

ON THE STABLE MANIFOLDS OF NONWANDERING SETS

KUPKA-SMALE DIFFEOMORPHISMS

KEN SAWADA 沢田 賢

A diffeomorphism f is said to have the Kupka-Smale property if all periodic point of f are hyperbolic and for any periodic points p and q , the stable manifold $W^s(p)$ intersects the unstable manifold $W^u(q)$ transversely.

The purpose of this paper is to give a simple example of a diffeomorphism f with the Kupka-Smale property for which the set of periodic points of f , $\text{Per}(f)$ is dense in the nonwandering set of f , $\Omega(f)$ but

$\bigcup_{p \in \text{Per}(f)} W^s(p)$ is not dense in $W^s(\Omega(f))$.

EXAMPLE

Our example of a diffeomorphism f is defined on the annulus, a product of the interval $I = (1, 2)$ and the circle S^1 , by composing three diffeomorphisms ρ , G and H on $I \times S^1$ i.e., $f = \rho \circ G \circ H$.

First we define a diffeomorphism H .

Let $\{t_n\}$ be a sequence of rational numbers in I such that $t_1 < t_2 < \dots < t_n < \dots$ and $t_n \rightarrow \sqrt{2}$ as $n \rightarrow \infty$. We construct a orientation preserving diffeomorphism h on I as follows :

(1) t_n is a hyperbolic fixed point with $h'(t_n) > 1$ (< 1) for even n (odd n)

(2) $\sqrt{2}$ is a fixed point with $h'(\sqrt{2}) = 1$

(3) $h^m(t) \rightarrow \sqrt{2}$ as $m \rightarrow \infty$ if $t \in (\sqrt{2}, 2)$

$h^m(t) \rightarrow t_n$ as $m \rightarrow \infty$ for some odd n if $t \in (1, \sqrt{2}) - \bigcup_n \{t_n\}$.

where $h'(t)$ is the derivative of h at t .

We define a diffeomorphism H by the formula $H(t, \theta) = (h(t), \theta)$ for any $(t, \theta) \in I \times S^1$.

Next we define a diffeomorphism G .

Let g_k be the time one map of a flow ϕ_t on S^1 such that $\phi_t(\theta + 2\pi/k) = \phi_t(\theta) + 2\pi/k$ for any $0 \leq \theta \leq 2\pi$ and the flow has $2k$ -hyperbolic fixed points (see Figure 1). Now we construct a smooth isotopy g :

$I \times S^1 \rightarrow S^1$ which satisfies the followings :

$$(4) \quad g(t_n, \theta) = g_{k(n)}(\theta) \quad \text{for any } n.$$

$$(5) \quad g(\sqrt{2}, \theta) = \theta$$

$$(6) \quad \left| \frac{\partial}{\partial t} g(t, \theta) \right| < 2\pi \quad \text{for any } (t, \theta) \in I \times S^1$$

where $\{k(n)\}$ is a sequence of integers such that $k(1) < \dots < k(n) < \dots$

... and $k(n) \cdot t_n$ is an integer for any n ,

We define a diffeomorphism G by the formula $G(t, \theta) = (t, g(t, \theta))$ for any $(t, \theta) \in I \times S^1$.

Finally let ρ be a map such that for any $(t, \theta) \in I \times S^1$, $\rho(t, \theta) = (t, \theta + 2\pi \cdot t)$.

LEMMA. A diffeomorphism f has the Kupka-Smale property.

PROOF. First we show $\text{Per}(f)$ is dense in $\Omega(f)$. By the definition of h , $\Omega(f) \subset \bigcup_n \{t_n\} \times S^1 \cup \{\sqrt{2}\} \times S^1$. Since $f|_{\{\sqrt{2}\} \times S^1}$ is the irrational $(2\pi\sqrt{2})$ rotation, all elements in $\{\sqrt{2}\} \times S^1$ are nonwandering

points but not periodic points. In $\{t_n\} \times S^1$, $f(t_n, \theta) = (t_n, g_{k(n)}(\theta) + 2\pi \cdot t_n)$. Since $g_{k(n)}$ commutes with $2\pi/k(n)$ -rotation and $2\pi \cdot t_n = 2\pi N/k(n)$ for some integer N , $f^{k(n)}(t_n, \theta) = (t_n, (g_{k(n)})^{k(n)}(\theta))$. By the definition of $g_{k(n)}$, there are no nonwandering points of f except for $2k(n)$ -periodic points of f . Therefore $\Omega(f) = \{\sqrt{2}\} \times S^1 \cup (\bigcup_n (t_n, \theta_{n_j}))$ where θ_{n_j} is a hyperbolic fixed point of $g_{k(n)}$. Since $k(n) \rightarrow \infty$ as $n \rightarrow \infty$, $|\theta_{n_j} - \theta_{n_{j+1}}| \rightarrow 0$. Therefore $\text{Per}(f)$ is dense in $\Omega(f)$.

Next we show that periodic points are hyperbolic. From now on, we identify the tangent space of $I \times S^1$ at $x = (t, \theta)$, $T_x(I \times S^1)$ with $T_t(I) \times T_\theta(S^1)$. Then the derivative of f at x , $Df(x)$ is represented by

$$Df(x) = \begin{bmatrix} A(x), & 0 \\ C(x), & B(x) \end{bmatrix}$$

where $A(x) = h'(t)$, $B(x) = \frac{2}{2\theta} g(x)$ and $C(x) = h'(t)(2\pi + \frac{2}{\partial t} g(x))$.

By definitions of h and g , and by (6), $A(x)$, $B(x)$ and $C(x) > 0$.

Let $p = (t_n, \theta_{n_j}) \in \text{Per}(f)$ and ℓ be the period of p . Then $|A(p)| = |h'(t_n)| \neq 1$ and $|B(p)| = |\frac{d}{d\theta} g_{k(n)}(\theta_{n_j})| \neq 1$ by definitions of h and $g_{k(n)}$. Since

$$(7) \quad Df^\ell(p) = \begin{bmatrix} (A(p))^\ell, & 0 \\ (*), & (B(p))^\ell \end{bmatrix},$$

where $(*) > 0$, the eigenvalues of $Df^\ell(p)$, $(A(p))^\ell$ and $(B(p))^\ell$ are ones of which absolute values are not equal to 1. Hence p is hyperbolic.

We finally show that for $p, q \in \text{Per}(f)$, $W^u(p)$ intersects $W^s(q)$ transversely. It suffices to show in the case of p and q are saddle points. Hence we may assume that $p \in \{t_n\} \times S^1$ and $q \in \{t_{n+1}\} \times S^1$ for some even n (if p and q are saddle points contained in the same invariant circle, then $W^u(p)$ and $W^s(q)$ have no intersection point). Then $A(p) > 1$ and $A(q) < 1$. Let ℓ and ℓ' be periods of p and q respectively. Since $T_p(W^u(p))$ is the eigenspace corresponding to the eigenvalue $(A(p))^\ell$ of $Df^\ell(p)$, the slope of $v = (v_t, v_\theta) \in T_p(W^u(p))$, $v_\theta/v_t = (*) / [(A(p))^\ell - (B(p))^\ell] > 0$ from (7). By the same argument, for $w = (w_t, w_\theta) \in T_q(W^s(q))$, $w_\theta/w_t < 0$. If $W^u(p)$ and $W^s(q)$ have a nontransversal intersection point x , then for any $v' = (v'_t, v'_\theta) \in T_x(W^u(p))$, the slope of $Df^{m \cdot a}(x)(v') \rightarrow w_\theta/w_t$ as $m \rightarrow \infty$ where $a = \ell \cdot \ell'$.

On the other hand, since

$$Df^{m \cdot a}(x) = Df(x_{ma-1}) \cdots Df(x) = \begin{bmatrix} A(x_{ma-1}) \cdots A(x), & 0 \\ (**), & B(x_{ma-1}) \cdots B(x) \end{bmatrix}$$

where $x_i = f^i(x)$ and $(**) > 0$, the slope of $Df^{ma}(x)(v') = [B(x_{ma-1}) \cdots B(x)/A(x_{ma-1}) \cdots A(x)] \cdot v'_\theta/v'_t + (**)$. Taking x sufficiently near to p , $v'_\theta/v'_t > 0$. hence the slope of $Df^{ma}(x)(v') \rightarrow \infty$ as $m \rightarrow \infty$ since $A(x_i) \rightarrow A(q) < 1$ and $B(x_i) \rightarrow B(q) > 1$. Therefore $W^u(p)$ intersects $W^s(q)$ transversely.

For f , $\bigcup_{p \in \text{Per}(f)} W^s(p)$ is not dense in $W^s(\Omega(f))$ since $(\sqrt{2}, 2) \times S^1 \subset W^s(\{\sqrt{2}\} \times S^1)$ by (3) and $\{\sqrt{2}\} \times S^1 \subset \Omega(f) - \text{Per}(f)$.

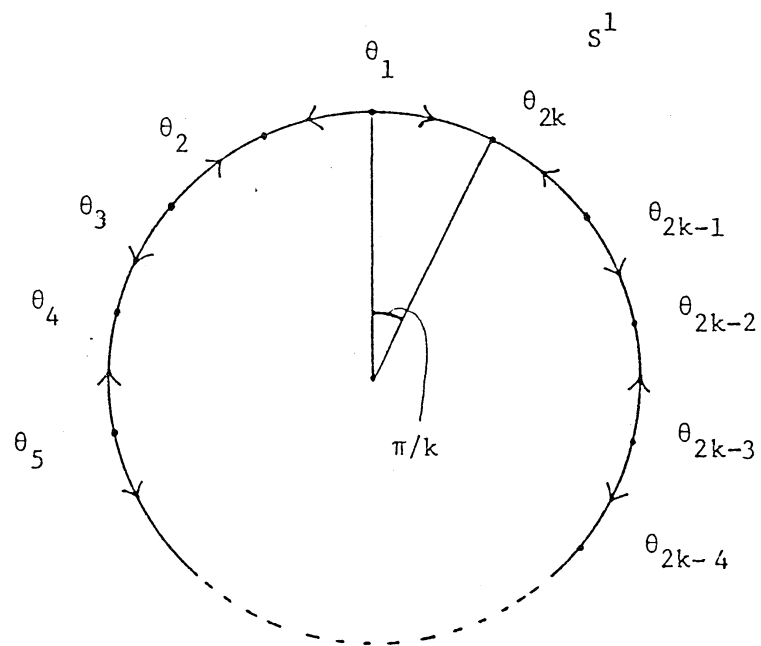


FIGURE 1