}, \dots, e_{it}) = (e_{i'1}, \dots, e_{i't})$ we get $a_i q_i^{2R_{k+1}} = a_{i'} q_{i'}^{2R_{k+1}}$ ($k \geq u$) for any $i, i' \in \{1, \dots, s\}$ and $\sum_{i=1}^s c_i x_i^{k-u+1} q_i^{R_{u+1}+\cdots+R_{k+1}} = 0$ ($k \geq u$). This completes the proof of the theorem. $\square$

Sketch of the proof of Main Theorem 2.3. Choosing a suitable complete set of representatives of the orbits U_A/D_{2N} , we can prove the theorem. $\square$

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