

A NEW CRITERION FOR CLOSE-TO-CONVEXITY

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ABSTRACT. R.Singh and S.Singh [2] obtained a new criterion for close-to-convexity. The aim of this paper is to obtain a generalized result of them.

1. Introduction.

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = 1, 2, 3, \dots)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p -valently starlike if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

A function $f(z)$ in $A(p)$ is said to be p -valently convex if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } E.$$

Further, a function $f(z)$ in $A(p)$ is said to be p -valently close-to-convex if there exists a p -valently starlike function $g(z) \in A(p)$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \text{in } E.$$

It is well known that a p -valently close-to-convex function is p -valent in E . [4.Theorem 1]

2. Preliminary.

Lemma 1 Let $f(z) \in A(p)$ and if there exists a $(p-k+1)$ -valently starlike function $g(z) = \sum_{n=p-k+1}^{\infty} b_n z^n$, ($b_{p-k+1} \neq 0$) that satisfies

$$\operatorname{Re} \frac{zf^{(k)}(z)}{g(z)} > 0 \quad \text{in } E,$$

then $f(z)$ is p -valently close-to-convex and therefore, $f(z)$ is p -valent in E .

We can obtain the above lemma from [3.Theorem 8.].

3. Main result.

Theorem 1 Let $f(z) \in A(p)$ and suppose

$$\operatorname{Re} \left[\frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{f'(z)}{z^{p-2}} \right] < \alpha \quad \text{in } E$$

for some α ($1 < \alpha$), then $f(z)$ is p -valently close-to-convex in E .

Proof. Since

$$\alpha - \frac{1 + (1 - 2\alpha)z}{1 - z} - \frac{f'(z)}{z^{p-2}} = (\alpha - 1)(1 + c_1z + \dots),$$

using Herglotz representation formula [1], we have

$$\alpha - \frac{1 + (1 - 2\alpha)z}{1 - z} - \frac{f'(z)}{z^{p-2}} = (\alpha - 1) \int_0^{2\pi} \frac{1 + ze^{it}}{1 - ze^{it}} d\mu(t), \quad (1)$$

where $\mu(t)$ is the probability measure on $[0, 2\pi]$, satisfying

$$\int_0^{2\pi} d\mu(t) = 1.$$

From (1) we get

$$(1 - z)^2 \frac{f'(z)}{z^{p-1}} = 2(\alpha - 1) \int_0^{2\pi} \frac{(1 - z)(1 - e^{it})}{1 - ze^{it}} d\mu(t). \quad (2)$$

Now letting

$$K(z, t) = \frac{(1 - z)(1 - e^{it})}{1 - ze^{it}}, \quad (z \in E; t \in [0, 2\pi])$$

then we verify that

$$\begin{aligned} \operatorname{Re} K(z, t) &= \operatorname{Re} \left[\frac{(1 - re^{i\theta})(1 - e^{it})}{1 - re^{i(\theta+t)}} \right] \\ &= \frac{(1 - \cos t)(1 - r^2)}{1 - 2r \cos(\theta + t) + r^2} \\ &\geq 0 \quad (z = re^{i\theta}) \end{aligned}$$

for all z in E and for all t in $[0, 2\pi]$.

From (2) we obtain that

$$\operatorname{Re} \left[(1 - z)^2 \frac{f'(z)}{z^{p-1}} \right] \geq 0 \quad \text{in } E.$$

Letting

$$g(z) = \frac{z^p}{(1 - z)^2},$$

we see that $g(z)$ is a p -valently starlike function.

Therefore, from Lemma 1 $f(z)$ is p -valently close-to-convex in E .

This completes our proof.

Corollary 1 Let $f(z) \in A(1)$ and suppose

$$\operatorname{Re} \left[\frac{1 + (1 - 2\alpha)z}{1 - z} + zf'(z) \right] < \alpha \quad \text{in } E$$

for some α ($1 < \alpha$), then $f(z)$ is close-to-convex in E .

Theorem 2 Let $f(z) \in A(p)$ and suppose

$$\operatorname{Re} \left[\frac{1 + (1 - 2\alpha)z}{1 - z} - \frac{f'(z)}{z^{p-2}} \right] > \alpha \quad \text{in } E$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ is p -valently close-to-convex in E .

Proof. Applying the same method as the proof of Theorem 1, we have

$$(1 - z)^2 \frac{f'(z)}{z^{p-1}} = 2(1 - \alpha) \int_0^{2\pi} \frac{(1 - z)(1 - e^{it})}{1 - ze^{it}} d\mu(t).$$

From the same reason as the proof of Theorem 1, we have

$$\operatorname{Re} \left[(1 - z)^2 \frac{f'(z)}{z^{p-1}} \right] > 0 \quad \text{in } E.$$

This shows that $f(z)$ is p -valently close-to-convex in E .

Corollary 2 ([2]) Let $f(z) \in A(1)$ and suppose

$$\operatorname{Re} \left[\frac{1 + (1 - 2\alpha)z}{1 - z} - zf'(z) \right] > \alpha \quad \text{in } E$$

for some α ($0 \leq \alpha < 1$), then $f(z)$ is close-to-convex in E .

Theorem 3 Let $f(z) \in A(1)$, $g(z)$ be analytic in E with $g(0) = 1$, $\operatorname{Reg}'(0) > -2(1 - \alpha)$ and $\operatorname{Reg}(z) \leq \alpha$ ($z \in E$) for some α ($\alpha > 1$).

If

$$\operatorname{Re}(g(z) - zf'(z)) < \alpha \quad \text{in } E,$$

then $f(z)$ is close-to-convex.

Proof. Applying the same method as the proof of Theorem 1, we have

$$(1 - z)^2 f'(z) = \int_0^{2\pi} (1 - z)^2 \left[\frac{g(z) - \alpha}{z} - \frac{(1 - \alpha)(1 + ze^{it})}{z(1 - ze^{it})} \right] d\mu(t).$$

Let

$$K(z, t) = (1 - z)^2 \left[\frac{g(z) - \alpha}{z} - \frac{(1 - \alpha)(1 + ze^{it})}{z(1 - ze^{it})} \right],$$

where $z \in E, t \in [0, 2\pi], \alpha > 1$.

Since

$$K(z, t) = (1 - z)^2 \left[\frac{g(z) - 1}{z} - \frac{2(1 - \alpha)e^{it}}{1 - ze^{it}} \right],$$

$$\begin{aligned} \operatorname{Re} K(0, t) &= \operatorname{Re} [g'(0) - 2(1 - \alpha)e^{it}] \\ &\geq \operatorname{Re} g'(0) + 2(1 - \alpha) \\ &> 0 \end{aligned}$$

Further, since

$$K(z, t) = \frac{(1 - z)^2}{z} \left[g(z) - 1 - \frac{2(1 - \alpha)ze^{it}}{1 - ze^{it}} \right],$$

$$\begin{aligned} \operatorname{Re} K(e^{i\theta}, t) &= 2(\cos \theta - 1) \operatorname{Re} \left[g(e^{i\theta}) - 1 - \frac{2(1 - \alpha)e^{i(t+\theta)}}{1 - e^{i(t+\theta)}} \right] \\ &= 2(\cos \theta - 1) [\operatorname{Re} g(e^{i\theta}) - 1 + (1 - \alpha)] \\ &= 2(\cos \theta - 1) [\operatorname{Re} g(e^{i\theta}) - \alpha] \\ &\geq 0 \end{aligned}$$

Therefore,

$$\operatorname{Re} [(1 - z)^2 f'(z)] > 0 \quad \text{in } E,$$

so $f(z)$ is close-to-convex in E .

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References

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