

Mackey functor and cohomology of finite groups

Akihiko Hida

Faculty of Education, Saitama University
(飛田明彦 埼玉大学教育学部)

1 Composition factors of Mackey functors

Let G be a finite group and k a field of characteristic $p > 0$. P. Symonds [4] determined the composition factors of the cohomology as a cohomological Mackey functor for G . We consider more detailed structure of $H^*(-, k)$.

First, we recall the definition of the Mackey functor for G .

Definition A cohomological Mackey functor M for G is the following specification.

- k -vector space $M(H)$ for $H \leq G$.
- k -linear map

$$I_K^H : M(K) \longrightarrow M(H)$$

$$R_K^H : M(H) \longrightarrow M(K)$$

$$c_g : M(H) \longrightarrow M({}^g H)$$

for $K \leq H \leq G$, $g \in G$ such that

(i) I_H^H, R_H^H, c_h ($h \in H$) : identity maps on $M(H)$.

(ii)

$$I_K^H I_J^K = I_J^H, R_J^K R_K^H = R_J^H, c_g c_h = c_{gh}$$

$$I_{{}^g K}^{{}^g H} c_g = c_g I_K^H, R_{{}^g K}^{{}^g H} c_g = c_g R_K^H$$

for $J \leq K \leq H \leq G$, $g, h \in G$.

(iii)

$$R_J^H I_K^H = \sum_{x \in J \backslash H/K} I_{J \cap {}^x K}^J R_{J \cap {}^x K}^{{}^x K} c_x$$

for $J, K \leq H \leq G$.

(iv)

$$I_K^H R_K^H = |H : K|$$

for $K \leq H$.

A global Mackey functor is a functor defined on all finite groups. Its restriction to a finite group G is a Mackey functor for G (see [7]).

Example (1) Let M be a kG -module. Then $H^n(-, M)$ is a cohomological Mackey functor for G .

(2) $H^n(-, k)$ is a global cohomological Mackey functor.

We can consider simple (global) Mackey functors and composition factors of Mackey functors. Simple cohomological Mackey functors for G are classified by Yoshida [8], Thévenaz-Webb [5]. They are parameterized by the pairs (P, V) , where P is a p -subgroup of G and V is a simple $kN_G(P)/P$ -module (up to iso. and conjugation). Let $S_{P,V}^G$ be the simple cohomological Mackey functor corresponding to the pair (P, V) .

On the other hand, simple cohomological global Mackey functors are classified by Webb [7]. They are parameterized by the pairs (P, V) , where P is a p -group and V is a simple $k(\text{Out}(P))$ -module (up to iso.).

Symonds [4] determines the composition factors of $H^*(-, k)$ as a (global) Mackey functor.

Theorem 1.1 ([4]) $H^*(-, k)$ contains every simple global cohomological Mackey functor as a composition factor.

Let P be a p -subgroup of G . If V be a simple $k(N_G(P)/PC_G(P))$ -module, then $N_G(P)/PC_G(P)$ is a subgroup of $\text{Out}(P)$ and there exists a simple $k(\text{Out}(P))$ -module W such that the restriction of W to $N_G(P)/PC_G(P)$ contains V as a composition factor. So we have the following corollary.

Corollary 1.2 Let P be a p -subgroup of G and V a simple $k(N_G(P)/P)$ -module. Then $H^*(-, k)$ contains the simple cohomological Mackey functor $S_{P,V}^G$ (for G) as a composition factor if and only if $C_G(P)$ acts trivially on V .

Remark (1) For Theorem 1.1, we need a result from topology :

$$A(P, P) \longrightarrow \{(BP_+)_p^\wedge, (BP_+)_p^\wedge\} \otimes k \longrightarrow \text{End}(H^*(P, k))$$

has nilpotent kernel, see [2].

(2) For Corollary 1.2, we have algebraic proofs (see [1], [3]).

2 Indecomposable direct summands of cohomology as a Mackey functor

Let G be a finite group. Cohomological Mackey functors for G are equivalent to the modules for a certain finite dimensional algebra, called cohomological Mackey algebra (see [6], [8]). We consider indecomposable direct summands of $H^n = H^n(-, k)$

as a cohomological Mackey functor for G .

Example 2.1 (1) Let G be a cyclic p -group. Then

$$H^n \simeq H^{n+2}$$

for $n > 0$.

(2) Assume that $p = 2$. Let G be a cyclic group of order 4. Then every conjugation is trivial and H^n is a module for the finite dimensional algebra Λ defined by the following quiver and relations :

$$0 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 1, \quad \alpha\beta = \beta\alpha = 0.$$

Then,

$$H^{2n-1} \simeq \begin{pmatrix} S_1 \\ S_0 \end{pmatrix}, \quad H^{2n} \simeq \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}$$

for $n > 0$, where S_i is a simple Λ -module corresponding to the vertex i .

Example 2.2 Let $p = 2$ and $G = C_2 \times C_2$. Then every conjugation and every transfer are trivial, so H^n ($n > 0$) is a module for the path algebra kQ ,

$$Q = \begin{array}{ccccc} & & 0 & & \\ & \swarrow & \downarrow & \searrow & \\ 1 & & 2 & & 3 \end{array}.$$

The cohomology algebra $H^*(G, k)$ is a polynomial algebra

$$H^*(G, k) = k[x_1, x_2]$$

where $\deg x_i = 1$. Let H_j ($j = 1, 2, 3$) be the subgroups of order 2. Then

$$H^*(H_j, k) = k[y_j]$$

where $\deg y_i = 1$. The restrictions from G to H_j are as follows :

$$\begin{array}{ccccc} & & x_1 & & x_2 \\ & \swarrow & & \searrow & \swarrow & \searrow \\ H^1 & & y_1 & & y_2 & & y_3 \\ & & & & & & \\ & & x_1^2 + x_1x_2 & & x_1x_2 & & x_1x_2 + x_2^2 \\ & & \downarrow & & \downarrow & & \downarrow \\ H^2 & & y_1^2 & & y_2^2 & & y_3^2 \end{array}$$

$$\begin{array}{ccccccccc}
& & x_1^3 + x_1^2 x_2 & x_1^2 x_2 & x_1 x_2^2 + x_2^3 & x_1^2 x_2 + x_1 x_2^2 & & & \\
H^3 & & \downarrow y_1^3 & \downarrow y_2^3 & \downarrow y_3^3 & \downarrow 0 & & & \\
& & x_1^4 + x_1^3 x_2 & x_1^3 x_2 & x_1 x_2^3 + x_2^4 & x_1^3 x_2 + x_1^2 x_2^2 & x_1^2 x_2^2 + x_1 x_2^3 & & \\
H^4 & & \downarrow y_1^4 & \downarrow y_2^4 & \downarrow y_3^4 & \downarrow 0 & \downarrow 0 & & \\
& & & & \dots\dots\dots & & & &
\end{array}$$

Let S_i ($0 \leq i \leq 3$) be simple kQ -modules. We have the following kQ -module structure of H^n .

$$\begin{aligned}
H^1 &\simeq \begin{pmatrix} S_0 & S_0 \\ S_1 & S_2 & S_3 \end{pmatrix} \\
H^2 &\simeq \begin{pmatrix} S_0 \\ S_1 \end{pmatrix} \oplus \begin{pmatrix} S_0 \\ S_2 \end{pmatrix} \oplus \begin{pmatrix} S_0 \\ S_3 \end{pmatrix} \\
H^3 &\simeq H^2 \oplus S_0 \\
H^4 &\simeq H^2 \oplus S_0 \oplus S_0 \\
&\dots\dots\dots
\end{aligned}$$

Definition Let Λ be a finite dimensional k -algebra and M_n ($n \geq 0$) finitely generated Λ -modules. Let $\text{Ind}(\bigoplus M_n)$ be the set of isomorphism classes of the indecomposable direct summands of M_n ($n \geq 0$). Namely

$$\text{Ind}(\bigoplus M_n) = \{ \text{indec. direct summands of } \bigoplus M_n \} / \simeq .$$

Remark To show that $\text{Ind}(\bigoplus H^n)$ is a finite set for a finite group G , we may assume that k is a finite field.

The fact that $\text{Ind}(\bigoplus H^n)$ is a finite set for the elementary abelian 2-group of order 4 (Example 2.2) is explained by the following Lemma.

Lemma 2.3 Let k be a finite field and Λ a finite dimensional k -algebra. Let $N_n \subseteq M_n$, ($n \geq 0$) be finitely generated Λ -modules. Suppose that $\text{Ind}(\bigoplus (M_n/N_n))$ is finite and there is $d > 0$ such that $\dim N_n \leq d$ for any n . Then $\text{Ind}(\bigoplus M_n)$ is a finite set.

Using this Lemma and its dual, we have the following.

Example 2.4 Let $G = C_p \times C_p$ or $G = C_p \times C_p \times C_p$. Then $\text{Ind}(\bigoplus H^n)$ is a finite set.

In this example, we do not know the explicit structure of H^n . On the other hand, for elementary abelian p -groups of arbitrary rank, we do not know even whether $\text{Ind}(\bigoplus H^n)$ is a finite set or not.

Question If G is an elementary abelian p -group, then $\text{Ind}(\bigoplus H^n)$ is a finite set?

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