

SOLVABILITY OF A CLASS OF DIFFERENTIAL OPERATORS IN $\mathcal{CO}$

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Introduction

Let V and Σ be an involutive submanifold and a lagrangian submanifold of $\sqrt{-1}T^*\mathbb{R}^n$ respectively given as follows:

$$V = \{(x; i\xi) \in \sqrt{-1}T^*\mathbb{R}^n; \xi_1 = \cdots = \xi_{n-1} = 0\}, \quad (1)$$

$$\Sigma = \{(x; i\xi) \in V; x_n = 0\}. \quad (2)$$

In [3], Grigis-Schapira-Sjöstrand obtained a result on the propagation of micro-analyticity of solutions along Σ for transversally elliptic operators P ; that is, the principal symbol $\sigma(P)$ of P satisfies

$$|\sigma(P)(x, i\eta)| \sim (|\eta'| + |x_n| \cdot |\eta_n|)^\ell \quad \text{near } \Sigma, \quad (3)$$

where ℓ is some positive integer and $\eta' = (\eta_1, \dots, \eta_{n-1})$. On the other hand, by using an elementary functorial construction of the sheaf $\tilde{\mathcal{C}}_V^2$ of small second microfunctions, the first author Funakoshi proved in [2] the solvability of those operators in the space of small second microfunctions as follows:

Theorem 1. *Let $P(x, \partial_x)$ be a differential operator with real analytic coefficients defined at $x = 0$. We suppose that*

$$|\sigma(P)(x, i\eta)| \sim (|\eta'| + |x_n|^k \cdot |\eta_n|)^\ell \quad \text{near } \Sigma \quad (4)$$

for some positive integers k, ℓ . Then we have a sheaf isomorphism:

$$\tilde{\mathcal{C}}_V^2 \xrightarrow{P} \tilde{\mathcal{C}}_V^2 \quad \text{on } \pi^{-1}(\Sigma).$$

Here, $\tilde{\mathcal{C}}_V^2$ is called the sheaf on $T_V^* \tilde{V}$ of small second microfunctions along V , which satisfies the following exact sequence:

$$0 \longrightarrow \mathcal{A}_V^2 \longrightarrow \mathcal{C}_{\mathbb{R}^n}|_V \longrightarrow \overset{\circ}{\pi}_* \tilde{\mathcal{C}}_V^2 \longrightarrow 0, \quad (5)$$

where $\overset{\circ}{\pi} : \overset{\circ}{T}_V^* \tilde{V} = T_V^* \tilde{V} \setminus V \rightarrow V$ is the canonical projection, $\tilde{V}$ is the partial complexification of V along each leaf of V , and

$$\mathcal{A}_V^2 := \mathcal{C}_{\tilde{V}}|_V = \mathcal{C}_{x_n} \mathcal{O}_{z'}|_V \quad (6)$$

is the sheaf on V of second analytic functions along V . Since any section of $\mathcal{A}_V^2$ has a unique continuation property along each leaf of V , Theorem 1 implies the above-mentioned result of Grigis-Schapira-Sjöstrand. However, to get a solvability result in microfunctions, Theorem 1 is not sufficient. We need a solvability result in $\mathcal{A}_V^2$. Though we have a general result due to Bony and Schapira [1] on solvability in $\mathcal{C}_{\tilde{V}}$ for non-micro-characteristic operators, our operators as in (4) do not fall in such a class of operators.

In this paper, we introduce some special class of differential operators satisfying the property (4), which admit the solvability in $\mathcal{A}_V^2|_\Sigma = \mathcal{C}_{\tilde{V}}|_\Sigma = \mathcal{C}_{x_n} \mathcal{O}_{z'}|_\Sigma$.

Our Main Results

Theorem 2. *Let $P(z, \partial_z)$ be a holomorphic differential operator written in the form:*

$$P(z, \partial_z) = \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \partial_{z'}^\alpha (z_n \partial_{z_n})^\beta. \quad (7)$$

Here the $C_{\alpha,\beta}$'s are complex constants satisfying

$$C_{0,m} \neq 0. \quad (8)$$

Then, the morphism

$$P : \mathcal{C}_{x_n} \mathcal{O}_{z'} \rightarrow \mathcal{C}_{x_n} \mathcal{O}_{z'}$$

is surjective on $\{x_n = 0\}$.

As a direct corollary of Theorem 1 and Theorem 2, we have

Theorem 3.

Let $P(z, \partial_z)$ be a holomorphic differential operator written as in (7). Suppose that

$$\left| \sum_{|\alpha|+\beta=m} C_{\alpha,\beta} (\eta')^\alpha (x_n)^\beta \right| \sim (|\eta'| + |x_n|)^m \quad (9)$$

for any real small vectors (η', x_n) . Then the morphism

$$P : \mathcal{C}_{\mathbb{R}^n} \rightarrow \mathcal{C}_{\mathbb{R}^n}$$

is surjective on Σ .

Indeed, condition (9) implies condition (8).

A Sketch of Proof of Theorem 2

Any germ f of $\mathcal{C}_{x_n} \mathcal{O}_{z'}$ at $(0, 0; i dx_n) \in \Sigma$ is written as a boundary value $F(z', x_n + i0)$ of a holomorphic function $F(z)$ in a domain

$$D_r = \{z \in \mathbb{C}^n; |z'| < r, |z_n| < r, \text{Im} z_n > 0\}. \quad (10)$$

Hence, our problem reduces to finding a holomorphic solution U of the following equation for any given $F(z)$ in a complex domain like D_r :

$$P(z, \partial_z)U(z) = F(z). \quad (11)$$

Step 1. Considering the Szegő kernel for a complex ball, we have a decomposition of $F(z)$ for a sufficiently small $r > 0$:

$$F(z) = \text{Const.} \int_{|w'|=r} \frac{F(w', z_n)}{(r^2 - z' \cdot \bar{w}')^n} dS(w'), \quad (12)$$

where $z' \cdot \bar{w}' = \sum_{j=1}^{n-1} z_j \bar{w}_j$ and dS is the surface element. Hence, it is sufficient to solve (11) for any holomorphic function F with continuous parameter w' of the following form:

$$F = G(z' \cdot \bar{w}', z_n; w'), \quad (13)$$

where $G(p, z_n; w')$ is a continuous function on

$$\{(p, z_n, w') \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}; |p| < r, |z_n| < r, \text{Im} z_n > 0, |w'| = r\} \quad (14)$$

depending holomorphically on (p, z_n) . Therefore, equation (11) reduces to the following one:

$$C_{0,m} \prod_{j=1}^m (z_n \partial_{z_n} - \varphi_j(w') \partial_p) \cdot U(p, z_n; w') = G(p, z_n; w'), \quad (15)$$

where $\{\varphi_j(w'); j = 1, \dots, m\}$ are the m -solutions of the algebraic equation

$$\sum_{|\alpha|+\beta=m} C_{\alpha,\beta} \bar{w}'^\alpha \phi^\beta = 0 \quad (16)$$

in ϕ .

Step 2. By solving first order equations in (15) successively, we get the final solution of (15). Hence, our problem is to solve the following first order equation:

$$(z_n \partial_{z_n} - \varphi_j(w') \partial_p) U(p, z_n; w') = G(p, z_n; w'). \quad (17)$$

In fact, if $\varphi_j(w') \neq 0$, we have a holomorphic solution of (17) of the form

$$U(p, z_n; w') = -\frac{1}{\varphi_j(w')} \int_{\tau(w')}^p G(s, z_n e^{(p-s)/\varphi_j(w')}; w') ds. \quad (18)$$

However this solution is not holomorphic in a domain like (14). To get a solution defined in a domain like (14), we must decompose G as

$$G(p, z_n; w') = G_+(p, z_n; w') + G_-(p, z_n; w'). \quad (19)$$

Here, roughly speaking, G_+ is holomorphic in

$$0 < \arg z_n < \pi + \epsilon$$

and G_- is holomorphic in

$$-\epsilon < \arg z_n < \pi$$

for some $\epsilon > 0$. Indeed taking $\tau_{\pm}(w')$ in the formula (18) as

$$0 < \mp \operatorname{Im} \left(\frac{\tau_{\pm}(w')}{\varphi_j(w')} \right) < \epsilon \quad (20)$$

respectively, we can show that the corresponding solutions $U_{\pm}$ are holomorphic in a domain like (14). The most difficult point of our problem is how to treat the case $\varphi_j(w') = 0$. This is not an exceptional problem because for almost all operators P the sets

$$\begin{aligned} & \{w' \in \mathbb{C}^{n-1}; |w'| = 1, \varphi_j(w') = 0 \text{ for some } j\} \\ &= \{w' \in \mathbb{C}^{n-1}; |w'| = 1, \sum_{|\alpha|=m} C_{\alpha,0} \bar{w}'^{\alpha} = 0\} \end{aligned}$$

are not void (but usually of real codimension ≥ 1). To overcome this difficulty, we use a good decomposition of G in (19) based on Hörmander's solution with L^2 -growth order for a $\bar{\partial}$ -equation in the whole $\mathbb{C}$. Before making such a decomposition we choose a better defining function $G(p, z_n : w')$. That is, by solving a Cousin problem on $\mathbb{C} \times \mathbb{P}^1$ with parameter w' , we can choose a better defining function $G(p, z_n : w')$, which is holomorphic on

$$\{p \in \mathbb{C}; |p| < r\} \times \{z_n \in \mathbb{P}^1; \operatorname{Im} z_n > 0 \text{ or } |z_n| > r\} \quad (21)$$

satisfying

$$G(p, \infty; w') = 0. \quad (22)$$

Here neglecting variables p, w' we consider a holomorphic function

$$H(\tau) = G(p, e^\tau; w') \quad (23)$$

in τ defined on

$$\{\tau \in \mathbb{C}; 0 < \operatorname{Im} \tau < \pi\}$$

with a growth order

$$|H(\tau)| < C e^{-\operatorname{Re} \tau}$$

as $\operatorname{Re} \tau$ goes to $+\infty$. Now we apply Hörmander's Theorem to the decomposition of H .

Hörmander's Theorem.

Let $\varphi(\tau)$ be a subharmonic function on $\mathbb{C}$. Then for any measurable function $h(\tau)$ satisfying

$$\iint_{\mathbb{C}} |h(\tau)|^2 e^{-\varphi(\tau)} dv(\tau) < \infty \quad (24)$$

we have a weak solution $f(\tau)$ of

$$\frac{\partial}{\partial \bar{\tau}} f(\tau) = h(\tau) \quad (25)$$

satisfying

$$\iint_{\mathbb{C}} |f(\tau)|^2 e^{-\varphi(\tau) - 2 \log(|\tau|^2 + 1)} dv(\tau) < \infty. \quad (26)$$

The detailed proof will be published elsewhere.

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